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Reducing the negative effects of random tie-breaking in student allocation mechanisms

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Contents

Abstract

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This thesis introduces two new methods to reduce the negative effects that are caused by using randomness as a selection criterion in a centralized allocation problem of students to schools. The first method, the Waste-Reducing Lottery Design (WRLD) procedure, reduces the number of available seats that are not assigned to any student due to random tie-breaking. The second method, the Maximin decomposition, reduces the uncertainty about the total number of students that will be assigned to a school. Both methods obtain their objective by determining the probability with which an allocation of students to schools will be selected as the final allocation. As both methods are applicable to all mechanisms that adopt random tie-breaking, they provide a general framework for decision-makers to improve upon the currently used mechanisms. The performance of the introduced methods is evaluated both on real-world data from Antwerp and Ghent, and on generated data.

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List of abbreviations

DA	Deferred Acceptance
IC	Improvement Cycle
IP	Integer Programming
LB	Lower bound
LOP	Lokaal overlegplatform
LP	Linear Programming
MTB	Multiple Tie-Breaking
PS	Probabilistic Serial
RDA	Randomized Deferred Acceptance
r-PSP	r-partially strategy-proofness
RSD	Random Serial Dictatorship
SCC	Strongly Connected Component
SIC	Stable Improvement Cycle
STB	Single Tie-Breaking
TTC	Top Trading Cycle
UB	Upper bound
URBI	Uniformly relatively bounded indifference
WCD	Worst-case difference
WRLD	Waste-Reducing Lottery Design

Chapter 1

Introduction

Today, many aspects of life are already being decided for us by algorithms: the music we listen to, the movies we watch or how we can get from point A to B in the best possible way. But in recent years, algorithms have also been used to decide which school a student can attend if school capacities are insufficient. This approach offers multiple advantages, such as an increase in the levels of transparency and of fairness and a decrease in segregation. However, as the final allocation that is determined by the algorithm plays a crucial role in students' future, it is important that the way in which this decision is taken and the possibilities in which it could be improved are investigated thoroughly.

The student allocation problem has received broad attention in recent years, both in the academic world and in the popular press. In 2012, for example, the Nobel Prize in Economic Sciences was awarded to Alvin Roth and Lloyd Shapley, two scholars who have extensively studied the problem of matching different agents in the best possible way (The Royal Swedish Acadamy of Sciences, 2012). Also in the Flemish political debate, the centralized allocation system of students to schools has been a widely discussed topic.

The aim of this thesis is twofold. On the one hand, this thesis aims to provide an overview of the existing literature to evaluate how and to what extent the currently used mechanisms can be improved upon. On the other hand, this thesis proposes two new methods to reduce the negative effects that are caused by using randomness as a selection criterion if schools are indifferent between students. In practice, the allocation mechanisms that are adopted for secondary education in Flanders rely heavily on randomness because other criteria such as prior grades or the distance between the school and the student's house or the parent's workplace are prohibited. Randomness is also used for primary education, but to a lesser extent as distance is the main selection criterion in this context. The main argument in favour of using randomness as a selection criterion is the fairness it implies, as two students who submit the same preference list will have the same chance of being assigned to the school of their choice. Unfortunately, randomness also implies uncertainty about the final result, as certain final allocations will be more preferred by students than others.

The two methods introduced in this thesis each tackle a different negative implication of using randomness. The first method, the *Waste-Reducing Lottery Design* (WRLD) procedure, aims to reduce the number of available seats that are not assigned to any student by a mechanism that adopts random tie-breaking. The aim of the second method, the Maximin decomposition, is to reduce the uncertainty about the final number of assigned students by an allocation mechanism that adopts random tie-breaking. Both methods obtain these results by determining the probability with which each possible way of randomly breaking ties between students will be selected as the final way of breaking ties, which will then determine which students can go to which schools. Moreover, both methods can be applied to all allocation mechanisms that use random tie-breaking.

This thesis is structured in the following way. The remainder of Chapter 1 provides a literature review and an overview of the regulation on school choice in Flanders. Chapter 2 contains a formal problem statement and introduces two methods to reduce the negative effects of using randomness as a selection criterion, namely the WRLD procedure and the Maximin decomposition. The performance of both methods is evaluated in Chapter 3, based on real-world data sets of Antwerp and Ghent and on generated data. Chapter 4 contains some considerations on the implementation of both methods and Chapter 5 concludes.

1.1 Problem statement

The problem that will be considered in this thesis is how to assign students to schools. A first input required to make this decision is the submitted *preference lists* of the students over the schools. In an example[1] with four students and three schools, in which each school has only one available seat, the preferences of the students could be represented as:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$	$>_{c_4}$
s_1	s_3	s_3	s_3
s_2	s_1	s_2	0
s_3	0	0	0

The preference list of student i is represented by $>_{c_i}$, in which the element on the first row represents the first choice of the student, the element on the second row the second choice, etc. The element 0 in the preference list indicates that a student prefers the outside option to being assigned to the schools that are not present in the list. This outside option could refer to a school that is not included in the centralized allocation system or to homeschooling. Student 2, for example, submitted school 3 as the first choice, school 1 as the second choice and prefers the outside option to being assigned to school 2.

[1] This example is almost identical to an example from Erdil (2014).

Secondly, schools can also have *priorities* over the students. In the same example, these priorities could be represented as:

$>_{s_1}$	$>_{s_2}$	$>_{s_3}$
c_2	c_3	c_1
c_1	c_1	c_4
c_3	c_2	c_2
c_4	c_4	c_3

Similarly to the preference lists, $>_{s_j}$ denotes the priority list of school j. For example, school 1 prefers student 2 to be assigned to the school, rather than student 1, etc. In this example, it is assumed that schools are never indifferent between students (*strict* priorities). In Flanders, however, schools do not have strict priorities as they are not allowed to prioritize students based on distance or prior grades (Onderwijs Vlaanderen, 2012*b*). Therefore, this assumption will be relaxed further on.

For this particular example, one possible final *matching* between students and schools could be represented as:

student	**school**	**preference**
c_1	s_1	1
c_2	s_3	1
c_3	s_2	2
c_4	0	0

(1.1)

Equivalently, a more compact notation of this matching is $(s_1, s_3, s_2, 0)$. This matching assigns student 1 to school 1, which is his/her first choice. Student 2 is assigned to school 3, student 3 to school 2 and student 4 is not assigned to any school in this solution.

In designing a method that obtains such a matching, several desirable criteria related to the welfare of the students could be aimed for. First of all, a *Pareto efficient* matching is a matching in which it is impossible for two or more students to exchange their allocated schools and all be better off. The matching in (1.1), for example, is Pareto efficient as no student can be allocated to a school of higher preference without making at least one student worse off. However, the following matching is not Pareto efficient as it is possible for students 1 and 2 to exchange their allocated schools and both be better of.

student	**school**	**preference**
c_1	s_3	3
c_2	s_1	2
c_3	s_2	2
c_4	0	0

(1.2)

Secondly, a *stable* matching is a matching in which no justified envy exists. This means that there is no student who prefers another school to his/her current assignment and who has a higher priority on that school than at least one of the admitted students (Abdulkadiroğlu and Sönmez, 2003). It can be noted that the matching in (1.2) is stable. The matching in (1.1), on the other hand, is not stable as student 4 prefers being assigned to school 3 to not being assigned at all, while, at the same time, student 4 has a higher priority on school 3 than the currently assigned student 2.

So far, only the final matchings have been discussed. A method that obtains such a matching is called a *mechanism*. If both student preferences and school priorities are taken into consideration in a mechanism, it is called a *two-sided mechanism*, whereas a mechanism that only considers student preferences is called a *one-sided mechanism*. A mechanism is called Pareto efficient (stable) if it always results in a Pareto efficient (stable) matching.

Lastly, a mechanism is *strategy-proof* if it is a dominant strategy for the students to submit their true preferences, regardless of the priorities of the schools and the preferences submitted by the other students.

It has been shown, however, that it is impossible to design a mechanism that satisfies all desirable properties and that trade-offs will have to be made. As illustrated in this example, for instance, Pareto efficiency and stability are generally not compatible (Roth, 1982; Abdulkadiroğlu and Sönmez, 2003). In general terms, the aim of this thesis is to discuss and propose different mechanisms to realize improvements with respect to certain desirable properties compared to the allocations found by the *traditional* algorithms, which will be discussed in Section 1.2.1. These improvements, however, will generally come at the cost of a decrease in another desirable property. Therefore, in order to be able to assess the attractiveness of the discussed improvement mechanisms, the size of these negative implications will be clearly discussed in Chapter 3.

1.2 Literature review

1.2.1 Traditional algorithms

Abdulkadiroğlu and Sönmez (2003) were the first to address the shortcomings of widely used student assignment mechanisms such as the *Boston* mechanism (described in detail in Appendix A.1). They showed that this mechanism results in a matching that is Pareto efficient, but neither stable, nor strategy-proof, as students have a strong incentive to give a high preference to schools for which they have a high chance of getting accepted.[2] In order to tackle these issues, two algorithms were proposed for practical implementations, namely the *Deferred Acceptance* (DA) mechanism, developed by Gale and Shapley (1962), and the *Top Trading Cycle* (TTC) mechanism, developed by Shapley and Scarf (1974).

[2]See also Ergin and Sönmez (2006), Pathak and Sönmez (2008) and Dur et al. (2018).

Gale and Shapley (1962) originally developed the Deferred Acceptance (DA) mechanism in the context of the *Stable Marriage* problem, a one-to-one matching problem where men and women are matched based on their preferences. Nevertheless, the mechanism can be easily extended to a many-to-one matching (sometimes called *Hospitals/Residents* problem (Manlove, 2013)), in which one student is assigned to at most one school, whereas each school can potentially be assigned multiple students. Two versions of the Deferred Acceptance (DA) mechanism were developed, namely the Student-proposing and the School-proposing DA. The former maximizes the welfare of the students among all stable matchings, whereas the latter focuses on the welfare of the schools (Gale and Shapley, 1962). In the context of Flanders, however, because school priorities are often created artificially (see Sections 1.2.2 and 1.3), student welfare is perceived as more important than school welfare. Therefore, DA will simply refer to the Student-proposing DA in the remainder of this thesis.[3]

The Deferred Acceptance (DA) mechanism proceeds in the following way (Kesten, 2010):

- In the first step, each student applies to his/her most preferred school. If the number of applicants on a certain school is higher than the capacity of that school, the students with the highest priorities among the applicants are *temporarily* allocated to that school and the others are rejected.
- In general, in the k-th step, each student who was rejected at step $k-1$ applies to his/her school of next choice. If the number of applicants and temporarily allocated students on a certain school is higher than the capacity of that school, the students with the highest priorities among both the applicants and the temporarily allocated students are temporarily allocated to that school and the others are rejected.

The DA algorithm terminates when no student is rejected in a certain step. This is the case when all students are either assigned to a school from their preference list, or have been rejected by all schools on their preference list and have no more schools to apply to.

Example 1.2.1. To illustrate the DA algorithm, consider the example from Section 1.1. The following table displays the intermediate matchings in every step of the algorithm. For every step, the first column represents the school to which the corresponding student has applied and the second column represent the position of that school in his/her preference list. When a student is temporarily assigned to a school in a certain step, that school is shown in a box.

student	**step 1**		**step 2**		**step 3**		**step 4**		**result**	
c_1	$\boxed{\mathbf{s_1}}$	**1**	s_1	1	s_2	2	$\boxed{\mathbf{s_3}}$	**3**	$\boxed{\mathbf{s_3}}$	**3**
c_2	s_3	1	$\boxed{\mathbf{s_1}}$	**2**	$\boxed{\mathbf{s_1}}$	**2**	$\boxed{\mathbf{s_1}}$	**2**	$\boxed{\mathbf{s_1}}$	**2**
c_3	s_3	1	$\boxed{\mathbf{s_2}}$	**2**	$\boxed{\mathbf{s_2}}$	**2**	$\boxed{\mathbf{s_2}}$	**2**	$\boxed{\mathbf{s_2}}$	**2**
c_4	$\boxed{\mathbf{s_3}}$	**1**	$\boxed{\mathbf{s_3}}$	**1**	$\boxed{\mathbf{s_3}}$	**1**	s_3	1	0	0

[3]The School-Proposing Deferred Acceptance mechanism is described in detail in Appendix A.2.

In the first step, student 4 will be temporarily allocated to school 3 as his/her priority on that school was the highest of all applying students, and student 1 will be temporarily allocated to school 1, as (s)he was the only applicant to school 1 and capacities were not violated. This causes rejected students 2 and 3 to apply to their school of second choice in the second step, etc. The algorithm terminates after the fourth step as student 4 was rejected on all schools of his preference list and prefers the outside option to applying to another school.

The key difference between DA and the Boston mechanism is that in the Boston mechanism, every assignment is permanent. In DA, on the other hand, schools verify in every step whether there are applicants with a higher priority than one of the *temporarily* assigned students. In this example, this difference is present in step 2, as in the Boston algorithm, student 2 would not have replaced student 1 on school 2.

The DA mechanism is widely used because of its favourable properties. When both student preferences and school priorities are strict, i.e. no ties exist, the mechanism produces the unique matching that is Pareto-optimal from the perspective of the students in the set of stable matchings (Gale and Shapley, 1962). This means that every student weakly prefers his/her assigned school under DA to the result of every other stable matching. Moreover, Dubins and Freedman (1981) and Roth (1982) showed that DA is strategy-proof as truthful revelation of student preferences is a dominant strategy, even when ties exist and an arbitrary tie-breaking rule is adopted. However, the matchings obtained by DA are not Pareto efficient in general, as possibilities may exist for two or more students to exchange their allocated schools and all be better off.

The Top Trading Cycle (TTC) mechanism, on the other hand, was developed by Shapley and Scarf (1974) and further studied in the context of student assignment by Abdulkadiroğlu and Sönmez (2003). TTC is a strategy-proof mechanism that produces a Pareto efficient matching. However, the produced matching is not stable and given the fact that stability is an important criterion from a juridical point of view, this mechanism is less often adopted in practice. A detailed description of TTC can be found in Appendix A.3.

1.2.2 Ties and tie-breaking

The strict school priorities, as described in the previous section could be the result of, for example, previously obtained grades or the distance from the school to the student's house or to the parent's workplace. In general, however, some schools might be indifferent between certain groups of students. As a consequence, ties between students might exist and school priorities will no longer be strict. In Flanders, for example, both criteria are prohibited (Onderwijs Vlaanderen, 2012*b*), as will be discussed in more detail in Section 1.3. In order to apply mechanisms such as DA or TTC that require strict school priorities, ties in the priority lists have to be broken by fixing an order of the students. In practice, this order is often randomly chosen, as this is generally perceived as the most fair method (Bogomolnaia and Moulin, 2001). In the context of DA, this procedure is also referred to as *Randomized Deferred Acceptance* (RDA) (Erdil, 2014).

As a tie-breaking procedure creates artificial stability constraints that will never be violated by stable allocation mechanisms (such as DA), it may harm student welfare. It might be possible that a matching is Pareto-dominated by another matching which is stable with respect to the true, non-strict priority structure of the schools, but not with respect to the strict priority structure after randomly breaking the ties.

This loss in student welfare can be illustrated by the example from Section 1.1. Imagine that school 3 is in fact indifferent between all students. In order to apply DA, priorities have to be strict and, therefore, ties have to be broken (randomly). Suppose that the resulting priorities of school 3 are $>_{s_3}$ as shown in Section 1.1. The matching $(s_3, s_1, s_2, 0)$, as shown in (1.2) is the result of DA on this artificially created priority structure. However, the matching $(s_1, s_3, s_2, 0)$, as shown in (1.1), Pareto-dominates this matching while still being stable with respect to the true, non-strict priority structure before ties were randomly broken. In this example, the loss in student welfare caused by random tie-breaking is equal to the fact that student 1 and 2 are allocated to their third and second preference, respectively, instead of being allocated to their most preferred school.

Several authors have studied the negative effects of tie-breaking and have proposed attempts to overcome them. First of all, Erdil and Ergin (2008) showed the possibly significant welfare consequences of tie-breaking and developed a polynomial-time algorithm to find Pareto efficient improvement for stable matchings that preserve stability by finding so-called *stable improvement cycles*. In this manner, students are re-allocated to more preferred schools (based on Gale's Top Trading Cycle mechanism (Shapley and Scarf, 1974)) while making sure that, for each school, all re-allocated students have a priority (after tie-breaking) that is higher than that of the student with the lowest assigned priority on that school. A different algorithm to achieve the same objective has been proposed by Kesten (2010). In his solution, students who *block* a possible Pareto efficient exchange of assigned schools among other students because their priority on one of these schools is higher, can consent to abandon their place on the priority list of that school. This decision will cause the consenting student no harm, but it may facilitate Pareto efficient improvements for the other students.

However, Erdil (2014) has proven that if a strategy-proof method to improve the efficiency of a strategy-proof mechanism (e.g. DA with random tie-breaking) exists, it must allocate strictly more students. Therefore, a method that relies on re-allocating students, such as the methods proposed by Erdil and Ergin (2008) or Kesten (2010), can never be strategy-proof.

In recent years, many articles have appeared on the design of a tie-breaking mechanism. If the ordering of the students is identical for all schools, it is denoted as *Single Tie-Breaking* (STB), whereas a different ordering of the students on each school is called *Multiple Tie-Breaking* (MTB). Firstly, both Abdulkadiroğlu et al. (2009) and de Haan et al. (2015) gave empirical evidence, based on real-life data, that STB causes more students to be assigned to their top choice than MTB, while at the same time STB causes more students to not be assigned at all. Furthermore, Ashlagi and Nikzad (2016) argue that the results depend on the market conditions. Their results indicate that in a market where the number of school seats exceeds the number of students, MTB is more equitable than STB and efficiency trade-offs exist. In a market where school capacities are binding, on the other hand, STB outperforms MTB. Ashlagi and Nikzad therefore suggest to adopt a common ordering of students on popular schools and different student orderings on non-popular schools. In the context of TTC, however, Pathak and Sethuraman (2011) argue that STB is equivalent to MTB, although MTB is perceived as *more fair* by the students as the final matching does not depend on one single draw.

All mechanisms that have been discussed above are referred to as *two-sided* mechanisms, as they take into consideration both student preferences and school priorities. Another way to deal with indifferences in school priorities that is closely related to random tie-breaking, is to apply *one-sided* matching mechanisms. These mechanisms only consider the preferences of the students and no longer take into account the priorities of the schools as they are assumed not to exist or to be of lesser importance than student preferences. When the schools have fixed capacities, this type of problem is often referred to in the literature as the *Capacitated House Allocation* problem (Manlove, 2013).

The most widely used one-sided matching mechanism in the context of student allocation is the *Random Serial Dictatorship* (RSD)[4] mechanism, introduced by Abdulkadiroğlu and Sönmez (1998). This mechanism produces a Pareto efficient and strategy-proof matching by assigning the randomly ordered students one by one to the first school in their preference list that still has seats available.

Example 1.2.2. Reconsider, for instance, the example from Section 1.1. Imagine the random order of the students is $c_1 > c_2 > c_3 > c_4$. The following table displays every step of the RSD algorithm for this order. For every step, the first column represents the school to which the corresponding student was assigned and the second column represent the position of that school in his/her preference list.

student	**step 1**		**step 2**		**step 3**	
c_1	s_1	1	s_1	1	s_1	1
c_2	0	0	s_3	1	s_3	1
c_3	0	0	0	0	s_2	2
c_4	0	0	0	0	0	0

[4]Sometimes also referred to as the Random Priority (RP) mechanism (e.g. Bogomolnaia and Moulin (2001)).

Firstly, students 1 and 2 will be assigned to their first choices, schools 1 and 3, respectively. Student 3 is the next student in the order, but as school 3, which is his/her first choice, has no more seats available, (s)he will be assigned to school 2. As all available seats are assigned, student 4 cannot be assigned to any school.

Pathak and Sethuraman (2011) and Carroll (2014) have proven that RSD is equivalent to TTC with random tie-breaking (STB or MTB). Moreover, RSD is equivalent to DA when ties are broken in the same way for all schools (STB).

For further information, Manlove (2013) provides an elaborate overview of the different types of matching problems and the possible algorithms to tackle these from a more computational perspective.

Most mechanisms that are being used in practical student allocation problems are (variants of) the mechanisms that have already been described in this thesis, possibly adapting one of the efficiency improvements mentioned in this section. It has been shown, however, that these mechanisms are still subject to more subtle efficiency losses than the ones already mentioned. These alternative efficiency losses, and different ways to overcome them, will be the topic of the following section.

1.2.3 Probabilistic assignment mechanisms

When ties are broken randomly, students actually face certain probabilities of being allocated to a school. Therefore, the previously mentioned mechanisms with random tie-breaking are also referred to as *lottery mechanisms*, as they induce a probability distribution over deterministic assignments (Kesten et al., 2017).

Unlike lottery mechanisms, *probabilistic mechanisms*[5] are mechanisms that obtain these allocation probabilities directly, and not as a weighted average over all deterministic assignments. A probabilistic mechanism typically consists of three mains steps. Firstly, a probability matrix is obtained that contains the allocation probabilities for all student-school pairs. Different methods exist to obtain this probability matrix and they will be discussed below. In a second step, this probability matrix is rewritten as a weighted sum of deterministic assignment matrices. This transformation is called a Birkhoff-von Neumann decomposition (Birkhoff (1946); von Neumann (1953)) and their theorem states that any matrix in which both the sum of each row and the sum of each column are equal to one (a bistochastic matrix), can be decomposed into a weighted sum of deterministic assignment matrices in which each row, as well as each column contains at most one element that is equal to one. Kojima and Manea (2010) showed that this result can be extended to the context of student assignment where school capacities are larger than one. Therefore, every allocation probability matrix can be rewritten as a (not necessarily unique) weighted sum of deterministic assignments. In the last step, a lottery is

[5] Sometimes also referred to as *stochastic mechanisms* (e.g. Erdil (2014)).

performed over these deterministic assignments to determine the final matching. In this lottery, the selection probabilities of the previously obtained deterministic assignments are equal to the corresponding weights.

Example 1.2.3. To illustrate this procedure, reconsider the example that was introduced in Section 1.1. To obtain the allocation probabilities of RSD, all $4! = 24$ possibilities in which ties can be broken have to be considered. This would result in the following allocation probability matrix, in which element (i, j) represents the probability that student i is allocated to school j:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 9/12 & 3/12 & 0 \\ 3/12 & 0 & 4/12 \\ 0 & 8/12 & 4/12 \\ 0 & 0 & 4/12 \end{pmatrix} \end{array}$$

This allocation probability matrix can be decomposed into the following deterministic assignments:

$$4/12 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 4/12 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 3/12 \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 1/12 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

As a matter of fact, when ties are broken randomly, the previously mentioned *traditional* algorithms such as RDA and RSD could be considered as probabilistic mechanisms in which no decomposition is required because the initial probability matrix is already the weighted sum of deterministic assignments. However, Bogomolnaia and Moulin (2001) noted that, although every matching that is obtained by RSD is Pareto-optimal, possibilities for efficiency improvements exist when considering the allocation probability matrix. Consider an example[6] with four students, two schools with one available seat each and the following preferences:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$	$>_{c_4}$
s_1	s_1	s_2	s_2
s_2	s_2	s_1	s_1

When considering all 24 possibilities of breaking ties, the resulting allocation probabilities of RSD are equal to:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{pmatrix} 5/12 & 1/12 \\ 5/12 & 1/12 \\ 1/12 & 5/12 \\ 1/12 & 5/12 \end{pmatrix} \end{array}$$

[6]Budish et al. (2013)

It can be noted that, under the RSD mechanism, all students have a positive probability of being assigned to their school of second choice. If it would be possible to exchange shares of allocation probabilities, all students would be better off if students 1 and 2 exchanged their $1/12$ shares of s_2 with the $1/12$ shares of s_1 of students 3 and 4. In that case, the resulting allocation probabilities would be equal to:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{pmatrix} \end{array}$$

Bogomolnaia and Moulin (2001) concluded that a probabilistic assignment is *ordinally efficient*[7] if no other probabilistic assignment exists that is preferred by all students. Although RSD is *ex-post Pareto efficient*, as it will always produce Pareto efficient matchings, the example shows that it is not ordinally efficient. To tackle this issue, they created the *Probabilistic Serial* (PS) mechanism, which will always produce an ordinally efficient allocation probability matrix. In the PS mechanism, all students are considered to *eat* fractions of the schools with the same *eating speed*. This eating speed is equal to the uniform consumption of one school seat in one time unit. The algorithm then proceeds in the following way:

> Time runs continuously from 0 to 1. At each point in time, every student *eats* with a uniform eating speed from his/her most preferred school among those that have not yet been completely eaten up. At time $t = 1$, the resulting fractions of the schools that have been eaten by a student can be interpreted as his/her allocation probabilities to these schools (Bogomolnaia and Moulin, 2001; Budish et al., 2013).

Example 1.2.4. To illustrate the working of the PS algorithm, reconsider the example from Section 1.1. Below, the intermediate allocation probability matrices are shown at the points in time when a school was eaten up entirely and some students had to start eating from another school. The recently finished school is indicated by a box in the column title, and the schools from which students have been eating right before the indicated points in time are displayed in bold in the matrix.

$$\mathbf{t = 0.33} \qquad \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & \boxed{s_3} \end{array} \\ \begin{pmatrix} \mathbf{1/3} & 0 & 0 \\ 0 & 0 & \mathbf{1/3} \\ 0 & 0 & \mathbf{1/3} \\ 0 & 0 & \mathbf{1/3} \end{pmatrix} \end{array}$$

$$\mathbf{t = 0.67} \qquad \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} \boxed{s_1} & s_2 & s_3 \end{array} \\ \begin{pmatrix} \mathbf{2/3} & 0 & 0 \\ \mathbf{1/3} & 0 & 1/3 \\ 0 & \mathbf{1/3} & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix} \end{array}$$

$$\mathbf{t = 1} \qquad \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & \boxed{s_2} & s_3 \end{array} \\ \begin{pmatrix} 2/3 & \mathbf{1/3} & 0 \\ 1/3 & 0 & 1/3 \\ 0 & \mathbf{2/3} & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix} \end{array}$$

[7]Sometimes referred to as *sd-efficiency* (e.g. Kesten et al. (2017)). In a more general setting (e.g. with cardinal instead of ordinal preference structures), this concept is also referred to as *ex-ante efficiency* (e.g. Hylland and Zeckhauser (1979) or Kesten and Ünver (2015)).

At $t = 0.33$, school 3 is eaten up entirely. This causes students 2 and 3 to start eating from their school of second choice and student 4 to stop eating as school 3 was the only school in his/her preference list. The same reasoning can be applied for $t = 0.67$. The final allocation probabilities are obtained at $t = 1$.

The PS mechanism, however, is not entirely strategy-proof, as will be discussed in more detail in Section 1.2.5. In order to obtain a final matching from these probabilities obtained by PS, a decomposition followed by a lottery, as mentioned in the beginning of this section, must be performed.

A mechanism that also considers efficiency with respect to the allocation probability matrix was previously developed by Hylland and Zeckhauser (1979). However, their proposal was designed for a context in which objects are valued on a certain scale (referred to as cardinal or von Neumann-Morgenstern preferences), rather than simply ordered from most to least preferred (ordinal preferences), as is the case in the student assignment problem.

In contrast to previous results, Erdil (2014) found that it is possible to find strategy-proof efficiency improvements for the matchings resulting of DA and RSD when ties are broken randomly. As mentioned in Section 1.2.2, he proved that the only possible strategy-proof improvement over a strategy-proof mechanism can be made by allocating strictly more students, in contrast to the non-strategy-proof improvements based on re-allocating students by Erdil and Ergin (2008) and Kesten (2010). Moreover, he states that such an improvement can only be realized if a stochastic assignment is *wasteful*, namely if the sum of the allocation probabilities for a certain school is smaller than the available capacity and there exists at least one student who prefers that school to another school (or the outside option) to which (s)he is assigned with a strictly positive probability.

Example 1.2.5. This can be illustrated by considering, once again, the initial example from Section 1.1. The allocation probabilities of RSD, when considering all 24 possible ways of tie-breaking, are equal to:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.75 & 0.167 & 0 \\ 0.25 & 0 & 0.33 \\ 0 & 0.625 & 0.33 \\ 0 & 0 & 0.33 \end{pmatrix} \end{array}$$

Note that the sum of the allocation probabilities for school 2 is smaller than one, which means that with a probability of 20.83% or $5/24$ no student is assigned to school 2. However, both students 1 and 3 face a positive probability of not being assigned to any school, while they both prefer being assigned to school 2 to not being assigned at all. Therefore, in this example, RSD with random tie-breaking is wasteful. Martini (2016) showed that non-wastefulness is an ex-ante efficiency concept that is weaker than the concept of ordinal efficiency from Bogomolnaia and Moulin (2001).

Erdil's solution consists of replacing certain random orderings of students, and the corresponding matchings, by others orderings, in such a way that no student's allocation probabilities decrease and that the allocation probabilities of at least one student are improved. Consider the following ways of tie-breaking and the resulting matchings from RSD:

Allocation 1	**Allocation 3**
$c_4{>}c_3{>}c_2{>}c_1$ or $c_4{>}c_2{>}c_3{>}c_1$	$c_3{>}c_2{>}c_1{>}c_4$ or $c_3{>}c_2{>}c_4{>}c_1$
$(0, s_1, s_2, s_3)$	$(s_2, s_1, s_3, 0)$
Allocation 2	**Allocation 4**
$c_3{>}c_1{>}c_2{>}c_4$ or $c_3{>}c_1{>}c_4{>}c_2$	$c_4{>}c_1{>}c_3{>}c_2$ or $c_4{>}c_1{>}c_2{>}c_3$
$(s_1, 0, s_3, 0)$	$(s_1, 0, s_2, s_3)$

If the indicated ways of breaking ties that lead to allocations 1 and 2 would be replaced by the ones that lead to allocations 3 and 4, then students 2, 3 and 4 would not experience a difference. Student 3, for example, will be assigned to school 2 in two of the four considered allocations and to school 3 in the other two, under both scenarios. Student 1, on the other hand, would experience an improvement, as (s)he will now be assigned to school 2 in two additional random draws. By replacing the random orders that lead to allocations 1 and 2 by the ones that lead to allocations 3 and 4, while keeping all other random orders unchanged, the new allocation probability matrix is:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.75 & \mathbf{0.25} & 0 \\ 0.25 & 0 & 0.33 \\ 0 & 0.625 & 0.33 \\ 0 & 0 & 0.33 \end{pmatrix} \end{array}$$

Because of this replacement, student 1 will be assigned to school 2 in two additional random draws compared to the initial situation. As the total number of possible ways to break ties is equal to 24, the improvement in the allocation probability of student 1 to school 2 is, therefore, $^2/_{24}$ or 8.33%.

Erdil notes that, in order for this improvement to be strategy-proof, the preference structure of the other students should be *symmetric*. In Section 2.2, a method to find efficiency improvements for wasteful probabilistic assignments will be introduced, but the constraint of symmetric preferences of the other students will be relaxed for two main reasons. First of all, Erdil does not provide a formal definition of when the preference structure of the other students can be considered to be symmetric with respect to a student's preference list. Moreover, regardless of the exact definition, symmetric preferences of the other students are rare in large instances. As a relaxation of this constraint implies a loss of strategy-proofness, however, the size of this loss will be evaluated in Section 3.6.

1.2.4 Trade-offs among properties

Throughout the previous sections, it has become clear that trade-offs among desirable properties exist. As a matter of fact, a vast collection of articles has been published on impossibility results for mechanisms with respect to certain desirable properties. In this section, the most relevant results for the *traditional* algorithms will be mentioned, followed by a discussion on the trade-offs that exist for probabilistic mechanisms.

First of all, as could be seen in the introductory example in Section 1.1, ex-post Pareto efficiency and stability are generally not compatible (Roth (1982); Abdulkadiroğlu and Sönmez (2003)). Nevertheless, as previously mentioned, DA is strategy-proof and results in the most ex-post Pareto efficient matching among all stable matchings.

However, as schools tend to be indifferent between groups of students or even between all students, the question can be raised how valuable the concept of stability still is as it will only protect artificially created stability constraints. Instead, another desirable property that could be aimed for is *fairness*[8]: a mechanism is *fair* if students with identical preferences are treated equally. Zhou (1990) proved that, in a context with cardinal preference structures, no mechanism exists that satisfies ex-ante Pareto efficiency, fairness and strategy-proofness. Moreover, Bogomolnaia and Moulin (2001) found a similar result for the context with ordinal preferences. Recently, Martini (2016) strengthened this result by proving the impossibility of obtaining non-wastefulness, fairness and strategy-proofness. Nonetheless, it is possible to design a mechanism that satisfies two of these desiderata:

- The Random Serial Dictatorship (RSD) mechanism is strategy-proof and fair, but has been proven to be wasteful (Erdil, 2014). Despite its wastefulness, RSD is ex-post Pareto efficient.

- The Probabilistic Serial (PS) mechanism is fair and ordinally efficient, which implies non-wastefulness. However, it is not strategy-proof.

- The Serial Dictatorship mechanism, in which students are assigned to their most preferred school with an available seat in a fixed order (e.g. alphabetically), is strategy-proof and non-wasteful, as non-wastefulness is equivalent to ex-post Pareto efficiency in a deterministic mechanism. However, it is clearly not fair.

[8]Sometimes also referred to as *symmetry* (e.g. Zhou (1990)) or *anonymity* (e.g. Bogomolnaia and Moulin (2001)) or *equal treatment of equals* (e.g. Martini (2016))

An additional result was obtained by Liu and Pycia (2016); they showed that all asymptotically efficient, symmetric and asymptotically strategy-proof mechanisms are allocationally equivalent to RSD. This has implications for the possible improvements over RSD that can be gained by the mechanisms described in Section 1.2.3. The mechanism proposed by Erdil (2014), for example, is asymptotically efficient, symmetric and strategy-proof. Therefore, for *large* instances, the final improvements over RSD will be negligible. The same is true for PS, which is efficient, symmetric and asymptotically strategy-proof (see Section 1.2.5).[9] It has to be noted, however, that their result only holds asymptotically. More specifically, it only holds if the number of seats in the schools approaches infinity, while maintaining the same ratio of students who have preferences over these schools. Therefore, despite this result, it is still worth examining possible improvement mechanisms for real-world applications, as every realised improvement will have an impact on at least one student's life.

1.2.5 Strategy-proofness results

In the previous sections, the concept of strategy-proofness has been approached in a rather binary way: either reporting true preferences is a dominant strategy, or it is not. However, it might be possible that, in real-world applications, some mechanisms that are not strategy-proof in theory are less sensitive to manipulation than others.

Kojima and Manea (2010) were the first to show that, although the Probabilistic Serial mechanism is not strategy-proof, reporting true preferences is a weakly dominant strategy if the instance of the problem is *sufficiently large*. More specifically, they obtain this result for a setting in which the number of schools is constant, but the number of seats on each school is increased according to the same ratio as the number of students that have preference structures over these schools (also referred to as a *replica economy*).

Therefore, several proposals have been made to introduce a relaxation of strategy-proofness that is less stringent than perfect strategy-proofness, but would be perceived as not manipulable in real-world applications. Mennle and Seuken (2014) introduced an axiomatic approach that characterizes perfect strategy-proofness by three axioms, which will be explained in more detail in Section 3.6, and they suggested the notion of *partial strategy-proofness*, which is obtained by dropping the least intuitive of the three axioms. In this way, it is possible to calculate how different the valuations by a student for different schools should be in order for the mechanism to still be partially strategy-proof. This measure, referred to by the authors as r-partial strategy-proofness, can be interpreted as the degree of strategy-proofness of a mechanism, or the extent to which it can be manipulated. They found that PS is r-partially strategy-proof and they introduced an r-partially strategy-proof adaptation of the non-partially strategy-proof Boston mechanism.

[9] See also: Che and Kojima (2010).

Other notable relaxations of strategy-proofness are, for example, the concept of *strategy-proofness in the large* by Azevedo and Budish (2018) and the concept of *vulnerability to manipulation* by Pathak and Sönmez (2013). However, in the remainder of this thesis, the notion introduced by Mennle and Seuken (2014) will be preferred over these two concepts, as it also applies to instances that are not *large*, as opposed to Azevedo and Budish (2018), and as their axiomatic approach provides insight into which preferences of the students can benefit from misreporting, in contrast to the concept of Pathak and Sönmez (2013).

Moreover, a relevant question is whether the designer of a student allocation mechanism should truly aim for a perfectly strategy-proof mechanism. Budish and Cantillon (2012), for example, empirically showed that, in the context of allocating students to courses, adopting a non-strategy-proof mechanism may lead to higher student welfare than using an ex-post Pareto efficient and strategy-proof mechanism such as RSD.

1.2.6 An alternative approach: optimization techniques

Lastly, instead of solving the problem of allocating students to schools with a step-by-step algorithm, it could also be solved by formulating it as an optimization problem, in which a certain objective function is optimized while satisfying certain constraints. This approach is less adopted in practical applications as, in general, the solution method is less transparent.

In her master's thesis, D'haeseleer (2016) showed that the same allocations from mechanisms like, for instance, DA can be obtained by solving an Integer Programming (IP) formulation. Moreover, she extended these formulations by evaluating different methods for obtaining a certain desired level of social mix in the schools. She concluded that the use of optimization techniques is not beneficial over the *traditional* methods when school priorities are determined by one single criterion (e.g. distance or random tie-breaking). However, when two criteria to determine school priorities are adopted, an improvement in the objective function can be noted compared to the *traditional* methods. If, for example, distance and random tie-breaking would be used, a certain proportion of the students will be allocated to a school because the distance is small and another proportion because ties were broken in their favor. This method, however, will no longer be strategy-proof.

1.3 School choice regulation in Flanders

This section will firstly discuss the evolution of the Flemish regulation on school choice for secondary schools. Furthermore, both the current state of affairs in Flanders and its implementation in the major Flemish cities will be explained.

1.3.1 History

In Belgium, the language communities have the authority to regulate the school choice. In Flanders, the GOK decree of 2002 ("*Gelijke Onderwijskansen*") specified the first regulations with respect to application systems for schools, based on the idea of an unrestricted school choice and equal education opportunities for all students (Vlaamse overheid, 2002). This unrestricted school choice implies that schools are legally prohibited from refusing students for any other reason than a lack of available capacity.

Before the introduction of a centralized student allocation mechanism, students were accepted based on a *first-come, first-served* principle in schools that could not accept all applicants. The arrival times in this system were determined by, for example, camping in front of the school entrance or by trying to call in at a specific moment in time. However, this *first-come, first-served* priority criterion might favour students from a better socio-economic background, as their parents can spare the time to camp and can count on a broader network. Moreover, the growing queues in front of popular schools caused many parents to complain.[10]

In 2006, the Flemish government decided to create local coordination committees, called LOPs ("*lokale overlegplatforms*"), that help to ensure equal education opportunities for students in a certain region. As the school enrollment procedures impact the social diversity and the level of segregation in schools, coordinating this student application process is part of the responsibilities of the LOPs (Cantillon, 2009). LOPs consist of the representatives of all schools in the area, parents' associations and key social associations, such as the CLB ("*Centrum voor leerlingenbegeleiding*").

As capacity constraints became tighter over the years, the school choice regulations became more and more elaborated. For primary education, the LOPs of the major cities in Flanders decided to adopt an online centralized application system in the academic years of 2009-2010 (Ghent) and 2010-2011 (Antwerp and Brussels) (Wouters and Groenez, 2014*a*). In the context of primary education, schools have priorities over the students based on distance, combined with random tie-breaking.

For secondary education, on the other hand, a centralized application system was only introduced several years later. In the academic year of 2013-2014, Leuven was the first major city in Flanders to adopt a centralized application system (KSLeuven, 2016), followed in 2018-2019 by Antwerp, Ghent and the Dutch-speaking schools in Brussels (Stassijns, 2017; Salumu, 2017; Hubo, 2017). Not all schools in these cities took part in the system, however, as participation was not mandatory. The details of these mechanisms will be clarified in Sections 1.3.2 and 1.3.3.

[10]Throughout the years, camping in front of school entrances has been a widely discussed topic in the Flemish popular press, e.g. De Herdt (2010) or Debruyne (2016).

The implementation of a centralized application system entails two main advantages, namely an increase in transparency and legal security for the parents, and the possibility to use the application mechanism as a tool to obtain a proportional distribution of *minority* and *majority* students, compared to the school's surrounding area. In Flanders, students can be distinguished as minority students, called "*indicatorleerlingen*", if either their mother has not obtained a diploma of secondary education or if their family receives an education allowance (Onderwijs Vlaanderen, 2012*b*). A majority student is a student who does not meet any of these two criteria. As several studies have indicated the positive effects of low school segregation levels,[11] the Flemish Government decided to introduce a *double quota system*, named "*dubbele contingentering*", obligatory for all schools in a LOP-area in the academic year of 2013-2014 (Wouters and Groenez, 2015). Wouters and Groenez (2015) showed that the introduction of this system led to a decrease in school segregation for the first time in 10 years. The working of the double quota system will be discussed in Section 1.3.2.

Despite the overall improvement in the number of allocated students and the decrease in the waiting lines because of the introduction of a centralized application system for secondary schools in 2018-2019, the parents' reactions after the announcement of the assignments were, overall, rather negative. The main complaints were about students who were not assigned to any school at all, the existence of Pareto-improving exchanges and twins who submitted the same preference list, but were assigned to different schools.[12] Caused by the increased attention in the press, a proposal of decree was approved by the Flemish Parliament in October 2018 (Vlaams Parlement, 2018). This proposal of decree made the use of an online application system mandatory for all schools with capacity constraints, introduced one common application date for all schools in Flanders and removed the requirement to adopt the double quota system for secondary schools. Moreover, the proposal states that all LOPs in Flanders would make use of the same *standard algorithm*, but the specific properties of this algorithm were not discussed (Vlaams Parlement, 2018).

In December 2018, however, the French Community Commission (COCOF), who are responsible for the French-speaking community in the Brussels-Capital Region, submitted a conflict of interest against the proposal of decree (Belga, 2018). Their claim was that the proposed 10% increase in the percentage of seats in Brussels schools for which Dutch-speaking students were prioritized, would cause an increase in the proportion of students who speak neither Dutch nor French in the French-speaking schools.

However, when the conflict of interest expired in April 2019, the Flemish Parliament nevertheless approved the proposal of decree, but due to the conflict of interest, the implementation has been delayed from academic year 2019-2020 to 2020-2021 (Onderwijs

[11] See, for example, Thrupp et al. (2002) or Sacerdote (2011) for an overview of studies on the effects of school composition and peer effects.

[12] See, for example, newspaper articles such as Cools (2018) or Gordts (2018).

Vlaanderen, 2019). In some major cities, such as Antwerp, Ghent and Leuven, however, all secondary schools accepted to already participate in the central allocation system in 2019-2020 (Braeckman, 2019; Mouchalleh, 2019).

1.3.2 Current regulation for enrollment in secondary education in Flanders

The discussion of the current regulations on enrollment procedures for secondary schools in this section will be mainly based on the circular that was initially distributed in 2012 by the Flemish administration in charge of Education (Onderwijs Vlaanderen, 2012*b*). The circular has been updated ever since and is still applicable at the time of writing. As the legislation is written in Dutch, the Dutch terminology will be added in parentheses in this section.

The *principles of enrollment* ("*inschrijvingsrecht*") are a set of rules to ensure that students can enroll in a school of their choice in a transparent and legally secure way. The main objectives of the Flemish principles of enrollment are (Onderwijs Vlaanderen, 2012*b*):

- The realisation of optimal study and development opportunities for all students;
- The avoidance of exclusion, segregation and discrimination;
- The stimulation of cohesion and of a good social mix;
- Additionally for Brussels, the protection of equal education and enrollment opportunities for Dutch-speaking students.

One of the measures to obtain these objectives is the definition of *priority groups* ("*voorrangsgroepen*"). Students who belong to one of these priority groups can apply to the school of their preference before all other students, and experience, therefore, a significant increase in enrollment probabilities. Each priority group has an entitled *priority period* ("*voorrangsperiode*"), during which the students can benefit from their priority on the school. In order to be able to benefit from this measure, students must fulfill at least one of the following criteria, listed in chronological order of the corresponding priority periods:

- The student is the brother or sister (same living group) of a student who is already enrolled in the school;
- The student is the child of an employee at the school;
- Only for Brussels, the student has at least one Dutch-speaking parent.

In order to obtain a proportional distribution of minority and majority students in comparison to the school's environment (as defined in Section 1.3.1), a *double quota system* is adopted. In this system, each school divides all places that are still available after the assignment of the priority groups into two *contingents* of predefined size that will be used for the simultaneous enrollment of minority and majority students. In both of these contingents, the available places are filled according to the adapted priority criterion (see below). The places that have not been filled in this way will be made available to the other group of students. The size of each contingent can be determined based on, for instance, the relative presence of each group in the school's surroundings or on other criteria, as long as it helps to obtain the objective of the double quota system, namely to improve the social mix in the school and to reduce segregation (Onderwijs Vlaanderen, 2012*b*).

After the assignment of the priority groups, the remaining places have to be allocated. In order to do this in a legally transparent way, this will be done based on the priorities of the schools. As the prioritized students are already enrolled at this stage of the procedure, in general, schools are indifferent between large groups of students and ties will have to be broken within these groups. According to the principles of enrollment, the only criteria that can be adapted to determine the priorities of secondary schools in Flanders and Brussels, are (Onderwijs Vlaanderen, 2012*b*):

(i) The chronology of application, without considering the moment of physical application;

(ii) Randomness, only to be used in combination with (i) or (iii);

(iii) The position of the school in the preference list of the student, only to be used in combination with (i) or (ii).

Note that the use of distance as a priority criterion is not allowed in secondary education, as opposed to the context of primary education (Onderwijs Vlaanderen, 2012*a*). It can be argued that, in the short-run, using distance as a priority criterion would naturally lead to a better correspondence between the school composition and the neighborhood composition. In the long-run, however, people will take this regulation into account when making housing decisions, which would lead to a rise in the housing prices in the area around the schools that are perceived as good. In this scenario, neighborhood segregation will increase as only families from a higher socio-economic background will be able to live in the surroundings of good schools, which, in turn, would mean that school segration will increase as well.[13]

Normally, the introduction of a central application mechanism is the result of a local consensus, but in the case of severe capacity constraints, the government can impose the introduction of a centralized application mechanism on a group of schools or an LOP (Onderwijs Vlaanderen, 2012*b*). In the context of primary education, this obligation already exists for Antwerp, Brussels and Ghent, but, at the time of writing, no such obligation exists for secondary education.

[13] Black and Machin (2011) provided an overview of the studies on the effect of school quality on housing prices. Wouters and Groenez (2014*b*) discussed the relevance of this issue for the Flemish context.

1.3.3 Implementation in Flemish cities

The Flemish regulation at the time of writing leaves some authority to the LOPs on the implementation of the centralized application system, for example with respect to the choice of the allocation mechanism. In this section, the particularities of the application systems for secondary education in Antwerp, Brussels, Leuven and Ghent will be briefly discussed. All four cities use randomness as a priority criterion, in combination with the position of the schools in the preference lists of the students.

Firstly, in Leuven and in the Dutch-speaking schools in Brussels, the Boston mechanism (described in detail in Appendix A.1) with Single Tie-Breaking (STB) is used, but it is adapted in such a way that it can comply with the double quota system mentioned in Section 1.3.2 (KSLeuven, 2019; Quartier, 2017; Inschrijven in Brussel, 2019). The advantages of the Boston mechanism are the ex-post Pareto efficient matching, the relatively simple procedure and the fact that it assigns the maximum number of students to their first choice, given a certain random draw. However, as discussed in Section 1.2.1, the Boston mechanism is not stable nor strategy-proof as students have an incentive to give a high preference to schools on which they have a high chance of getting accepted (Abdulkadiroğlu and Sönmez, 2003). The only difference between the systems of the two cities is the way in which the mechanism is presented to the parents. In Leuven, the mechanism is presented as described in Appendix A.1, but with Single Tie-Breaking (STB). In Brussels, on the other hand, each school is said to have its own priority list, on which students are ranked based on the position of the school in their preference lists. This means that all students who listed a certain school as their first choice appear at the top of the list of that school, followed by the students who submitted that school as their second choice, etc. Within these groups, ties are broken by the unique random number that is assigned to each student (STB). Afterwards, the first iteration of the allocation algorithm assigns students to their school of first choice in correspondence with their randomly generated unique numbers and the available capacities. In the second round, it is checked whether the students who have not been assigned to their first choice can be assigned to their school of second choice, based on the priority lists and the remaining capacities, etc. (Quartier, 2017). This iteration procedure essentially boils down to the Boston algorithm as described in Appendix A.1. Therefore, it could be argued that the mechanism in Brussels is presented in a slightly more complicated way than in Leuven, although they are equivalent.

Secondly, Antwerp and Ghent have adopted the School-proposing Deferred Acceptance mechanism with Multiple Tie-Breaking (MTB) (described in detail in Appendix A.2), and it is adapted in such a way that it allows for the use of the double quota system (Meld je aan Antwerpen, 2019; Meld je aan Gent, 2019). The School-proposing DA results in a matching that is stable with respect to the randomly drawn school priorities, but that is not ex-post Pareto efficient (see the discussion on Student-proposing DA in Section 1.2.1). In contrast to the Student-proposing DA, however, the School-proposing DA is not entirely strategy-proof (Balinski and Sönmez, 1999), but the possibilities for manipulation are less obvious than in the Boston mechanism (discussed in more detail in Section 3.6.2). Moreover, the School-proposing DA finds the stable matching that is preferred to any other stable matching from the perspective of the schools, but it is possible that this matching is ex-post Pareto dominated from the perspective of the students by the matching from the Student-proposing DA. Two differences between the application procedures in Antwerp and Ghent exist for the academic year of 2019-2020. Firstly, in Ghent, the schools with insufficient capacity were contacted to increase their capacity after a simulation (Salumu, 2019). Secondly, Ghent adapted the final matching by removing improvement cycles in order to obtain an ex-post Pareto efficient matching. As discussed in Section 1.2.2, however, this second measure implies an additional loss of strategy-proofness.

Chapter 2

Models

Section 2.1 defines the problem and the concepts that have been introduced in the introduction in a more formal way. In Section 2.2, an alternative method using optimization techniques is proposed to reduce the negative effects on student welfare caused by random tie-breaking. Section 2.3 introduces a new method using optimization techniques to reduce the uncertainty about the final number of allocated students caused by random tie-breaking. Lastly, Sections 2.4 and 2.5 discuss possible solutions for improving the performance and the computation time of both proposed methods.

2.1 Formalized problem statement

2.1.1 General terminology

Let $C = \{c_1, c_2, ..., c_n\}$ denote a set of n *students*, and $S = \{s_1, s_2, ..., s_m\}$ a set of m *schools*. The *capacity* of school s_j is denoted by $q_j \in \mathbb{N}$. The *preference list* $>_{c_i}$ of student c_i is a strict ranking of the elements in $S \cup \{0\}$, in which student c_i is said to *prefer* school s_j to school s_k if $s_j >_{c_i} s_k$ and to prefer the *outside option*, which is not being assigned to any school in the system, to being assigned to school s_r if $0 >_{c_i} s_r$. The set of all student preferences is denoted by $>_C$. Student preferences are assumed to always be strict, which means that a student will never be indifferent between two schools.[1] Despite the fact that, in practical applications, students only report such preference lists (*ordinal* preferences), it is convenient to assume that all students in C actually experience a specific *utility* of being assigned to a school (*cardinal* or *von Neumann-Morgenstern* preferences). Let $u_i(s_j) \in \mathbb{R}^+$ denote the normalized utility for student c_i of being assigned to school s_j. These underlying utilities are unknown, but are assumed to be *compatible* with the preference profiles, meaning that $(u_i(s_j) > u_i(s_k)) \Rightarrow (s_j >_{c_i} s_k)$ for all $c_i \in C$ and $s_j, s_k \in S$.

Similarly to student preferences, all schools in S have a *priority list* of the students

[1]This is a reasonable assumption as only the schools that are preferred to the outside option will be taken into consideration in assignment mechanisms and the remaining schools can be ranked arbitrarily.

in C. If school s_j is not indifferent between any two students, the strict *priorities* are denoted by $>_{s_j}$. In practice, however, there might be *ties* in school priorities if schools are indifferent between students. In that case, the *weak* priorities of school s_j are denoted by $\geq_{s_j}$. If strict priorities are required, ties have to be *broken* by a *tie-breaking rule* τ. Consider $\mathcal{T}$ to be the set of all possible tie-breaking rules, given the weak priority structures of the schools $\geq_S = \{\geq_{s_1}, ..., \geq_{s_m}\}$. Each tie-breaking rule $\tau \in \mathcal{T}$ will transform $\geq_S$ into a strict priority structure $>_S^\tau = \{>_{s_1}^\tau, ..., >_{s_m}^\tau\}$.

The *student allocation* problem can be considered as finding a *deterministic assignment*[2] from the students in C to the schools in S. Such a deterministic assignment can be represented by an $(n \times m)$ matrix $M = [m_{ij}]$, in which $m_{ij} = 1$ if student c_i is assigned to school s_j and $m_{ij} = 0$ otherwise. A matching is *feasible* if each student is assigned to at most one school and the capacities of the schools are not violated:

(i) $\sum_{s_j \in S} m_{ij} \leq 1$ for all students $c_i \in C$;

(ii) $\sum_{c_i \in C} m_{ij} \leq q_j$ for all schools $s_j \in S$.

In a feasible matching, $M(c_i)$ denotes the school to which student c_i is assigned, and $M(c_i) = 0$ if student c_i is not assigned to any school (outside option). Similarly, $M(s_j)$ is the set of students that are assigned to school s_j. A matching is *individually rational* if students are never assigned to a school that they prefer less than the outside option: $M(c_i) >_{c_i} 0$ for all $c_i \in C : M(c_i) \neq 0$. Let $\mathcal{M}$ represent the set of all feasible and individually rational matchings.

A *probabilistic assignment* generalizes the idea of a deterministic assignment and specifies the allocation probabilities for all student-school pairs. It can be represented by an $(n \times m)$ matrix $P = [p_{ij}]$, in which $p_{ij} \in [0, 1]$ indicates the probability that student c_i is assigned to school s_j. In order for a probabilistic assignment to be feasible, the same feasibility criteria apply as for a deterministic assignment, namely $\sum_j p_{ij} \leq 1$ for all $c_i \in C$ and $\sum_i p_{ij} \leq q_j$ for all $s_j \in S$. A probabilistic assignment P is individually rational if no student has a strictly positive probability of being assigned to a school that they prefer less than the outside option: $p_{ij} > 0 \Rightarrow s_j >_{c_i} 0$ for all $c_i \in C, s_j \in S$. Let $\mathcal{P}$ represent the set of all feasible and individually rational probabilistic assignments. As discussed in Section 1.2.3, a generalized version of the Birkhoff-von Neumann theorem (Birkhoff, 1946; von Neumann, 1953; Budish et al., 2013) guarantees that this probabilistic assignment can be rewritten as a (generally not unique) weighted sum of feasible deterministic assignments, in which the weight of matching $M_t \in \mathcal{M}$ is equal to λ_t:

$$P = \sum_{M_t \in \mathcal{M}} \lambda_t \cdot M_t \qquad \text{where} \qquad \sum_{M_t \in \mathcal{M}} \lambda_t = 1, \lambda_t \geq 0.$$

[2] In the remainder of this thesis, the term *matching* will be refer to a deterministic assignment, whereas the term *assignment* will refer to a probabilistic assignment.

In such a decomposition, matching $M_t \in \mathcal{M}$ is said to be *selected* if $\lambda_t > 0$. This leads to the following link between probabilistic and deterministic assignments: when students are allocated to schools using a probabilistic assignment method, the final assignment can then be determined by executing a *lottery* over the deterministic assignments in $\mathcal{M}$ in which each matching $M_t \in \mathcal{M}$ is selected as the final matching with probability λ_t.

In order to differentiate between the final matching and the method that is used to obtain this matching, the concept of a *mechanism* is used. More specifically, a mechanism is a method to obtain a final matching of students to schools. A *deterministic mechanism* obtains this allocation directly, whereas the final matching in a *probabilistic mechanism* is only obtained after generating a probabilistic assignment, followed by a lottery over the decomposed deterministic assignments. Moreover, a *two-sided mechanism* considers both student preferences and school priorities, whereas a *one-sided mechanism* only takes the preferences of the students into account.

2.1.2 Properties

Since not every matching in $\mathcal{M}$ is equally desirable, a mechanism should be designed in such a way that it always results in a matching satisfying certain desirable properties. This section defines a selection of the most important assignment properties with respect to the welfare of the students.

Firstly, the extent to which the priorities of the schools for the students have been respected in two-sided matchings can be evaluated. A student-school pair (c_i, s_j) is called a *blocking pair* in a deterministic assignment $M \in \mathcal{M}$ if student c_i prefers school s_j to his/her current assignment $M(c_i)$ and school s_j has assigned a seat to another student c_k who has a lower priority for that school than student c_i. A matching is *stable* if no blocking pairs exist. In a stable matching, all school priorities have been respected.

Secondly, a mechanism is *strategy-proof* if truthful preference reporting maximizes the expected utility for every student. Students experience uncertainty over their final assigned school because of random tie-breaking when school priorities are not strict, or because they might have limited information about the preferences of the other students. Denote the expected utility for student $c_i \in C$ of reporting preference list $>_{c_i}$, under the considered mechanism, by $\mathbb{E}(u_i(>_{c_i}))$. This mechanism is then strategy-proof if, for each student $c_i \in C$, reporting an alternative preference list $>'_{c_i}$ results in an expected utility that is not higher than that from reporting truthfully, ceteris paribus:

$$\mathbb{E}(u_i(>_{c_i})) \geq \mathbb{E}(u_i(>'_{c_i})) \qquad \forall >'_{c_i} \text{ and } \forall\, c_i \in C.$$

As students only submit their ordinal preferences, and not their utility functions, these utility functions are generally unknown. This means that a mechanism based on ordinal preferences is only strategy-proof if it is strategy-proof for all utility functions that are compatible with the true preference list.

Thirdly, concepts related to *efficiency* will be discussed. A matching $M \in \mathcal{M}$ *ex-post Pareto dominates* another matching $M' \in \mathcal{M}$ if it assigns all students to a school that is at least as preferred as their assigned school in M', and at least one student to a more preferred school than in M': $M(c_i) \geq_{c_i} M'(c_i)$ for all $c_i \in C$ and $M(c_k) >_{c_k} M'(c_k)$ for some $c_k \in C$. A matching $M \in \mathcal{M}$ is *ex-post Pareto efficient* if no matching $M' \in \mathcal{M}$ exists such that M' ex-post Pareto dominates M. A probabilistic assignment is considered to be ex-post Pareto efficient if it can be decomposed into ex-post Pareto efficient matchings.

As mentioned in Section 1.2.3, Bogomolnaia and Moulin (2001) introduced a stronger notion of efficiency. Let $\psi_i^c = [\psi_{ip}^c]$ denote the *cumulative preference profile* of student c_i for assignment $P \in \mathcal{P}$, in which ψ_{ip}^c is the expected probability for student c_i of being assigned to one of his/her first p choices. If ordinal preferences are submitted, a probabilistic assignment $P \in \mathcal{P}$ *stochastically dominates* another probabilistic assignment $P' \in \mathcal{P}$ if, given preferences $>_C$, for each student, the probability of being assigned to one of his/her first $p \in \{1, ..., m\}$ choices is at least as large in P as in P', and strictly larger for at least one student and some value of p. Denoting the cumulative preference profiles of student c_i in P and P' by ψ_i^c and $\psi_i'^c$, respectively, these two criteria can be rewritten as:

(i) $\psi_{ip}^c \geq \psi_{ip}'^c$ for all $c_i \in C,\ p \in \{1, ..., m\}$;

(ii) $\psi_{kp}^c > \psi_{kp}'^c$ for some $c_k \in C\ p \in \{1, ..., m\}$.

A probabilistic assignment $P \in \mathcal{P}$ is *ordinally efficient* if no probabilistic assignment $P' \in \mathcal{P}$ exists such that P' stochastically dominates P. Note that ordinal efficiency implies ex-post Pareto efficiency as it will always be possible to decompose an ordinally efficient probabilistic assignment into a weighted sum of ex-post Pareto efficient deterministic assignments (Bogomolnaia and Moulin, 2001). As $\mathcal{M} \subset \mathcal{P}$, ordinal efficiency can also be defined for deterministic assignments, but making the distinction between ordinal efficiency and ex-post Pareto efficiency in this context is not relevant as both concepts will be equivalent.

As ordinal efficiency is a very strong notion of efficiency, intermediate concepts have been proposed, such as *wastefulness*, that are weaker than ordinal efficiency but still stronger than ex-post Pareto efficiency (Erdil, 2014). A probabilistic assignment $P \in \mathcal{P}$ is *wasteful* if there exists a school $s_j \in S$ for which the sum of the allocation probabilities is smaller than the available capacity q_j and there exists at least one student $c_i \in C$ who prefers that school to another school (or the outside option) $s_k \in S \cup \{0\}$ to which it is assigned with a positive probability $p_{ik} > 0$. More formally,

$$s_j >_{c_i} s_k, p_{ik} > 0, \sum_{c_l \in C} p_{lj} < q_j.$$

Using the notation of this definition, student c_i and school s_j are said to *experience waste*, and an assignment in which no student or school experiences waste is called *non-wasteful.* Moreover, ordinal efficiency implies non-wastefulness, but the reverse is not true in general (Martini, 2016). For deterministic assignments, however, non-wastefulness is equivalent to both ordinal efficiency and ex-post Pareto efficiency.

Lastly, a probabilistic mechanism is *fair* if it treats students with the same preference lists in an identical way. Consider students $c_i, c_k \in C$ with $>_{c_i} = >_{c_k}$, then a mechanism that results in a certain probabilistic assignment $P \in \mathcal{P}$ is fair if $p_{ij} = p_{kj}$ for all $s_j \in S$.

2.2 Improving wasteful mechanisms

2.2.1 Intuition and procedure

As has been illustrated in Section 1.2.3, a mechanism that results in a Pareto efficient deterministic assignment (ex-post Pareto efficient), can actually be wasteful when random tie-breaking rules are adopted (ex-ante inefficient). Erdil (2014) showed that this is the case for, amongst others, the Randomized Deferred Acceptance (RDA) and Random Serial Dictatorship (RSD) mechanisms. The method he proposed to reduce this waste in a strategy-proof way is to replace certain tie-breaking rules and corresponding matchings by others, in order to obtain a probabilistic assignment that stochastically dominates the initial assignment (see Example 1.2.5 in Section 1.2.3). This means that, for all student-school pairs, the allocation probabilities after the improvements will be at least as high as before, and strictly higher for at least one student-school pair.

The improvements he proposed, however, are only strategy-proof under very specific conditions, namely if for each student $c_i \in C$ that experiences an improvement in his/her allocation probabilities, the preferences of the other students, $>_{C_{-i}} = \{>_{c_1}, ..., >_{c_{i-1}}, >_{c_{i+1}}$ $, ..., >_{c_n}\}$, are *symmetric* with respect to the preference list $>_{c_i}$ of student c_i. Erdil claims that this symmetry ensures that, for each alternative submitted preference list $>'_{c_1}$, student 1 would benefit from an increase in the allocation probabilities that is never larger than the increase when true preferences are submitted. Although $>_{C_{-1}}$ from Example 1.2.5 appears to be symmetric with respect to $>_{c_1}$, Erdil does not formally specify when an arbitrary set of preference lists $>_{C_{-i}}$ can be considered to be symmetric with respect to $>_{c_i}$. Moreover, Erdil does not specify a general and computationally efficient method to find strategy-proof improvements.

Therefore, in order to be able to find improvements for wasteful mechanisms in real-world problem instances, Erdil's proposal will be adapted in two ways. First of all, the constraint that other students' preference structures have to be symmetric will be dropped. As a consequence, the guarantee that the found improvements are strategy-proof is no longer valid, but the extent to which the strategy-proofness of the mechanism is harmed will be discussed extensively in Section 3.6. Secondly, instead of simply exchanging certain tie-breaking rules and corresponding matchings by others to find a lottery that stochastically dominates the wasteful mechanism, each matching $M_\tau \in \mathcal{M}$ will be assigned a certain weight $\lambda_\tau \in [0,1]$, which represents the probability that tie-breaking rule τ and the corresponding matching $M_\tau \in \mathcal{M}$ are selected as the final matching.

The adjusted procedure, which will be referred to as *Waste-Reducing Lottery Design* (WRLD), consists of the following four steps:

1. In large instances, the number of possible tie-breaking rules can become very large. Therefore, only a subset of the tie-breaking rules $\tilde{\mathcal{T}} \subset \mathcal{T}$ will be considered, with $|\tilde{\mathcal{T}}| = N$. For a certain mechanism, the matching that results from tie-breaking rule $\tau \in \tilde{\mathcal{T}}$ is denoted by M_τ, and the set of all matchings resulting from the tie-breaking rules in $\tilde{\mathcal{T}}$ is denoted by $\tilde{\mathcal{M}} \subset \mathcal{M}$.

2. Based on the matchings in $\tilde{\mathcal{M}}$, the initial allocation probabilities for the students are calculated, denoted by $P^0 = [p^0_{ij}]$. Initially, each matching $M_\tau \in \tilde{\mathcal{M}}$ has the same weight, namely $\lambda^0_\tau = \frac{1}{N}$. Therefore, the probability that student $c_i \in C$ will be allocated to school $s_j \in S$ can be calculated as: $p^0_{ij} = \sum_{\tau=1}^{N} \lambda^0_\tau m^\tau_{ij}$.

3. For each matching $M_\tau \in \tilde{\mathcal{T}}$, we want to obtain new weights λ^1_τ, in such a way that the probabilistic assignment $P^1 = [p^1_{ij}]$, in which $p^1_{ij} = \sum_{\tau=1}^{N} \lambda^1_\tau m^\tau_{ij}$, stochastically dominates the initial probabilistic assignment P^0. These weights are found by solving a Linear Programming (LP) formulation that maximizes the expected number of assigned students under the constraint that, for all student-school pairs, the new allocation probabilities are not smaller than the initial allocation probabilities: $p^1_{ij} \geq p^0_{ij}$ for all $c_i \in C, s_j \in S$. This LP-formulation will be described in detail in Section 2.2.2.

4. Lastly, a final matching is obtained by performing a lottery over the matchings in $\tilde{\mathcal{M}}$. In this lottery, the probability that matching $M_\tau \in \tilde{\mathcal{M}}$ is selected as the final matching is equal to the weight that has been assigned to it by solving the LP-formulation, namely λ^1_τ.

Note that Step 3 is actually a combination of finding a new probabilistic assignment, and of finding the weights for a Birkhoff-von Neumann (Birkhoff, 1946; von Neumann, 1953; Budish et al., 2013) decomposition of that assignment.

Moreover, this procedure is compatible with any type of deterministic mechanism in which ties are broken randomly. A discussion on whether or not some mechanisms experience more improvements from this procedure than others, will be held in Section 3.3.1.

2.2.2 Linear Programming model

By determining the weights of each matching in $\tilde{\mathcal{M}}$ directly, instead of replacing certain matchings by others as proposed by Erdil (2014), the optimization problem in Step 3 can be formulated as an LP-formulation, rather than as an Integer Programming (IP) formulation. This advantage of this approach is that the running time to solve an LP is generally significantly smaller than the one to solve an IP.

In the following LP-formulation, the decision variable x_τ represents the weight λ^1_τ that is assigned to matching $M_\tau \in \tilde{\mathcal{M}}$ in the new probabilistic assignment P^1 (WRLD-LP formulation):

$$\max \quad \sum_{\tau \in \tilde{\mathcal{T}}} \Big(\sum_{c_i \in C} \sum_{s_j \in S} m^\tau_{ij} \Big) \cdot x_\tau$$

Subject to:

$$\sum_{\tau \in \tilde{\mathcal{T}}} m^\tau_{ij} \cdot x_\tau \geq p^0_{ij} \qquad \forall\, c_i \in C, s_j \in S; \tag{2.1}$$

$$\sum_{\tau \in \tilde{\mathcal{T}}} x_\tau = 1; \tag{2.2}$$

$$x_\tau \geq 0 \qquad \forall\, \tau \in \tilde{\mathcal{T}}. \tag{2.3}$$

The objective function maximizes the expected total number of students that is assigned to a school in the new probabilistic assignment P^1. Constraint (2.1) states that the new probability of student c_i being assigned to school s_j is at least as large as the initial probability p^0_{ij}, for all student-school pairs. Constraint (2.2) ensures that the weights of all matchings $M_\tau \in \tilde{\mathcal{M}}$ sum up to one, and constraint (2.3) ensures that all those weights are non-negative.

2.2.3 Example

To illustrate the working of the Waste-Reducing Lottery Design (WRLD) procedure, reconsider the initial example from Section 1.2.3. It will be checked whether improvements upon the RSD solution can be realized. RSD requires a tie-breaking rule to be a simple order of the students. Therefore, the total number of tie-breaking rules is limited ($4! = 24$), which makes it possible to consider the entire set of tie-breaking rules and corresponding matchings: $\tilde{\mathcal{T}} = \mathcal{T}$ and $\tilde{\mathcal{M}} = \mathcal{M}$.[3] The preferences of the students and the initial allocation probabilities P^0 of RSD when all tie-breaking rules in $\tilde{\mathcal{T}}$ are considered, are equal to:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$	$>_{c_4}$
s_1	s_3	s_3	s_3
s_2	s_1	s_2	0
s_3	0	0	0

$$P^0 = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.75 & 0.167 & 0 \\ 0.25 & 0 & 0.33 \\ 0 & 0.625 & 0.33 \\ 0 & 0 & 0.33 \end{pmatrix} \end{array}$$

[3] Equivalently, the improvements for DA with Multiple Tie-Breaking (MTB) could be considered, but as each tie-breaking rule for DA-MTB contains a different order of the students *for each school*, the total number of tie-breaking rules is significantly larger ($(4!)^3 = 13{,}824$). This would cause the working of the procedure to become less clear in the example.

As observed in Section 1.2.3, P^0 is wasteful as students 1 and 3 experience waste on school 2. The solution of Erdil (2014) was to replace the two tie-breaking rules that lead to matching $(0, s_1, s_2, s_3)$ by two that lead to $(s_1, 0, s_2, s_3)$, and by replacing two of the tie-breaking rules that lead to $(s_1, 0, s_3, 0)$ by two that lead to $(s_2, s_1, s_3, 0)$. The resulting allocation probabilities are:

$$P^E = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.75 & \mathbf{0.25} & 0 \\ 0.25 & 0 & 0.33 \\ 0 & 0.625 & 0.33 \\ 0 & 0 & 0.33 \end{pmatrix} \end{array}$$

Despite the fact that the probability of student 1 being assigned to school 2 has increased by 1/12 or 8.33%, P^E is still wasteful as student 3 still experiences waste on school 2. Therefore, the WRLD procedure will be applied to the initial allocation probabilities P^0. Below, the resulting probabilistic assignment P^1 is shown, together with the subset of the matchings $\tilde{\mathcal{M}}_s \subset \tilde{\mathcal{M}}$ that have received a strictly positive weight, and the corresponding weights λ^1:

$$P^1 = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.75 & \mathbf{0.25} & 0 \\ 0.25 & 0 & 0.33 \\ 0 & \mathbf{0.67} & 0.33 \\ 0 & 0 & 0.33 \end{pmatrix} \end{array} \qquad \tilde{\mathcal{M}}_s = \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{pmatrix} s_1 & s_3 & s_2 & 0 \\ s_1 & 0 & s_2 & s_3 \\ s_2 & s_1 & s_3 & 0 \\ s_1 & 0 & s_3 & 0 \end{pmatrix} \end{array} \qquad \lambda^1 = \begin{pmatrix} 8/24 \\ 8/24 \\ 6/24 \\ 2/24 \end{pmatrix}$$

Compared to P^E, the probability with which student 3 is assigned to school 2 has increased with 1/24 (4.17%) in P^1. Note that probabilistic assignment P^1 is non-wasteful, as both students 1 and 3 are now assigned to a school in each matching in $\tilde{\mathcal{M}}_s$, whereas this was not the case in P^0 and P^E.

To summarize, Table 2.2.3 displays the resulting weights λ^1_τ of the LP-formulation, for each matching $M_\tau \in \tilde{\mathcal{M}}$, together with the weights λ^0_τ that were used in P^0 and the weights λ^E_τ that were used in P^E. Compared to the initial allocation probabilities of the RSD mechanism, the WRLD procedure is expected to assign 0.125 students more, which is equivalent to one supplementary student in eight final matchings. This increase is caused solely by the decrease of 1/8 (12.5%) in the weight of matching $(s_1, 0, s_3, 0)$, which is the only matching in $\tilde{\mathcal{M}}$ that assigns only two students to a school. The changes in the weights of the other matchings make sure that, while the allocation probabilities on school 2 for both students 1 and 3 increase, all other allocation probabilities remain unchanged.

Table 2.1: Weights assigned to the matchings in $\tilde{\mathcal{M}}$ by the initial decomposition (λ_τ^0), by Erdil's procedure (λ_τ^E) and by the WRLD procedure (λ_τ^1)

$\mathbf{M}_\tau$	$\lambda_\tau^{\mathbf{0}}$	$\lambda_\tau^{\mathbf{E}}$	$\lambda_\tau^{\mathbf{1}}$
$(s_1, s_3, s_2, 0)$	8/24	8/24	8/24
$(s_1, 0, s_2, s_3)$	5/24	7/24	8/24
$(s_2, s_1, s_3, 0)$	3/24	5/24	6/24
$(\mathbf{s_1}, \mathbf{0}, \mathbf{s_3}, \mathbf{0})$	**5/24**	**3/24**	**2/24**
$(0, s_1, s_2, s_3)$	2/24	0	0
$(s_2, s_1, 0, s_3)$	1/24	1/24	0
Expected number of assigned students	2.7917	2.8750	2.9167

2.3 Maximin decomposition

2.3.1 Intuition and procedure

Because the Birkhoff-von Neumann decomposition of a probabilistic assignment into a weighted sum of deterministic assignments is generally not unique, the selection of the final set of deterministic assignments with strictly positive weights, or the *selected* matchings in the decomposition, can be partly decided upon. From an individual student's point of view, all possible decompositions are equally preferred, as each decomposition will perfectly respect the allocation probabilities of the probabilistic assignment. From the perspective of overall student welfare, on the other hand, some decompositions might be preferred to others. One of the elements that determine overall student welfare, is the total number of assigned students in a matching. Therefore, this could be one possible criterion to take into account while selecting a decomposition of a probabilistic assignment. Although the average number of assigned students will be equal in each decomposition of a probabilistic assignment, a mechanism designer might be risk-averse and might prefer to maximize the lowest number of students that is assigned in any matching of the decomposition.[4] This decomposition will be referred to as the *Maximin decomposition*.

The idea behind the Maximin decomposition is closely related to the *original position* theory of Rawls (1971). This is a thought experiment to find a fair way of allocating

[4]Although the non-uniqueness of the Birkhoff-von Neumann decomposition has been studied extensively, most of the research has focused on algorithms to find the minimum number of different matchings with a strictly positive weight (e.g. Brualdi (1982) or Dufossé and Uçar (2016)). I am not aware, however, of any article that investigates the problem of finding a decomposition that maximizes the lowest number of elements equal to one in any matching that is selected in the decomposition.

certain resources, such as money, in which an individual has to decide on a distribution mechanism from behind a *veil of ignorance*. This veil of ignorance hides from the decision maker which share of the resources (s)he will receive and will only be removed after the decision has been taken. Rawls argued that, in this case, the decision maker would allocate the resources in such a way that the smallest of all shares will be as large as possible.

The Maximin decomposition proceeds in a way that is similar to the WRLD procedure. Based on a subset of tie-breaking rules $\tilde{\mathcal{T}} \subset \mathcal{T}$, with $|\tilde{\mathcal{T}}| = N$, and the corresponding matchings $\tilde{\mathcal{M}} \subset \mathcal{M}$, the initial allocation probabilities P are calculated for a certain mechanism. P is then decomposed into a weighted sum of deterministic assignments in such a way that the minimum number of assigned students in any selected matching is maximized. In this decomposition, the weight of matching $M_\tau \in \tilde{\mathcal{M}}$ is equal to λ_τ^1. These weights are obtained by the Mixed Integer Linear Programming (MILP) formulation that is described in detail in Section 2.3.2. Lastly, to obtain the final matching, a lottery over the matchings in $\tilde{\mathcal{M}}$ is performed, in which the probability that matching $M_\tau \in \tilde{\mathcal{M}}$ is selected as the final matching is equal to λ_τ^1.

Note that the Maximin decomposition can be performed for every mechanism that obtains a probabilistic assignment. For mechanisms such as RDA, RSD or the WRLD procedure, in which the allocation probabilities P are already a weighted sum of deterministic assignments (lottery mechanisms), the set of considered matchings $\tilde{\mathcal{M}}$ is identical to the set of matchings that is used to obtain P. For mechanisms such as PS, on the other hand, which obtain the allocation probabilities P directly (probabilistic mechanisms), the set $\tilde{\mathcal{M}}$ has to be obtained separately by simulating different tie-breaking rules in mechanisms such as RDA or RSD. The selection of the set $\tilde{\mathcal{M}}$ will be discussed in more detail in Section 2.4.

Moreover, the Maximin decomposition does not harm strategy-proofness in any way, as the allocation probabilities of P are perfectly respected.

2.3.2 Mixed Integer Linear Programming model

Consider an arbitrary probabilistic assignment $P = [p_{ij}]$, that is a weighted sum of the deterministic assignments $M_\tau \in \tilde{\mathcal{M}}$, with $p_{ij} = \sum_{\tau=1}^{N} \lambda_\tau^0 m_{ij}^\tau$. Imagine we want to find an alternative decomposition for P that maximizes the minimum number of assigned students in any matching that is selected in the decomposition.

This decomposition will be found by the Mixed Integer Linear Programming (MILP) formulation that is described below, in which the following decision variables are used:

- x_τ is a decision variable that represents the weight that will be assigned to matching $M_\tau \in \tilde{\mathcal{M}}$ in the Maximin decomposition.
- y_τ is a binary decision variable that is equal to one if matching $M_\tau \in \tilde{\mathcal{M}}$ has a strictly positive weight in the Maximin decomposition, and zero otherwise.

– z is a decision variable that represents the lowest number of students that is assigned in any of the matchings in $\tilde{\mathcal{M}}$ with a strictly positive weight.

The Maximin problem can then be formulated as (MILP-formulation):

$$\max \quad z$$

Subject to:

$$\sum_{\tau \in \tilde{\mathcal{T}}} m_{ij}^{\tau} \cdot x_{\tau} = p_{ij} \qquad \forall c_i \in C, s_j \in S; \tag{2.4}$$

$$\sum_{\tau \in \tilde{\mathcal{T}}} x_{\tau} = 1; \tag{2.5}$$

$$x_{\tau} \geq 0 \qquad \forall \tau \in \tilde{\mathcal{T}}; \tag{2.6}$$

$$x_{\tau} \leq y_{\tau} \qquad \forall \tau \in \tilde{\mathcal{T}}; \tag{2.7}$$

$$\sum_{c_i \in C} \sum_{s_j \in S} m_{ij}^{\tau} + D \cdot (1 - y_{\tau}) \geq z \qquad \forall \tau \in \tilde{\mathcal{T}}; \tag{2.8}$$

$$y_{\tau} \in \{0; 1\} \qquad \forall \tau \in \tilde{\mathcal{T}}. \tag{2.9}$$

The objective function in this model maximizes the minimum number of students assigned in all matchings that are selected in the Maximin decomposition. Constraint (2.4) states that the new probability of student c_i being assigned to school s_j should be equal to the initial probability p_{ij}. Constraint (2.5) ensures that the weights of all matchings $M_{\tau} \in \tilde{\mathcal{M}}$ sum up to one, and constraint (2.6) ensures that all those weights are non-negative. Decision variable y_{τ} is set equal to one for all matchings $M_{\tau} \in \tilde{\mathcal{M}}$ for which $x_{\tau} > 0$ by constraint (2.7). In constraint (2.8), the value of z is set equal to the lowest number of assigned students among all selected matchings. In this constraint, $D \in \mathbb{N}$ represents a big number, and it will be set equal to the difference in the number of assigned students between the matching $M_{\tau_M} \in \tilde{\mathcal{M}}$ that assigns the largest number of students and the matching $M_{\tau_m} \in \tilde{\mathcal{M}}$ that assigns the smallest number of students. Lastly, constraint (2.9) ensures that all decision variables y_{τ} are binary.

2.3.3 Example

In this subsection, the working of the Maximin decomposition will be visually illustrated. Consider an arbitrary example with 50 students and 5 schools, generated by the data-generator that will be discussed in Section 3.1.2. In the context of the RSD mechanism, the set of all possible tie-breaking rules $\mathcal{T}$ contains all possible orders of the students. As it is not possible to consider all $50! \approx 3 \cdot 10^{64}$ possible orders, a randomly selected subset $\tilde{\mathcal{T}} \subset \mathcal{T}$ and the corresponding set of matchings $\tilde{\mathcal{M}} \subset \mathcal{M}$ will be considered, with $|\tilde{\mathcal{T}}| = 1000$. The blue curve in Figure 2.1 displays the distribution of the number of allocated students and the purple line represents the expected number of allocated students over all tie-breaking rules in $\tilde{\mathcal{T}}$.

When using RSD, on average 46.94 out of the 50 students are allocated. In one of the matchings in $\tilde{\mathcal{M}}$, however, only 44 students are allocated. To verify whether the expected number of allocated students can be increased, the WRLD procedure from Section 2.2 can be applied to the results of RSD. In Figure 2.1, the resulting distribution of the number of allocated students of WRLD is shown in red, and the green line represents the expected number of allocated students after applying WRLD to RSD.

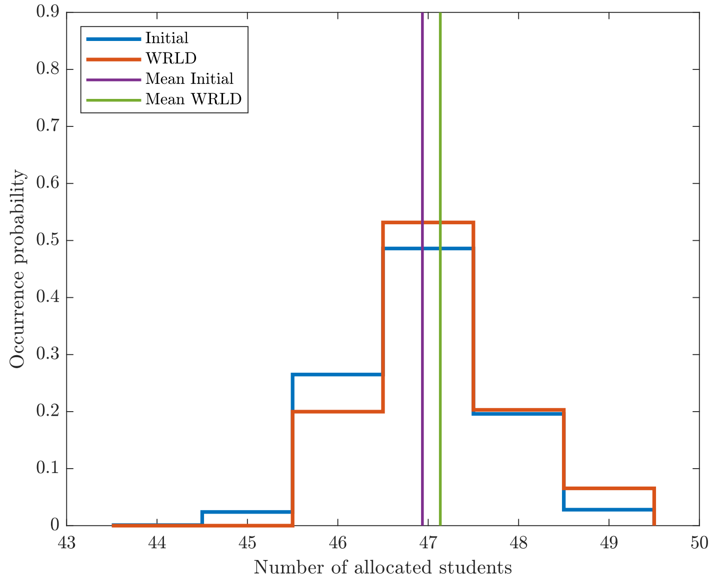


Figure 2.1: Distribution of the number of allocated students RSD & WRLD ($N = 1000$)

The WRLD procedure causes the expected number of allocated students to rise to 47.13, which is an increase of 0.19 student compared to RSD, or approximately one student in every five random draws. Moreover, at least 46 students will be allocated in any of the resulting matchings of WRLD, compared to the worst-case scenario of 44 students for RSD.

As the decomposition to obtain the allocation probabilities of the WRLD procedure is not unique, the Maximin decomposition will verify whether it is possible to increase the number of allocated students in the worst-case scenario. Figure 2.2 shows the resulting distribution of the number of allocated students after the Maximin decomposition, while respecting the allocation probabilities of the WRLD procedure.

The Maximin decomposition guarantees that, in every final matching, at least 47 students will be assigned. Due to risk-aversion, however, the drawback of this decomposition is that no matching will assign 49 students to a school, whereas this was possible in RSD or WRLD. This shows that the Maximin decomposition will reduce the uncertainty

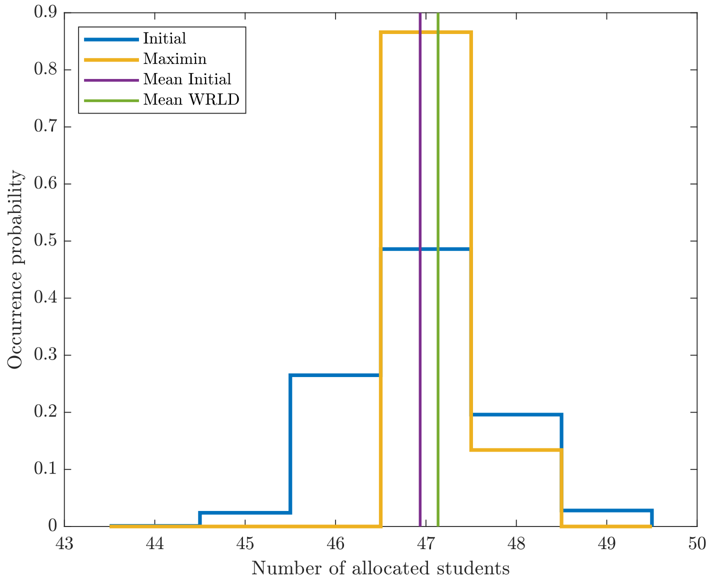


Figure 2.2: Distribution of the number of allocated students RSD & WRLD-Maximin ($N = 1000$)

on the final number of allocated students that is caused by random tie-breaking. An evaluation of the performance of the Maximin decomposition is included in Section 3.4.

2.3.4 Binary search method

As the computation times of the MILP-formulation in the previous section increase strongly for larger problem instances, this section proposes an alternative model to obtain the Maximin decomposition. The binary search method iteratively checks for different values of $k \in \mathbb{N}$ whether it is possible to decompose a probabilistic assignment $P \in \mathcal{P}$ by only using matchings that assign at least k students to a school.

Similarly to the previous sections, only a sample set of all matchings $\tilde{\mathcal{M}} \subset \mathcal{M}$ is considered. Denote the set of matchings in $\tilde{\mathcal{M}}$ that assign at least k students by $\tilde{\mathcal{M}}_k \subseteq \tilde{\mathcal{M}}$ and the set of the corresponding tie-breaking rules by $\tilde{\mathcal{T}}_k \subseteq \tilde{\mathcal{T}}$. To find the Maximin decomposition of a probabilistic assignment $P \in \mathcal{P}$, the binary search method will find the largest value of k for which a feasible decomposition of P with the matchings in $\tilde{\mathcal{M}}_k$ can be found by verifying feasibility iteratively for different values of k.

A feasible decomposition of a probabilistic assignment $P = [p_{ij}]$ with the matchings in $\tilde{\mathcal{M}}_k$ exists if a feasible solution for the weights $x_\tau \in \tilde{\mathcal{T}}_k$ can be found for the following system of equations and inequalities:

$$\sum_{\tau \in \tilde{\mathcal{T}}_k} m_{ij}^{\tau} \cdot x_\tau = p_{ij} \qquad \forall c_i \in C, s_j \in S; \tag{2.10}$$

$$\sum_{\tau \in \tilde{\mathcal{T}}_k} x_\tau = 1; \tag{2.11}$$

$$x_\tau \geq 0 \qquad \forall \tau \in \tilde{\mathcal{T}}_k. \tag{2.12}$$

These constraints are identical to constraints (2.4) - (2.6), with the small difference that they are only considered for $\tilde{\mathcal{T}}_k \subseteq \tilde{\mathcal{T}}$. One possible way to check the feasibility of this system of linear equations is defining an LP-model with an arbitrary objective function (e.g. a constant) and constraints (2.10) - (2.12).

The total computation time of this method depends partly on the order in which the feasibility for different values of k is checked. A lower bound (LB) for the optimal value of k, given a sample set of matchings $\tilde{\mathcal{M}}$, is simply be the minimum number of assigned students over all matchings in $\tilde{\mathcal{M}}$, whereas an upper bound (UB) can be defined by the expected number of assigned students in P, rounded down to the closest integer. This is a valid upper bound as a decomposition of a probabilistic assignment P in which each matching assigns strictly more students to a school than the expected number of assigned students in P does not exist.

Based on this lower and upper bound, an initial interval for the possible optimal values of k can be defined. Consecutively, the binary search method will first check whether a feasible solution exists when the value of k is set equal to the middle value in this interval. If a feasible decomposition with the matchings in $\tilde{\mathcal{M}}_k$ exists, the new lower bound is set to k. Otherwise, if no feasible decomposition exists, the new upper bound is set to $k-1$. Given the new interval of smaller size, the next value of k for which feasibility will be checked is again set equal to the middle value in this new interval. This procedure continues until the lower and the upper bound coincide and the maximum value of k for which it is possible to decompose a probabilistic assignment P such that all matchings in the decomposition assign at least k students has been found. The maximum number of iterations for the binary search method is equal to

$$\left\lceil \log_2(\text{UB} - \text{LB}) \right\rceil + 1.$$

2.4 *Smart* selection of the matchings in $\tilde{\mathcal{M}}$

In the previous sections, the sample of tie-breaking rules $\tilde{\mathcal{T}}$ was simply determined by randomly selecting tie-breaking rules in $\mathcal{T}$. However, some matchings $M_\tau \in \tilde{\mathcal{M}}$, that are the result of the tie-breaking rules $\tau \in \tilde{\mathcal{T}}$, are more likely to enable a better result by the WRLD procedure or the Maximin decomposition than others. Additionally, it might be possible that matchings that would have made it possible to achieve a better result are not included in $\tilde{\mathcal{M}}$.

Consider, for example, the situation in which two different tie-breaking rules $\tau_1, \tau_2 \in \tilde{\mathcal{T}}$ would result in the same matching $M_{\tau_1} = M_{\tau_2}$. As the same final solution can be attained if only one of these matchings is included in $\tilde{\mathcal{M}}$, in the remainder of this thesis, all duplicate matchings in $\tilde{\mathcal{M}}$ will be removed. Other methods that improve upon randomly sampling the tie-breaking rules in $\tilde{\mathcal{T}}$ exist. It might, for example, be possible to categorize matchings according to different properties and to observe whether matchings with certain properties are more likely to obtain good results than others for either the WRLD procedure or the Maximin decomposition. Consecutively, a *stratified sampling* method could be adopted in which the matchings in $\tilde{\mathcal{M}}$ are sampled from the sets of matchings that satisfy certain of these properties (see, for example, the *structured random sampling* method by van Campen et al. (2017)). Which properties would be important for the WRLD procedure or for the Maximin decomposition remains an open question and is left as a direction for further research.

2.5 A column generation approach

Next to sampling, another option to tackle the issue of the very large number of matchings in $\mathcal{M}$ would be to adopt a column generation approach. This section discusses how column generation could be applied to the WRLD-LP formulation from Section 2.2.2 or to the binary search formulation for the Maximin decomposition from Section 2.3.4 and which difficulties will be faced in designing such an approach.

Denote the problem that has to be solved for the set of all matchings $\mathcal{M}$ as the *master problem.* The main idea behind column generation is to restrict the set of the matchings for which the master problem is solved by only considering a subset of the matchings $\mathcal{M}_p \subset \mathcal{M}$. This reduced problem is called the *restricted master problem.* Starting from a feasible solution, in an iterative manner it is verified whether or not the found solution of the restricted master problem is optimal by checking the existence of a matching $M_{\tau'} \in \mathcal{M}$ that violates a constraint in the dual of the restricted master problem. If such a matching $M_{\tau'}$ exists, the found solution is not optimal and the restricted master problem will be solved again for the subset of matchings $\mathcal{M}_p \cup \{M_{\tau'}\}$. The problem of finding which matching should be added to the subset $\mathcal{M}_p$ is called the *pricing problem.* This process continues until the pricing problem can find no matching that violates a constraint in the dual. In this case, the optimal solution for the master problem over all matchings in $\mathcal{M}$ has been found (Bertsimas and Tsitsiklis, 1997).

The working of a column generation approach will be illustrated for the WRLD-LP formulation from Section 2.2.2. Note that the application of column generation to the binary search formulation for the Maximin decomposition from Section 2.3.4 will be very similar since that formulation can be reformulated in such a way that the constraints are identical to the constraints in the WRLD-LP formulation.[5] Denoting the dual variables of constraints (2.1) and (2.2) by the variables y_{ij} and z, the dual formulation of the WRLD-LP formulation is:

$$\min \quad z - \sum_{c_i \in C} \sum_{s_j \in S} p^0_{ij} \cdot y_{ij}$$

Subject to:

$$z - \sum_{c_i \in C} \sum_{s_j \in S} m^\tau_{ij} \cdot y_{ij} \geq \sum_{c_i \in C} \sum_{s_j \in S} m^\tau_{ij}; \qquad \forall\, M_\tau \in \mathcal{M} \qquad (2.13)$$

$$y_{ij} \geq 0. \qquad \forall\, c_i \in C, s_j \in S \qquad (2.14)$$

Suppose x^* is an optimal solution to the restricted WRLD-LP formulation with corresponding solutions y^* and z^* for the dual variables. By strong duality, this solution is only optimal if it is feasible in the dual, i.e. no matching $M_{\tau'} \in \mathcal{M}$ exists that violates constraints (2.13) and (2.14). More formally, no $M_{\tau'} \in \mathcal{M}$ should exist for which $\sum_{c_i \in C} \sum_{s_j \in S} (1 + y^*_{ij}) \cdot m^{\tau'}_{ij} > z^*$. To verify this, a pricing problem can be defined with decision variables m_{ij} and with the following objective function:

$$\max \quad \sum_{c_i \in C} \sum_{s_j \in S} (1 + y^*_{ij}) \cdot m_{ij}.$$

However, this pricing problem differs from a regular weighted matching problem, which can be solved efficiently, as the resulting matching should satisfy certain properties, depending on the desired matching mechanism. If, for example, the RSD mechanism is used, the resulting matching should be ex-post Pareto efficient. A solution for how this difficulty could be overcome and how the pricing problem could be solved remains an open question and is left as a direction for further research.

[5] This is done by changing the equality sign in Equation (2.10) to a greater than or equal to sign and by changing the objective function to minimizing $\sum_{c_i} \sum_{s_j} \sum_\tau (m^\tau_{ij} \cdot x_\tau - p_{ij})$. If the objective function equals zero, a feasible decomposition has been found.

Chapter 3

Results

This chapter evaluates the performance of the methods described in the previous chapter with respect to several desirable criteria. Section 3.1 introduces the real-world data from Antwerp and Ghent, as well as the working of the created data generator. Section 3.2 provides an overview of the performance of the different mechanisms for the data sets of Antwerp and Ghent. Sections 3.3 and 3.4 discuss the performance of the WRLD procedure and the Maximin decomposition. Section 3.5 compares the performances of the PS mechanism and the WRLD procedure and explores the possible application of the Maximin decomposition to the PS mechanism. Lastly, Section 3.6 evaluates the strategy-proofness of the WRLD procedure and of the PS mechanism.

3.1 Data

In order to compare the relative performance of the methods introduced in Chapter 2 to traditional mechanisms such as RDA or RSD, data on student preferences and school capacities are required. Both real-life data (described in Section 3.1.1) and generated data (described in Section 3.1.2) will be evaluated. In this chapter, all students will be considered to have the same priorities in the schools. This is in correspondence with the Flemish context for secondary education, in which the priority groups, such as brothers and sisters from students who are already enrolled or children of school employees, are enrolled in a period prior to the main application period (see Section 1.3.2). Moreover, in this chapter, mechanisms that aim to obtain a better social mix will not be taken into consideration; this issue will be the topic of Section 4.1.

3.1.1 Data of Antwerp and Ghent

Two real-world data sets will be considered in this thesis. The first data set is from LOP Antwerp and contains the applications for the primary schools in the academic year of 2014-2015. Due to privacy considerations, the preferences in the data set are not identical to the true preferences that were submitted by the students, but they are nevertheless similar. Secondly, the submitted preferences from LOP Ghent for the secondary schools in the academic year of 2018-2019 will be considered. In the remainder of this section, the submitted preferences will be assumed to be the true preferences of the students.[1] The main characteristics of both data sets are summarized in Table 3.1 and the distribution of the capacities and the lengths of the submitted preference lists can be found in Appendices B.1 and B.2.

Table 3.1: Main characteristics of the data sets of Antwerp and Ghent (standard deviation in parentheses)

	Antwerp	**Ghent**
Number of students	4236	3081
Number of schools	186	64
Ratio students/schools	22.77	48.14
Total capacity	4653	3687
Average capacity	25.02 (12.31)	57.61 (45.88)
Average length preference list	4.18 (2.66)	2.42 (1.05)

The data set of Antwerp is larger, but the number of available places on each school in Antwerp is smaller. This, in turn, causes the number of submitted preferences by the students to be higher in Antwerp than in Ghent (see Appendix B.2). Furthermore, the total capacity exceeds the number of students in both data sets. Nevertheless, it is not possible to assign all students to their school of first choice as in both data sets approximately 10% more seats would be required (see Appendix B.3).

With respect to the popularity of the submitted preferences, several observations can be made. The popularity of school $s_j \in S$ will be measured by the popularity ratio pop_j, which is defined as the total number of times school s_j appears in a preference list, divided by the capacity q_j of school s_j:

$$\text{pop}_j = \frac{\left|\left\{c_i \in C : s_j >_{c_i} 0\right\}\right|}{q_j}.$$

The distribution of the popularity ratios in Antwerp and Ghent can be found in Appendix B.4. Unless stated differently, the 10% schools with the highest popularity ratio will be considered as *popular* schools. On average, students who submit a shorter preference list have a lower probability of listing a popular school as their first choice. In Figure 3.1, for each possible number of submitted preferences, the proportion of the

[1] The mechanism that is used in both cities is the School-proposing DA. As discussed in Sections 1.3.3 and 3.6.2, this mechanism is not entirely strategy-proof, but the possibilities for manipulation are not straightforward (in contrast to e.g. the Boston mechanism), which means that the assumption that the submitted preferences correspond to the true preferences is reasonable.

students who submitted a popular school as their first choice is plotted. As a reference, the dashed lines represent the number of times a preference list of that length is submitted (these numbers are identical to the histograms in Appendix B.2). This observation could be explained by the fact that students who submit a popular school as their first choice are, in general, aware of the popularity of that school and therefore submit supplementary choices to decrease the probability of not being assigned to any school.

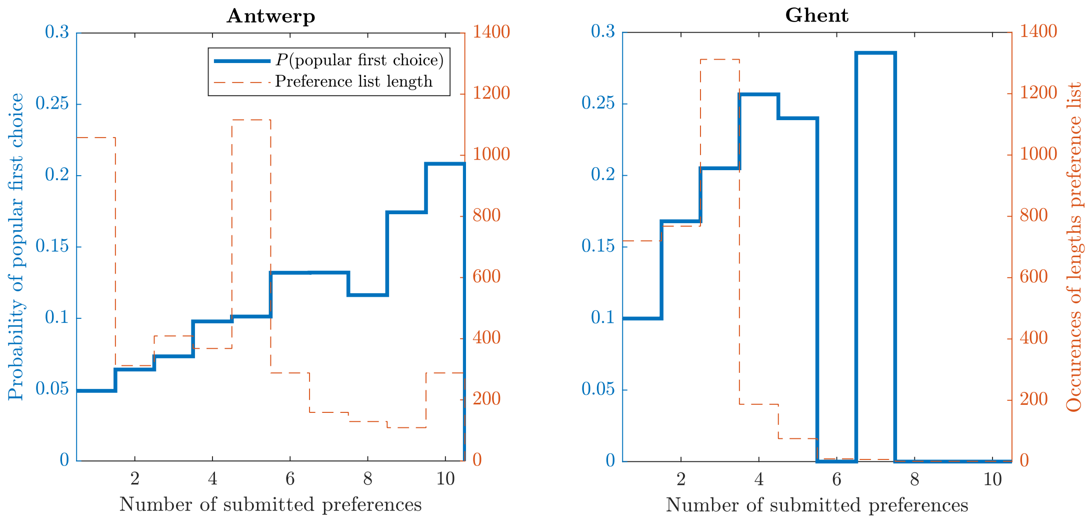


Figure 3.1: Probability of popular first choice with respect to the length of the preference list

Secondly, whereas in the data set of Antwerp, schools with a lower capacity clearly have a higher popularity ratio, this is not the case for the data set of Ghent. In Figure 3.2, the schools are sorted in 10 groups of increasing capacity and the average popularity ratio for each group of schools is plotted.[2] This observation is confirmed by the correlation between the capacity and the popularity, which is equal to -0.33 in Antwerp and to 0.21 in Ghent.

Lastly, it seems plausible that students who submit a popular school as their first choice would have a higher probability of submitting popular schools for their other choices as well. However, as shown in Appendix B.5, this effect is not large for Antwerp and Ghent.

[2]If schools with the same capacity had to be assigned to different groups, this selection was made randomly.

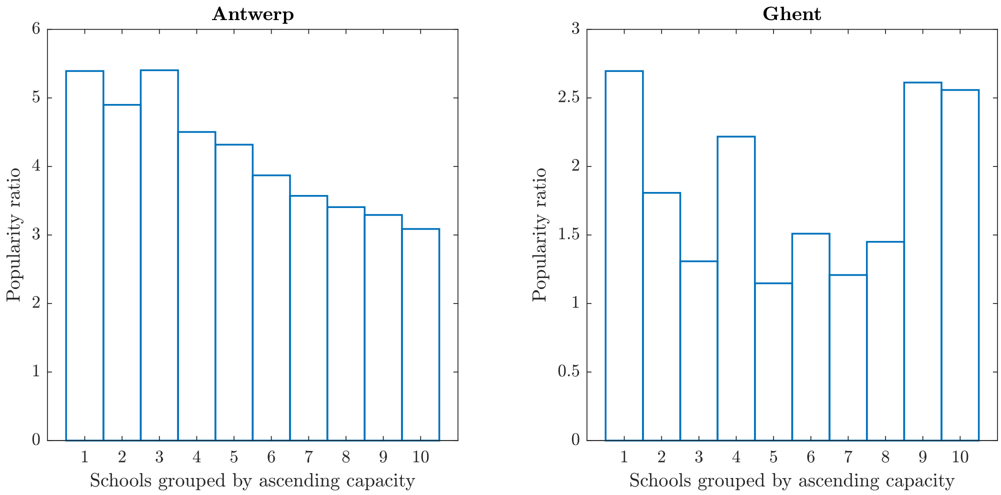


Figure 3.2: Popularity in function of increasing capacity

3.1.2 Data generation

If the effects of different methods would only be evaluated on the data sets of Antwerp and Ghent, it is possible that the resulting observations are not valid in general, but are only present for data sets with very specific characteristics. To check which characteristics influence the final results, the methods will be evaluated on generated data, as this allows us to parameterize certain properties. The selection of the set of parameters that can be controlled is motivated by the observations from Section 3.1.1, and is described in detail in Table 3.2. The values of these parameters for the data sets of Antwerp and Ghent are included in Appendix B.5.

Table 3.2: Data generation parameters

Parameter	**Description**
Nstudents	The number of students.
Nschools	The number of schools.
Capacity ratio	The ratio of the total capacity over the number of students.
ρ_{cp}	The correlation between the capacity and the popularity ratio.
μ_{pref}	The mean length of the preference lists.
σ_{pref}	The standard deviation of the length of the preference lists.
CV_c	The ratio of the standard deviation of the capacities over the mean capacity.

CV_p	The ratio of the standard deviation of the popularity ratio over the mean popularity ratio.
Δ_1	The difference in the popularity of the school of someone who submitted a preference list of average length, calculated as the mean of one and the maximum length of a preference list, compared to someone who submitted a preference list with only one school (see Figure 3.1).
Δ_2	The difference between the probability of submitting a popular school if the first choice was a popular school, compared to when the first choice was an unpopular school.
Popularity %	The percentage of the schools with the highest popularity ratio that will be defined to be *popular*.

The data generation process consists of five steps:

1. In the first step, correlated capacities and popularity ratios have to be generated. This is done by firstly generating independent capacities and popularity ratios from the standard normal distribution. These variables are then transformed by multiplying them with the upper triangle matrix that is obtained after the Cholesky decomposition of the predefined covariance matrix (Golub and Van Loan, 1996, p. 143). This procedure is described in detail in Appendix B.6.

2. Secondly, the lengths of the preference lists are generated from a normal distribution with the desired mean and standard deviation. The lengths are rounded to the closest integer. If this integer is smaller than one, it is set equal to one. Similarly, if it is larger than the number of schools, it is set equal to the number of schools.

3. In the next step, the capacities are transformed such that they correspond with the desired mean and standard deviation. Because of the use of the normal distribution in the first step, however, it is possible that some schools have a negative or very small capacity after this transformation. In that case, for each of these schools, the capacity is re-sampled, together with a new correlated popularity ratio, until all school capacities are feasible. After the re-sampling, the capacities are transformed once again to have the desired mean, and will be rounded to the closest integer.

4. Once the capacities and the lengths of the preference lists have been determined, the popularity ratios of all schools are transformed to correspond with the desired mean, namely the total number of expressed preferences over the number of schools. Similarly to the second step, if the popularity ratio of a school is negative or very small, it is re-sampled and transformed to have the desired correlation with the corresponding capacity of the school. After the re-sampling, the popularity ratios are transformed once more to meet the desired mean.

5. Lastly, the preference lists of the students are filled. This is done by determining, for each student and for each place in his/her preference list, the probability of choosing a popular school. Consequently, in the group of popular (unpopular) schools, each school is selected with a probability that is equal to the capacity of that school compared to the capacity of all popular (unpopular) schools. For the first choice, this probability is only determined by the difference in the probability of selecting a popular school that is caused by the length of the preference list (Δ_1, see Table 3.2) and by the proportion of seats in popular schools with respect to the total number of seats. For the other choices, the probability also depends on whether or not the first choice was popular (Δ_2, see Table 3.2).

Because of the re-sampling of both the capacities and the popularity ratios in the third and the fourth step, the final distributions of these two variables will no longer be perfectly normally distributed and the final correlation ρ_{cp} between the capacities and the popularity ratios will not be as requested.

To check the effect on the distribution of the capacities and the popularity ratios, it suffices to evaluate the standard deviations of both variables, as the means have been fixed to the desired values after the re-sampling. As shown in Appendix B.7, however, although the coefficient of variation is lower than the desired value for both the capacity and the popularity ratio due to the re-sampling, the difference is rather limited in size and is approximately constant, regardless of the number of students and schools. Moreover, Appendix B.8 shows that the proportion of the schools for which the capacity or the popularity ratio has to be re-sampled is rather limited as well, but that the percentage is slightly higher when the average number of students per school is low.

To check the effects of re-sampling on the final correlation, a Monte-Carlo simulation is performed to calculate, for different requested correlations ρ_{cp}, the average observed correlation $\tilde{\rho}_{cp}$ over 5,000 generated data sets with the same properties as the one from Ghent.[3] As illustrated in Figure 3.3, the observed correlation $\tilde{\rho}_{cp}$ is less strong than the requested correlation ρ_{cp}, regardless of the sign. Similar results are true for other choices of the parameters.

3.2 General comparison of mechanisms for Antwerp and Ghent

Before evaluating the effects of the WRLD procedure and the Maximin decomposition, the traditional mechanisms will be compared, together with the PS, on the data sets of Antwerp and Ghent to benchmark the performance of the current mechanisms in Flanders. Figure 3.4 shows the cumulative proportion of the students that are assigned

[3]As only the capacities, the lengths of the preference lists and the popularity ratios of the schools have to be generated to check the correlation between the capacities and the popularity ratios, all parameters except ρ_{cp}, Δ_1 and Δ_2 are set equal to the values in Ghent, which are mentioned in Appendix B.5.

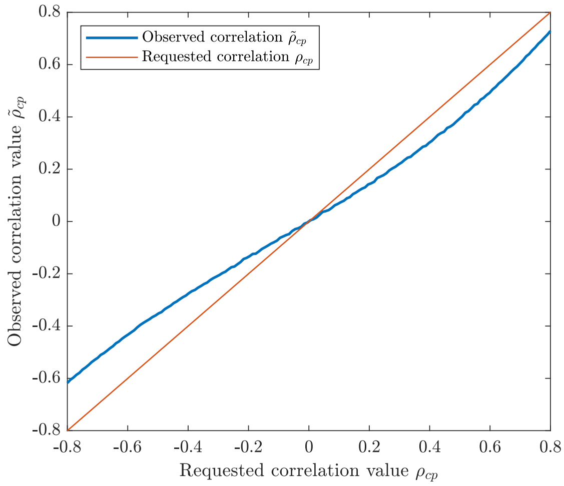


Figure 3.3: Monte-Carlo simulation ($N = 5000$) to obtain the average observed correlation $\tilde{\rho}_{cp}$ for different values of ρ_{cp}

to one of their p most preferred schools for the different mechanisms, both in Antwerp and in Ghent. The exact percentages can be found in Appendices B.9 and B.10. As the School-proposing DA performs only slightly worse than the Student-proposing DA, both curves would practically coincide in Figure 3.4. Therefore, only the curve of the Student-proposing DA is shown in Figure 3.4 and only the Student-proposing DA will be discussed in the remainder of this chapter.

The expected total number of assigned students in Antwerp is approximately the same for all mechanisms, whereas the difference in the number of students that is assigned to their school of first choice is smaller for the DA(MTB) compared to the RSD or the Boston mechanism. Nevertheless, DA(MTB) is not ex-post Pareto dominated by any of these mechanisms, as it assigns more students to one of their top p choices for $p \geq 3$. In Ghent, on the other hand, there is a clear trade-off between the proportion of students that are assigned to their first choice and the total number of assigned students. Moreover, as can be seen in Appendices B.9 and B.10, both in Antwerp and in Ghent, the PS mechanism will always assign more students to their top p choices than the RSD for any p. However, as the difference is small, the curves of both mechanisms almost coincide in Figure 3.4.

Next to the cumulative profiles, another relevant criterion to judge the desirability of a mechanism is whether or not students face the possibility to exchange their allocated school with a student or a group of students in such a way that all students who are involved in an exchange are better off. These exchanges are also called *improvement cycles* (ICs). In this context, the distinction should be made between all possible ICs

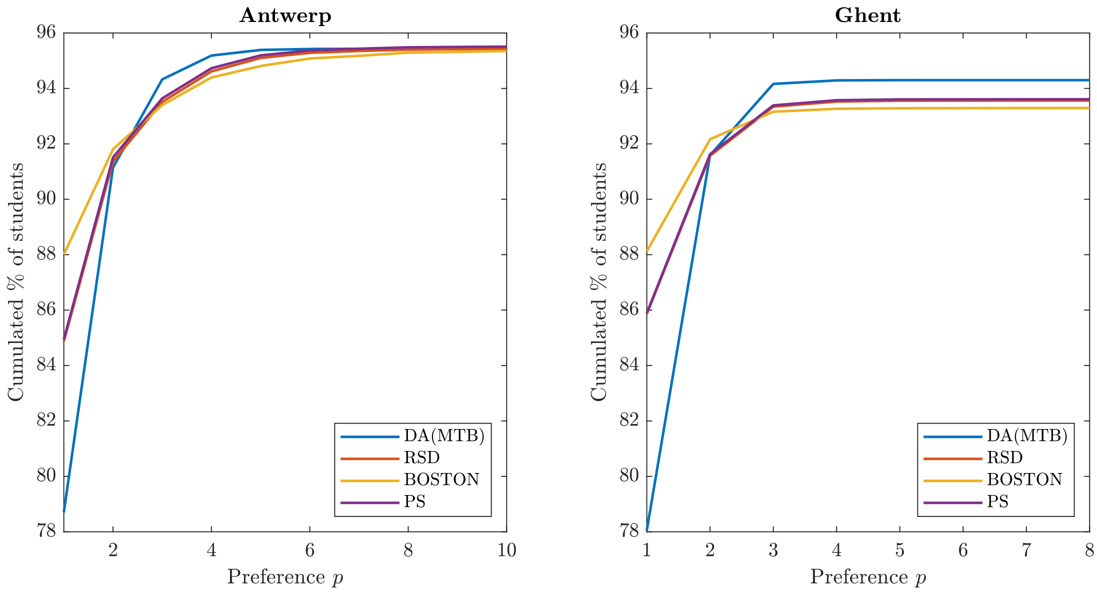


Figure 3.4: Cumulative proportion of the students that is assigned to one of their p most preferred schools (average of 2000 tie-breaking rules)

that are experienced by the students and the *stable improvement cycles* (SICs), in which all involved students will still have a higher priority on their new school than all other students who prefer that school to their current assignment. In the context of Flanders, however, schools do not prioritize over the students (Section 1.3.2), and, therefore, all improvement cycles will be stable.

In order to quantify the degree to which stable improvement cycles are present in a given matching, the total number of students that face the possibility of benefiting from a SIC will be calculated. This number will serve as an upper bound for the final number of students that will be involved in an SIC. This upper bound can be found by firstly constructing an *envy graph*. The envy graph to find all SICs is a directed graph with one node for each student and an edge from student $c_i \in C$ to student $c_k \in C$ if $M(c_k) >_{c_i} M(c_i)$.

Once the envy graph is constructed, it will be decomposed into *strongly connected components* (SCCs). [4] For each student who is part of an SCC with more than one element, there exists at least one stable improvement cycle. The average number of students who

[4] A set of nodes G is *strongly connected* if for each node $g \in G$ it is possible to reach any other node $h \in G$ by following a sequence of directed edges, or a *path* (Tarjan, 1972). The decomposition of a graph into SCCs is the process of finding all SCCs. In this thesis, this decomposition will be obtained by the algorithm that was proposed by Tarjan (1972).

are part of at least one possible SIC in Antwerp and Ghent for the DA with MTB are shown in Table 3.3. Both for Antwerp and for Ghent there is a lot of variance in the number of students who could benefit from an SIC. Moreover, it is remarkable that the proportion, and even the absolute number, of the students who are part of at least one SIC is lower in Antwerp than in Ghent.

As the RSD and the Boston mechanisms are ex-post Pareto efficient, the number of SICs is, by definition, equal to zero in both mechanisms.

Table 3.3: Upper bound on the number of student (proportion of the students) that are involved in an SIC in Ghent and Antwerp (average over 2,000 tie-breaking rules)

	Antwerp			**Ghent**		
Mechanism	**Students in SIC**	**MIN**	**MAX**	**Students in SIC**	**MIN**	**MAX**
DA(MTB)	224.5 (5.30%)	115 (2.71%)	325 (7.67%)	258.1 (8.38%)	166 (5.39%)	324 (10.52%)

3.3 Waste-Reducing Lottery Design (WRLD)

To check the size of the improvements of the Waste-Reducing Lottery Design (WRLD) procedure, an upper bound on the waste that can be captured will be defined. As defined in Section 2.1.2, an assignment is wasteful if the sum of the allocation probabilities for a certain school is smaller than the available capacity and if there exists at least one student who prefers that school to another school (or the outside option) to which (s)he is assigned with a strictly positive probability. Consider a probabilistic assignment $P = [p_{ij}]$ and denote the set of students that experience waste on school $s_j \in S$ by $W_j \subset C$. For each school $s_j \in S$, the upper bound UB_j on the waste that can be captured by the WRLD procedure is the minimum of the expected number of seats that is left unassigned on s_j and of the sum of the waste that is experienced by all students $c_i \in W_j$:

$$\text{UB}_j = \min\Big\{q_j - \sum_{c_k \in C} p_{kj}; \sum_{c_i \in W_j} \Big(1 - \sum_{s_l \in S} p_{il}\Big)\Big\}.$$

The total possible waste reduction UB is the sum of the possible waste reductions on each school, and the relative UB is the ratio of the upper bound over the number of students. It can be noted that, if a student experiences waste on multiple schools, double counting might occur in this definition of the upper bound. Nevertheless, the size of the double counting is limited as it will be possible to reduce the waste by an amount that is approximately equal to the upper bound if the number of considered tie-breaking rules in the sample $\tilde{\mathcal{T}}$ is sufficiently large, as will be shown in Section 3.3.3.

Given this observation and given the fact that obtaining the result of the WRLD procedure is a computationally intensive task, in the remainder of this section, the upper bound will be used as a proxy for the size of the waste reduction by the WRLD procedure in generated data sets with different parameters. The basic parameters will be set equal to the values in Ghent (see Appendix B.5). To evaluate the effect of one parameter, only the value of that parameter will be changed while keeping all others constant.

3.3.1 Possible improvements for different mechanisms

In general, the possibilities for waste reduction are the smallest for the DA mechanism with MTB and are comparable for the RSD and the Boston mechanisms. Table 3.4 shows the average upper bound on 100 data sets with the parameter values of Ghent, together with the standard deviations. Moreover, also the upper bounds for the waste improvements in Antwerp and Ghent are included. If, for example, the RSD mechanism would be used in Antwerp, the WRLD procedure will not be able to assign more than 4.64 extra students to a school. In the remainder of this section, the WRLD procedure

Table 3.4: Comparison of the average upper bound on the waste (and the standard deviation σ_{UB}) that can be reduced by the WRLD procedure in generated data (average over 100 generated data sets, 200 considered tie-breaking rules each) and in Antwerp and Ghent (200 considered tie-breaking rules)

Mechanism	UB	$\sigma_{\mathbf{UB}}$	$\mathbf{UB_{ANT}}$	$\mathbf{UB_{GHE}}$
DA(MTB)	1.04	0.43	3.32	0.36
RSD	1.53	0.64	4.64	1.72
BOSTON	1.62	0.68	3.49	2.31

will be applied to the probabilistic assignments that result from the RSD mechanism, as its small computation time will facilitate the execution of a larger number of iterations.

3.3.2 Impact of data generation parameters

To get a better idea of the drivers behind the size of the waste reduction possibilities by the WRLD procedure, the effects of several data properties will be evaluated in this subsection.

First of all, as shown in the left panel of Figure 3.5, the relative waste reduction possibilities with respect to the number of students are larger when the average number of students per school is low, which is equivalent to a relatively large number of schools. This can be explained by the fact that, in each school, the relative size of the possible waste, with respect to the capacity, is independent from, or even negatively correlated with the school's capacity (this correlation is equal to -0.02 in Antwerp and to -0.26

in Ghent). Therefore, a larger relative number of schools implies a higher total waste.[5] Moreover, the relative waste reduction possibilities seem to be slightly higher for a larger number of students.

Secondly, the possibilities for waste reduction are the largest when the total capacity is approximately equal to the number of students, as shown in the right panel of Figure 3.5. The logic behind this observation is that, when the overall capacity is very small (large), the degree of wastefulness will be small as most seats in schools (resp. most students) are assigned.

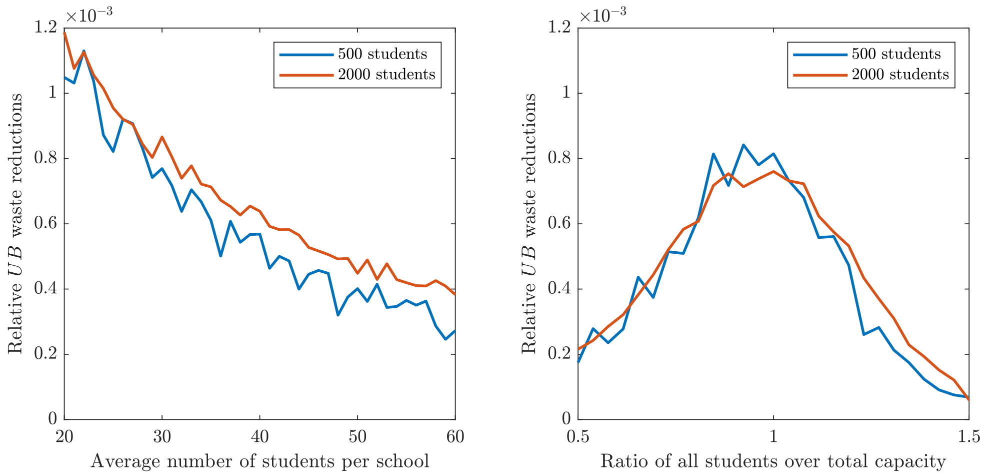


Figure 3.5: Average relative waste reduction possibilities with respect to the problem instance size (left panel) and to the ratio of all students over the total capacity (right panel) (average over 200 data sets, each with $|\tilde{\mathcal{T}}| = 200$)

Lastly, as shown in Appendix B.11, both the effect of the preference lists' length on the popularity of the submitted schools (Δ_1) and the correlation between the capacity and the popularity of the schools (ρ_{cp}) have no clear impact on the size of the possible waste reductions.

[5] Another explanation might be that the upper bound on the relative waste is analysed, in which double counting of waste is more likely if a student experiences waste on different schools, rather than the actual reduction by the WRLD procedure. However, as discussed in Section 3.3.3, the size of this double counting will not be large as for a sufficiently large sample of tie-breaking rules $\tilde{\mathcal{T}}$, the waste can be reduced by an amount that is approximately equal to the upper bound.

3.3.3 Impact of number of considered tie-breaking rules

So far, only the upper bound on the waste that can be captured has been considered. As the set of all tie-breaking rules $\mathcal{T}$ and corresponding matchings $\mathcal{M}$ is very large, however, it is only possible to consider a subset of tie breaking rules $\tilde{\mathcal{T}}$ and corresponding matchings $\tilde{\mathcal{M}}$ for the WRLD procedure. Therefore, the final proportion of the waste that can be captured depends on the properties of the matchings in $\tilde{\mathcal{M}}$. As, in this thesis, the matchings in $\tilde{\mathcal{M}}$ are simply a random sample of $\mathcal{M}$, the size of the sample will determine the proportion of waste that is captured, as a larger sample implies a larger number of possible decompositions.

As can be seen in the left panel of Figure 3.6, the marginal increase in the proportion of captured waste by adding more matchings to the sample is very small when the size of the sample is approximately equal to five times the number of students. Of course, a larger sample size will always lead to better results, but, as shown in the right panel of Figure 3.6, larger samples require a longer computation time.[6] Therefore, a sample size of five times the number of students can serve as a rule of thumb that balances the trade-off between the proportion of waste captured and the computation time.

With respect to the number of students, the left panel of Figure 3.6 shows that, when the number of students increases, a larger relative number of considered tie-breaking rules is needed to capture the same proportion of waste. At the same time, a larger number of students implies a strong increase in the computation times, as shown in the right panel of Figure 3.6.

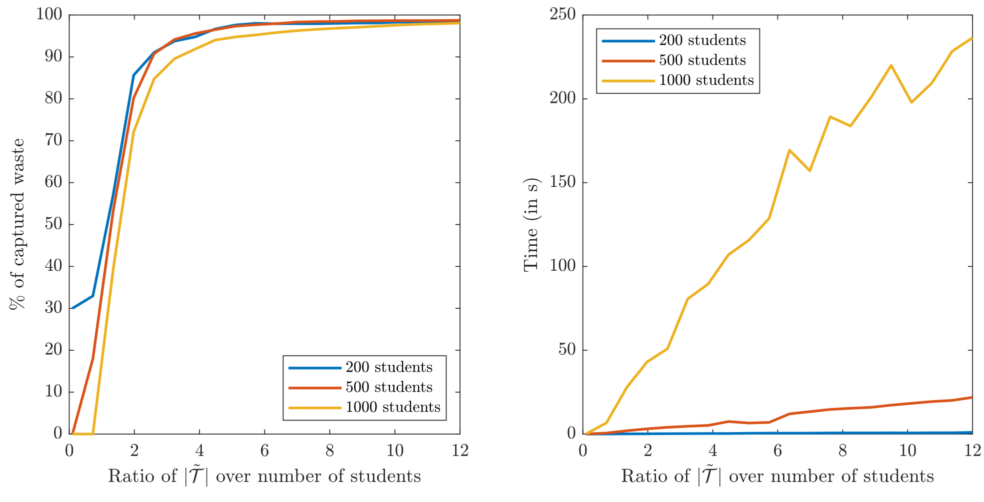


Figure 3.6: Proportion of captured waste (left panel) and time (right panel) for the WRLD procedure (average over 10 data sets)

[6]The model was implemented using CPLEX for MATLAB and was run on a Dell Latitude E6530 PC with an Intel(R) Core(TM) i5-3340M 2.7GHz processor and 8 GB RAM, equipped with Windows 10.

3.3.4 Profile of WRLD assignment for Antwerp and Ghent

Because of the way in which the WRLD procedure is defined, it will always result in a probabilistic assignment that stochastically dominates the probabilistic assignment to which the procedure was applied. Table 3.5 displays the expected number of students the WRLD procedure assigns additionally to each preference in comparison to the RSD mechanism, together with the required time to obtain the improvements and the number of tie-breaking rules in the sample set $\tilde{\mathcal{T}}$. For Ghent, the number of tie-breaking rules in $\tilde{\mathcal{T}}$ is set equal to 15,000, following the observations from Section 3.3.3. For Antwerp, however, the results could not be obtained for two reasons. First of all, a smaller number of samples has to be used, because the memory needed to store the binary variable m_{ij}^{τ}, which equals one if in matching $M_\tau \in \tilde{\mathcal{M}}$ student $c_i \in C$ is assigned to school $s_j \in S$, exceeds the available RAM memory.[7] Secondly, even for a smaller sample size of 8,000 tie-breaking rules, the CPLEX solver could not find an optimal solution in a period of 18 hours because of the large size of the problem instance.

Table 3.5: Expected number of additionally assigned students by the WRLD procedure in comparison to the RSD mechanism in Ghent (with $|\tilde{\mathcal{T}}| = 15{,}000$)

Preference	Ghent
1	0.630
2	0.760
3	0.008
4	0.089
⋮	⋮
UNASSIGNED	-1.485
Waste captured %	83.68%
time (in min)	79.9
Number of beneficial students	367 (11.91%)

In Ghent, the majority of the efficiency gains are students who are extra assigned to their school of first or second choice. Moreover, approximately 12% of the students would experience an expected increase in their overall allocation probabilities, but these increases are very small on average.

[7]Defining this variable as a sparse matrix would solve this issue, but this has not been done because MATLAB does not support three-dimensional sparse matrices.

3.4 Maximin decomposition

The main performance measure for the Maximin decomposition will be the *worst-case difference* (WCD), which equals the difference between the minimum number of assigned students in any of the decomposition's matchings between the Maximin decomposition and the initial decomposition. Consider a probabilistic assignment $P \in \mathcal{P}$ that can be decomposed into a weighted sum of the matchings in $\tilde{\mathcal{M}}$. Denote the set of matchings $M_\tau \in \tilde{\mathcal{M}}$ that have a strictly positive weight in the decomposition by $\tilde{\mathcal{M}}_s \subset \tilde{\mathcal{M}}$, and let χ be the minimum number of assigned students in all matchings in $\tilde{\mathcal{M}}_s$:

$$\chi = \min_{M_\tau \in \tilde{\mathcal{M}}_s} \Big\{ \sum_{c_i \in C} \sum_{s_j \in S} m_{ij}^\tau \Big\}.$$

The worst-case difference of an alternative decomposition of P, in which the set of matchings with a strictly positive weight is denoted by $\tilde{\mathcal{M}}'_s$ and the minimum number of assigned students in $\tilde{\mathcal{M}}'_s$ by χ', is defined as:

$$\text{WCD}(\tilde{\mathcal{M}}'_s, \tilde{\mathcal{M}}_s) = \chi' - \chi.$$

The relative worst-case difference between decompositions is the ratio of the worst-case difference over the number of students.

In any probabilistic assignment $P \in \mathcal{P}$, χ will never be larger than the expected number of assigned students in P, rounded down to the closest integer. Let χ_0 be the minimum number of assigned students in an initial decomposition of P with the matchings in $\tilde{\mathcal{M}}_s$, then an upper bound on the improvements in the worst-case difference by the Maximin decomposition can be defined as:

$$\text{UB}_{\text{WCD}}(\tilde{\mathcal{M}}_s) = \Big\lfloor \sum_{c_i \in C} \sum_{s_j \in S} p_{ij} \Big\rfloor - \chi_0.$$

An upper bound that is equal to 20, for example, would mean that it is not possible to find a decomposition in which each matching will assign at least 21 students more than the matching of the original decomposition that assigns the least students. In Section 3.4.3, the extent to which this improvement can be captured will be discussed.

Similarly to Section 3.3, the data to evaluate the performance of the Maximin decomposition will be generated by using the parameters of the data set of Ghent.

3.4.1 Possible improvements in WCD for different mechanisms

Table 3.6 shows that, as expected from the comparisons of the mechanisms on the data of Antwerp and Ghent (Appendices B.9 and B.10), the minimum number of assigned students is the highest among the traditional mechanisms for the DA with MTB. Moreover, a clear increase in the minimum number of assigned students χ is present because of the WRLD procedure, although the increase in the average number of assigned students by the WRLD procedure is rather small. This implies that the possible improvements in the worst-case differences are the largest for the traditional mechanisms. Moreover, Table 3.6 shows the possible improvements in Antwerp and Ghent.[8] The upper bound for the increase in χ by the Maximin decomposition when the DA with MTB is used in Ghent, for example, is equal to 23 students.

Table 3.6: Average minimum (χ) and expected number of assigned students, upper bound for Maximin decomposition (UB) for different mechanisms (average over 20 generated data sets with 500 students and 15 schools, each with $|\tilde{\mathcal{T}}| = 2{,}000$) and the upper bounds for Antwerp and Ghent ($|\tilde{\mathcal{T}}| = 15{,}000$)

Mechanism	χ	**Average assigned**	**UB**	$\sigma_{\mathbf{UB}}$	$\mathbf{UB_{ANT}}$	$\mathbf{UB_{GHE}}$
DA(MTB)	468.0	477.8	9.35	2.28	25	23
WRLD(DA(MTB))	469.0	478.0	8.30	2.10	-	23
RSD	464.6	474.8	9.70	2.15	33	25
WRLD(RSD)	466.1	475.0	8.35	2.13	-	25
BOSTON	463.6	473.5	9.35	2.35	33	23
WRLD(BOSTON)	465.0	473.7	8.20	2.02	-	20

Similarly to Section 3.3, in the remainder of this section, the Maximin decomposition will be applied to the probabilistic assignments that result from the RSD mechanism.

3.4.2 Impact of data generation parameters

The effects of the number of students and schools on the maximum relative improvements by the Maximin decomposition differs from the WRLD procedure (Section 3.3.2) in two respects. Firstly, the upper bound on the relative improvements is lower for a larger number of students, as shown in the left panel of Figure 3.7. This could be explained by the fact that the relative difference between the smallest and the largest number of assigned students, over all matchings in the decomposition, decreases with the number of students (illustrated in Appendix B.12). Secondly, the average number of students on each school seems to have a small negative impact on the relative WCD when the number of students is small, but not for a larger number of students.

The effect of a change in overall capacity, on the other hand, is similar to the effect

[8] As mentioned in Section 3.3.4, obtaining the improvements by the WRLD procedure for Antwerp with a large enough set of samples is computationally intensive. Therefore, the results for the WRLD procedure could not be obtained.

for the WRLD procedure: the possibilities for improvements in relative WCD are the largest when the number of students is slightly larger than the total capacity (right panel Figure 3.7). In a problem instance with a very small (large) overall capacity, most seats in schools (resp. most students) will be assigned, regardless of the random tie-breaking rules in $\tilde{\mathcal{T}}$. This causes a decrease in the variability of the number of assigned students over all matchings, which in turn decreases the upper bound on the relative worst-case difference.

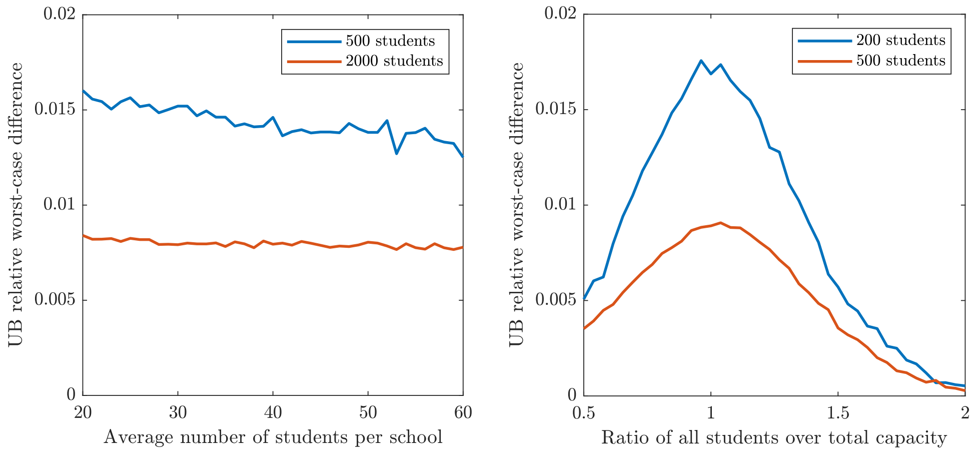


Figure 3.7: Average relative worst-case difference with respect to the problem instance size (left panel) and to the ratio of all students over the total capacity (right panel) (average over 200 data sets, each with $|\tilde{\mathcal{T}}| = 200$)

Lastly, as displayed in Appendix B.13, the upper bound for the relative WCD is slightly lower when the popularity of the submitted schools is more dependent on the length of the preference list (Δ_1). The correlation between a school's capacity and its popularity, on the other hand, has no clear impact on the relative WCD.

3.4.3 Impact of number of considered tie-breaking rules

Similarly to the WLRD procedure, the quality of the Maximin decomposition depends on the sample of considered tie-breaking rules $\tilde{\mathcal{T}}$ and corresponding matchings $\tilde{\mathcal{M}}$. To illustrate this, consider the Maximin decomposition of an arbitrary probabilistic assignment $P \in \mathcal{P}$ for two sets of considered tie-breaking rules of different sizes, as shown in Figure 3.8. This example clearly illustrates that the Maximin decomposition also causes a strong decrease in the proportion of the final matchings that assign many students to a school, as it would otherwise not be possible to assign the same expected number of students as in P.

The left panel in Figure 3.9 shows that, in comparison to the WRLD procedure, a larger

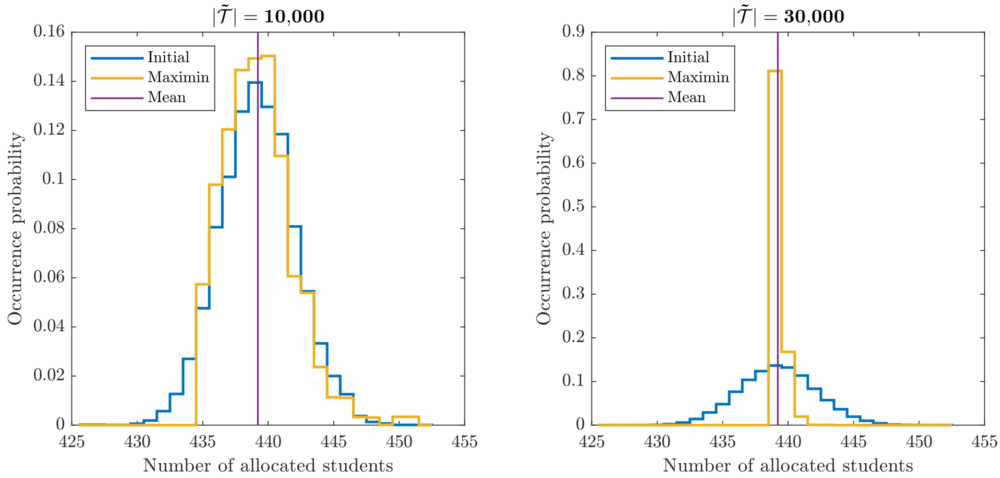


Figure 3.8: Distribution of the number of allocated students for RSD and Maximin decomposition (500 students)

sample size is required to obtain a relative worst-case difference that is close to the upper bound. Moreover, increasing the number of students strongly affects the gap between the upper bound and relative worst-case difference of the found solution. This implies that larger instances require a larger sample size. Unfortunately, the computation times increase with the ratio of the sample size over the number of students and, therefore, no clear rule of thumb can be defined on the sample size.

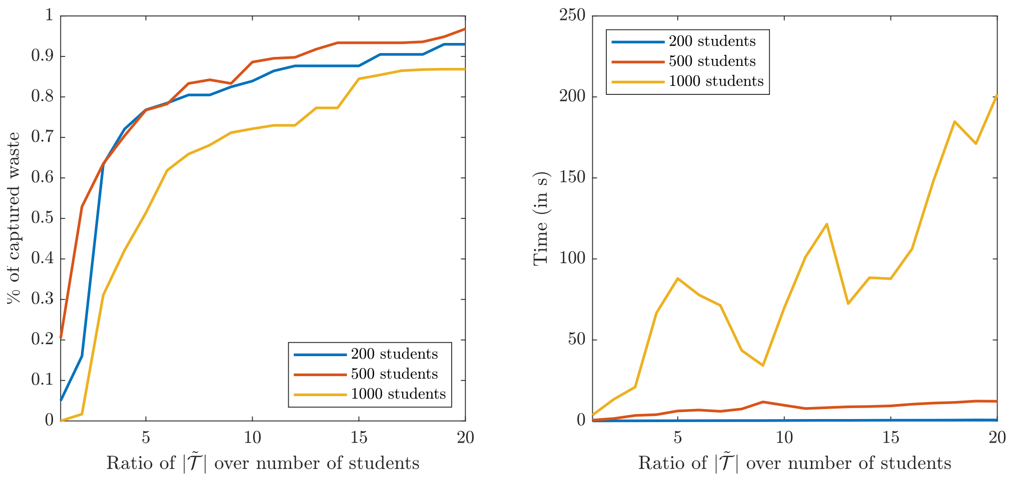


Figure 3.9: Proportion of realized WCD (left panel) and time (right panel) with respect to the relative size of $\tilde{\mathcal{T}}$ for the Maximin decomposition (binary search method) (average over 10 data sets)

3.4.4 MILP-formulation vs. binary search

In Sections 2.3.2 and 2.3.4, two different models to obtain the Maximin decomposition of a probabilistic assignment P have been discussed. The first method uses a Mixed Integer Linear Programming (MILP) model and the second a binary search method. Both methods obtain the same result, but, as shown in Figure 3.10, the computation time of the binary search method increase at a slower pace than the one of the MILP-model. Therefore, it is advisable to obtain the Maximin decomposition of a probabilistic assignment with the binary search method.

If the binary search method would be applied to the data set of Ghent and for a sample set of 20,000 tie-breaking rules, an improvement in the worst-case difference of eight students can be realised. As each of the five iterations of the binary search method requires approximately one hour of computation time, the overall time to find the optimal solution in Ghent for the given $\tilde{\mathcal{T}}$ is about 4.5 hours. For the reasons explained in Section 3.3.4, the Maximin decomposition for the data set of Antwerp for a sufficiently large sample of tie-breaking rules could not be found within a reasonable time period.

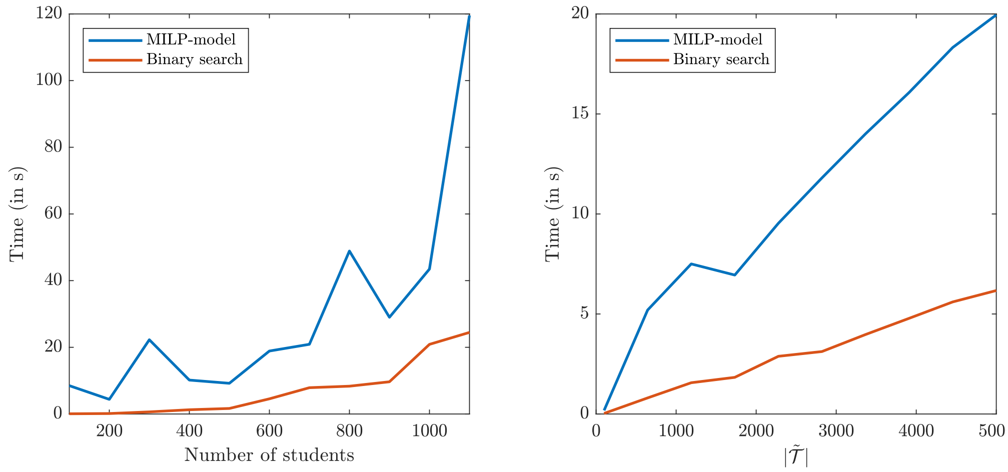


Figure 3.10: Time for different methods to obtain Maximin decomposition with respect to number of students (left panel, with $|\tilde{\mathcal{T}}| = 2{,}000$) and sample size (right panel, with 500 students) (average over 10 data sets)

3.5 Probabilistic Serial mechanism

This section will discuss the performance of the PS mechanism introduced by Bogomolnaia and Moulin (2001). Whereas the WRLD procedure reduces as much waste as possible, the PS mechanism always results in an ordinally efficient assignment, which implies non-wastefulness. Therefore, no improvement can be realised by applying the WRLD procedure to the assignment of the PS mechanism. Section 3.5.1 compares the profiles of both methods and Section 3.5.2 discusses the possibilities for applying the Maximin decomposition to the assignment of the PS mechanism.

3.5.1 WRLD vs. PS

The left panel of Figure 3.11 contains the average difference in the number of assigned students between (i) the PS and the WRLD procedure, (ii) the PS and the RSD and (iii) the WRLD procedure and the RSD. The PS mechanism does not stochastically dominate the assignment from the WRLD procedure on average: slightly more students will be assigned to their first choice under the PS, but the overall number of assigned students is slightly larger for the WRLD procedure. The reasoning behind this is that, by construction, the WRLD procedure reduces waste in a given wasteful assignment by maximizing the expected number of students, whereas the PS makes sure that no students would want to exchange allocation probabilities for different schools, which implies non-wastefulness.

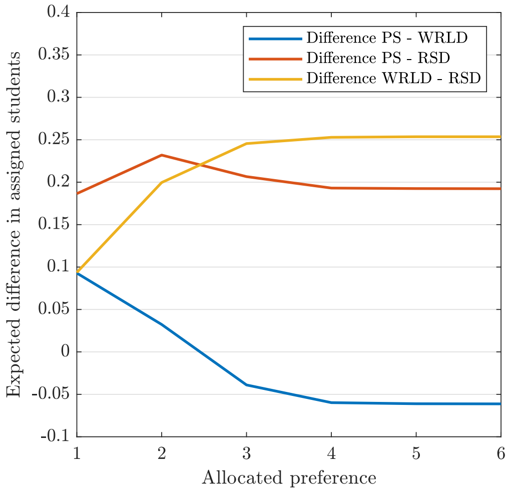
The right panel of Figure 3.11, in turn, shows in how many of the 50 data sets a mechanism's assignment stochastically dominated another mechanism's assignment. By construction, the WRLD procedure stochastically dominated the RSD in all data sets. The PS mechanism, however, only stochastically dominated the RSD in almost half of the data sets and was even dominated once. Although this seems counter-intuitive at first sight, this observation can be explained by the fact that only a sample set of 5,000 tie-breaking rules has been considered and not the entire set of all tie-breaking rules $\mathcal{T}$. Imagine, for example, that only one tie-breaking rule $\tau \in \mathcal{T}$ would be considered for the RSD mechanism. In that case, if the resulting matching $M_\tau \in \mathcal{M}$ assigns more students to a school than the expected number of assigned students by the PS mechanism, that assignment is not stochastically dominated by the PS. In fact, as shown in Appendix B.14, regardless of the sample size, PS will dominate the RSD in approximately half of the data sets.

Lastly, the PS stochastically dominated the WRLD procedure in four data sets, whereas the assignment of the WRLD procedure dominated the PS once. Given these observations, it cannot be concluded that one of the two mechanisms outperforms the other, as both mechanisms have their merits.

3.5.2 Maximin decomposition of PS

The Maximin decomposition can be applied to any probabilistic assignment, including the resulting assignment $P_{PS} \in \mathcal{P}$ from the PS mechanism. However, in contrast to the RSD or the RDA mechanism, P_{PS} is not an equally weighted average of the matchings in $\tilde{\mathcal{M}} \subset \mathcal{M}$ that are related to the sample of tie-breaking rules in $\tilde{\mathcal{T}} \subset \mathcal{T}$.[9] Hence, a feasible decomposition of the assignment P_{PS} with the matchings in $\tilde{\mathcal{M}}$ might not exist

[9] Assuming that duplicate matchings in $\tilde{\mathcal{M}}$, as discussed in Section 2.4, have not been removed.



	PS	WRLD	RSD
PS	-	(4,1)	(22,1)
WRLD		-	(50,0)
RSD			-

Figure 3.11: Difference in profile PS compared to WRLD and RSD (left panel) and the number of times a mechanism stochastically dominates/is stochastically dominated by another mechanism (right panel) (over 50 data sets with 500 students and $|\tilde{\mathcal{T}}| = 5{,}000$)

if the matchings in $\tilde{\mathcal{M}}$ are simply determined by random sampling, although a (generally not unique) decomposition with the set of all matchings in $\mathcal{M}$ is proven to exist by the Birkhoff-von Neumann theorem that has been described in Section 2.1.1. As shown in Appendix B.15, the proportion of the data sets in which a feasible decomposition of the probabilistic assignment by the PS mechanism can be found, decreases with the number of students. For a problem instance of only 1,000 students, for example, and a large sample of 20,000 tie-breaking rules a feasible decomposition only exists in half of the generated data sets.

To tackle this issue, an alternative approach, next to the possibility of column generation that has been discussed in Section 2.5, is to construct a feasible decomposition of the probabilistic assignment P_{PS}. Kesten et al. (2017), for example, proposed a method to construct an equal-weight decomposition of P_{PS} in which all individual matchings of the decomposition can be obtained by the RSD mechanism. After constructing such a feasible decomposition, the Maximin decomposition can be applied to this set of matchings, which could optionally be supplemented by other matchings in $\mathcal{M}$. A possible direction for further research consists of implementing this approach and investigating its performance.

3.6 Strategy-proofness

As has been mentioned in previous sections, the RSD and the DA are strategy-proof mechanisms and the Maximin decomposition does not affect strategy-proofness. The WRLD procedure, the School-proposing DA, the PS and the Boston mechanism, on the other hand, are not strategy-proof. Nevertheless, some mechanisms that are not strategy-proof might be more vulnerable to manipulation than others. This section will evaluate the size of the incentives to misreport for these non-strategy-proof mechanisms.

3.6.1 Strategy-proofness axioms

As discussed in Section 1.2.5, several relaxations and measurements of strategy-proofness have been proposed, but in this thesis, the axiomatic approach of Mennle and Seuken (2014) will be adopted because it can be applied to data sets that are not *large* (in the sense of Kojima and Manea (2010), see Section 1.2.5) and because the authors define strategy-proofness by three intuitive axioms that can be verified for a certain mechanism on a given data set.

Mennle and Seuken (2014) stated that a mechanism is strategy-proof if swapping the positions in the preference list of two adjacent schools $s_j, s_k \in S$ with $s_j >_{c_i} s_k$ does not affect the allocation probabilities of the other schools and will either leave the allocation probabilities of the swapped schools s_j and s_k unchanged, or will cause both a strict increase in the allocation probability of school s_k that is given a higher preference after the swap and a strict decrease in the allocation probability of school s_j that is given a lower preference. Appendix B.16 contains a detailed description of their theory.

They showed that, although the PS mechanism is not strategy-proof, swapping the positions in the preference list of two adjacent schools can only affect the allocation probabilities of less preferred schools in the PS and not of the swapped schools themselves or of more preferred schools. For the WRLD procedure, on the other hand, no such guarantee can be given as swapping two adjacent schools in the preference list might change the allocation probability of any school in the preference list (see Table B.5 for a comparison of all mechanisms), but the extent to which this affects students' incentives and their benefits of misreporting will be evaluated in the next subsection.

3.6.2 Incentives for misreporting in different mechanisms

Unlike for the Boston mechanism, the ways in which a student can benefit by misreporting are not clear in the PS, the School-proposing DA and in the WRLD procedure. To check whether or not it is beneficial for a student to submit their best possible alternative preference list $>'_{c_i}$, assuming that student c_i knows his/her best alternative, an assumption has to be made about the underlying utility function u_i of c_i. Considering a survey about the utility function in Ghent in 2013-2014, as shown in Appendix B.17 (D'haeseleer, 2016), a conservative assumption would be a utility function

$u_i = (1, 0.9, 0.81, ..., 0)$ for all students $c_i \in C$, in which the schools are ranked in order of decreasing preference. This is a conservative assumption since the incentives to misreport are higher when students are more indifferent between schools (see Example B.16.1). In u_i, the utility for the school of k-th choice is equal to 90% of the utility for the school of $(k-1)$-th choice and the utility for the outside option and for all schools that are less preferred than the outside option equals zero.

As considering all alternative preference lists is computationally intensive, only a subset of all preference lists will be considered. To determine the useful subset of alternative preference lists, the schools are firstly ordered according to the following rules:

- The school of first choice in the truthful preference list $>_{c_i}$ will be among the first three choices in $>'_{c_i}$;
- Only the positions of the five most preferred schools in $>_{c_i}$ are exchanged in $>'_{c_i}$ and the order of the other schools remains unchanged;
- To consider a selection of the preference lists that contain a school that is not in $>_{c_i}$, the most popular school that is not in $>_{c_i}$ is selected and different permutations of this school and the two most preferred schools in $>_{c_i}$ are created. The other places in $>'_{c_i}$ are filled with the remaining schools of $>_{c_i}$ without changing their order.

These ordered lists of schools are then truncated to obtain preference lists of different lengths. Denote the set of students who experience a *gain* in utility by submitting their best alternative preference list $>'_{c_i}$, instead of their true preference list $>_{c_i}$, by $C_g \subset C$.

As shown in Table 3.7, the average gains in utility that can be made by misreporting are very small for the PS mechanism and the WRLD procedure. Although the number of students that could possibly experience a gain in the WRLD procedure is relatively large, the size of all gains is limited and the maximum gain in utility that can be made by misreporting is even smaller than for the PS. It is clear, however, that the Boston mechanism is more vulnerable to manipulation by misreporting, as both the size and the number of gains are large. For the School-proposing DA, on the other hand, the possibilities for manipulation are extremely small.

Furthermore, Figure 3.12 shows that, for the PS mechanism, the incentives for misreporting decrease strongly when the relative number of schools decreases (i.e. when the average number of students per school increases).[10] In Antwerp and in Ghent, the average number of students per school is equal to 23 and 48 (Table 3.1), which shows that

[10] The graph is obtained for the PS mechanism, as the PS is computationally more efficient than the WRLD procedure or the Boston mechanism. Moreover, the number of students for which the graph is obtained is rather small as for each alternative preference list, a probabilistic assignment has to be calculated. As there are many possible alternative preference lists, this is a computationally intensive task.

Table 3.7: Utility gains by misreporting best alternative preference list (average over 10 data sets with 50 students and 10 schools)

	PS	**WRLD**	**BOSTON**	**School-prop. DA(MTB)**
Average gain over C	0.0004	0.0005	0.0087	0.0000
Average gain over C_g	0.0294	0.0025	0.0803	0.0002
Students with gain	0.6 (1.20%)	5.8 (11.60%)	5.4 (10.80%)	0.5 (1.00%)
Maximum gain	0.0817	0.0347	0.3364	0.0008
Average utility	0.9135	0.9137	0.9106	0.9122

the incentives for misreporting in the PS will be very low in both cities. This observation is in correspondence with the result by Kojima and Manea (2010) that it is a dominant strategy to report truthfully for the PS if the problem instance is *sufficiently large*.

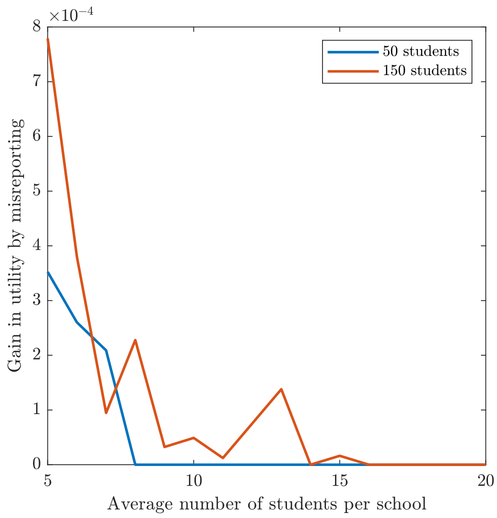


Figure 3.12: Average gain in utility by misreporting for PS for different problem instance sizes (average over 10 data sets)

In conclusion, it can be said that although the PS mechanism and the WRLD procedure are not strategy-proof, they are not highly vulnerable to manipulation by students, in contrast to the Boston mechanism. For the PS mechanism, submitting a false preference list in which two schools are swapped might only be beneficial for the schools that are less preferred than the swapped schools. For the WRLD procedure, on the other hand, this guarantee is not valid, but as shown in Table 3.7, the gains of submitting an alternative preference list are limited in size.

Chapter 4

Considerations on implementation

In real-life student assignment problems, several additional complications are present that have not been discussed in the previous chapters. This chapter briefly discusses how the proposed methods could be combined with measures to improve the social mix (Section 4.1), how the separation of twins could be avoided (Section 4.2) and to what extent the proposed methods will be perceived as transparent (Section 4.3).

4.1 Social mix

This thesis did not investigate which mechanism should be applied if the mechanism designer has the objective to reduce the level of school segregation between minority and majority students, also called *affirmative action* (see Section 1.3). Designing such a mechanism is not straightforward; it has been shown that in several mechanisms that involve affirmative action, some students might actually be worse off in comparison to the situation without affirmative action. If this is the case, the mechanism is said to violate the property of *minimal responsiveness* (Kojima, 2012). For the system with *majority quota*, in which no school can assign more majority students than its majority quota, these detrimental effects for minority student have been illustrated by Kojima (2012). In response to these results, Hafalir et al. (2013) proposed a system with *minority reserves*, in which each school reserves a certain number of seats for minority students, although majority students can also be assigned to those seats provided that no minority student prefers that schools to his/her assigned school. However, Doğan (2016) found that their proposal also violates minimal responsiveness, and he proposed a method that is minimally responsive, but not strategy-proof.

As the WRLD procedure and the Maximin decomposition that have been introduced in Chapter 2 can both be applied to all mechanisms that result in a probabilistic assignment, they can be applied to any mechanism that involves affirmative action. However, depending on the specific mechanism, some improvements can be made to the performance of the WRLD procedure with respect to the social mix. Suppose, for example, that school $s_j \in S$ with a capacity of q_j and a minority reserve of $r_j^m < q_j$ experiences a waste of $w_j \in [0, 1]$ and that both a minority student $c^m \in C$ and a majority student $c^M \in C$ prefer school s_j to the outside option and experience a waste $w > w_j$. If the expected number of assigned minority students to s_j is smaller than the minority reserve, and the difference is larger than the waste w_j on school s_j, then the waste of minority student c^m on school s_j would ideally be reduced by w_j. The WRLD procedure, however, will make no distinction between reducing the waste of student c^m or of student c^M. A possible direction for further research consists of developing a method to implement this in the WRLD procedure.

4.2 Avoiding twin separation

Twins who submitted the same preference lists but are not assigned to the same school are an often heard critique in the popular press.[1] One possible method to avoid this issue is to submit one single preference list for both twins, but to reduce the capacity of a school by two if the twins are assigned to that school. However, by doing this, twins will have a slightly lower chance of being assigned to a school than an individual student with the same preference list, as the twins will be rejected on a school if only one seat remains. Alternatively, a system could be put in place in which schools can increase their capacity by one if twins have the highest priority when only one seat remains. This system, in turn, would cause the allocation probabilities for twins to be slightly better than for an individual student with the same preferences. To illustrate the underlying intuition, consider an example with two available school seats for one individual student and one pair of twins. Whereas the individual student will only be assigned to the school if (s)he is selected for the first seat, the twins will always be assigned to this school by this alternative method, regardless of the tie-breaking rule.

As shown in Appendix B.18, which shows the difference in allocation probability to the school of first choice between twins and an individual student with the same preferences, the size of the difference becomes extremely small when the number of students per school increases for both methods. Further research might explore the existence of a general method to assign twins to schools without affecting the allocation probabilities compared to individual students and without violating the school capacities.

4.3 Transparency

A crucial element for the acceptance of any centralized application system is that parents perceive the system to be fair and transparent; parents should understand the allocation mechanism itself. Both the WRLD procedure and the Maximin decomposition, however, make abstraction of the specifics of the allocation mechanism and add an extra level of complexity. Moreover, both methods make use of a Linear Programming model, which is a technique that most parents are not familiar with.

[1] See, for example, Snoekx (2018).

Nevertheless, one could argue that as long as parent clearly understand the objective of a certain method and trust the method to achieve this objective, an understanding of the internal working of the method is not essential. The main idea behind both methods that have been introduced in this thesis could be explained as making a selection of the "best" final allocations of students to schools and assign weights to them. For the Maximin decomposition, the "best" final allocations are the ones in which the number of students that have not been assigned to a school are limited. For the WRLD procedure, on the other hand, explaining which are the "best" final allocations is less straightforward, as the presence of waste in the probabilistic assignments of, for example, the RSD or RDA mechanism is not intuitive.

Finally, in the case of the Maximin decomposition, a strong emphasis should be put on the fact that submitting true preferences is still a dominant strategy and that the Maximin decomposition will only make sure that the "bad" allocations, in which many students are not assigned to any school, can no longer be selected to determine the final matching.

Chapter 5

Conclusion

In this thesis, I investigated how student welfare can be improved in a central allocation system of students to schools when schools are indifferent between large groups of students and ties between students are broken randomly, as is the case for secondary education in Flanders. In the literature review, it has been shown that using randomness as a criterion to create artificial school priorities (*random tie-breaking*) in the traditional mechanisms such as the Random Serial Dictatorship (RSD) or the Randomized Deferred Acceptance (RDA) mechanism implies both a loss of efficiency and uncertainty about the final number of students that are assigned to a school. Two new methods to tackle these issues were introduced in this thesis. Both methods obtain their specific objective by assigning weights to all possible ways of randomly breaking ties between students (*tie-breaking rules*). These weights represent the probability with which a tie-breaking rule will be selected to obtain the final allocation of students to schools.

Firstly, it is possible that, because of the use of random tie-breaking, not all available seats on a school are assigned, whereas there are students who have a positive probability of not being assigned to any school and who would prefer being assigned to that school to not being assigned at all (*waste*). To tackle this issue, the Waste-Reducing Lottery Design (WRLD) procedure will increase the allocation probabilities of these student-school pairs by assigning specific weights to all tie-breaking rules. The WRLD procedure is an alternative to the Probabilistic Serial (PS) mechanism by Bogomolnaia and Moulin (2001) and, overall, it will generally assign a slightly higher number of students to a school, whereas the PS will assign slightly more students to their school of first choice. Both the WRLD procedure and the PS mechanism are not perfectly strategy-proof, but the possibilities for manipulation are very limited.

Secondly, the use of random tie-breaking implies uncertainty about the final number of assigned students. To reduce this uncertainty, the Maximin decomposition will maximize the total number of students that are assigned to a school in the worst-case scenario. In doing so, the Maximin decomposition does not harm strategy-proofness as it respects the allocation probabilities that were obtained by the mechanism to which it is applied.

Several possible directions for further research can be explored to improve the performance and to reduce the computation time of both the WRLD procedure and the Maximin decomposition. Firstly, as the total number of ways in which ties can be randomly broken is extremely large, in this thesis only a subset of these tie-breaking rules is considered for both methods. A possible direction for further research is to explore other methods of sampling that improve upon random sampling, as discussed in Section 2.4, and to examine the possibility of column generation, as discussed in Section 2.5. Moreover, it could be examined to what extent it is possible to geographically partition the student allocation problem by considering clusters of schools.

Lastly, I would like to emphasize the importance of the student allocation problem, as the effect of the attended school on the development of a student cannot be underestimated. Therefore, a further investigation of this problem is essential, in the hope that the obtained results will impact the implementation by well-informed decision makers.

Appendix A

Allocation mechanisms

A.1 Boston mechanism

The Boston mechanism proceeds in the following way:

– In the first step, each student applies to his/her most preferred school. If the number of applicants on a certain school is higher than the capacity of that school, the students with the highest priorities among the applicants are allocated to that school and the others are rejected.

– In general, in the k-th step, each student who was rejected at step $k-1$ applies to his/her school of next choice. If the number of applicants and temporarily allocated students on a certain school is higher than the capacity of that school, the students with the highest priorities among the applicants are allocated to that school and the others are rejected.

The Boston algorithm terminates when no student is rejected in a certain step. This is the case when all students are either assigned to a school from their preference list, or have been rejected by all schools on their preference list and have no more schools to apply to.

Example A.1.1. To illustrate the Boston algorithm, consider the example from Section 1.1. The following table displays the intermediate matchings in every step of the algorithm. For every step, the first column represents the school to which the corresponding student has applied and the second column represent the position of that school in his/her preference list. When a student is assigned to a school, that school is shown in a box.

student	**step 1**		**step 2**		**result**	
c_1	$\boxed{\mathbf{s_1}}$	**1**	$\boxed{\mathbf{s_1}}$	**1**	$\boxed{\mathbf{s_1}}$	**1**
c_2	s_3	1	s_1	2	0	0
c_3	s_3	1	$\boxed{\mathbf{s_2}}$	**2**	$\boxed{\mathbf{s_2}}$	**2**
c_4	$\boxed{\mathbf{s_3}}$	**1**	$\boxed{\mathbf{s_3}}$	**1**	$\boxed{\mathbf{s_3}}$	**1**

In the first step, all students apply to their most preferred school, but only students 1 and 4 are allocated. Rejected students 2 and 3, therefore, apply to their school of second choice and student 3 is allocated to school 2. Despite having a higher priority on school 1 than the allocated student 1, student 2 is rejected a second time as the available seat had already been assigned to student 1 in the first step. Because student 2 has been rejected on all schools of his/her preference list and has no more school to apply to, the algorithm terminates.

A.2 School-proposing Deferred Acceptance mechanism

The School-proposing Deferred Acceptance mechanism proceeds in the following way (Hafalir et al., 2013):

- In the first step, each school proposes all of their available seats to the students who have the highest priority in that school and who have included that school in their preference list. Consecutively, each student will only retain the proposed seat of the school (s)he prefers most among all proposing schools, and (s)he will reject the other proposals. If a student rejects a proposed seat, that student is removed from the priority list of that school.

- In general, in the k-th step, each school proposes all of the seats that have been rejected in the previous step to the students with the highest priority who are still present in the school's priority list and who are not temporarily accepted on that school. Consecutively, each student will only retain the proposed seat of the school (s)he prefers most among the seat (s)he temporarily accepted in the previous step and all proposing schools in this step. All other proposed seats are rejected. If a student rejects a proposed seat (or the seat that was temporarily accepted in the previous step), that student is removed from the priority list of that school.

The School-proposing Deferred Acceptance mechanism terminates when no school can propose to a student or when no proposed seats are rejected.

Example A.2.1. To illustrate the School-proposing Deferred Acceptance, consider the example from Section 1.1. As all schools only have one available seat, in the first step, each school proposes to the student with the highest priority. As no student is proposed to more than once, no seats are rejected and the algorithm terminates. The resulting matching is:

student	**school**	**preference**
c_1	s_3	3
c_2	s_1	2
c_3	s_2	2
c_4	0	0

A.3 Top Trading Cycle (TTC)

The Top Trading Cycle (TTC) mechanism proceeds in the following way (Abdulkadiroğlu and Sönmez, 2003):

– In the first step, assign a *counter* to each school to keep track of the number of seats that is still available on that school. This counter is initially set equal to the capacities of the schools. Create a graph that contains one node for every student and for every school. For each student, draw a directed edge from the student to his/her most preferred school. Similarly, for each school, draw a directed edge from the school to the student with the highest priority on that school. Since the number of students and schools are finite, there is at least one *cycle* constructed by these directed edges (Abdulkadiroğlu and Sönmez, 2003). As each node has at most one departing edge, each school and each student can be part of at most one cycle. All students that are part of a cycle are assigned to their most preferred school, i.e. the school to which the edge departing from that student's node pointed. The nodes corresponding to these students are removed from the graph. Moreover, the counter of each school in a cycle is reduced by one. If this causes the counter of a school to become zero, the node corresponding to this school is removed from the graph.

– In general, in the k-th step, draw a directed edge from each of the remaining students to their most preferred school among the remaining schools. Similarly, draw a directed edge from each of the remaining schools to the student with the highest priority among the remaining students. If no cycle exists, the algorithm terminates. Otherwise, all students in a cycle are assigned to the school to which their departing edge points, and the nodes corresponding to these students are removed from the graph. Moreover, the counter of each school in a cycle is reduced by one. If this causes the counter of a school to become zero, the node corresponding to this school is removed from the graph.

Example A.3.1. To illustrate the TTC mechanism, consider the example from Section 1.1. Both required steps to obtain a final allocation for this example, are represented in Figure A.1 as a graph in which a cycle is indicated by bold nodes and edges. As the capacities of each school are equal to one, the counters of the schools are not displayed.

In the first step, the graph contains one cycle, namely (c_1, s_1, c_2, s_3). Student 1 is, therefore, assigned to school 1 and student 2 is assigned to school 3. As this causes the number of available seats on both school 1 and school 3 to be equal to zero, the corresponding nodes are removed from the graph, as well as the nodes corresponding to the assigned students 1 and 2. In the second step, as school 3 has no more available seats, students 3 and 4 can no longer opt for their school of first choice. An edge is thus created from student 3 to his/her second choice, namely school 2. Student 4, however, has no school of second choice, which means that no edge departing from student 4 can be created. One cycle can be found, namely (c_3, s_2), and student 3 is assigned to

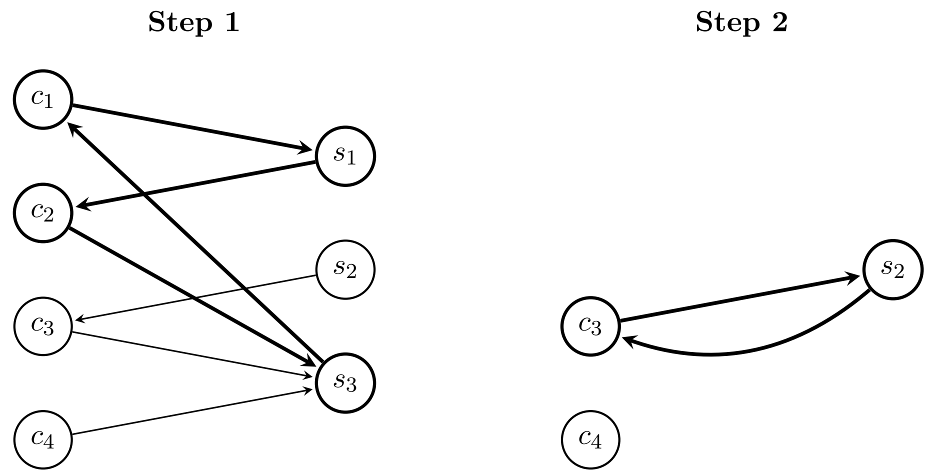


Figure A.1: Steps of TTC mechanism for Example from Section 1.1

school 2. The nodes corresponding to student 3 and school 2 will be removed and no further cycles can be found as no school has any available seats left.

To summarize, the following table displays the intermediate matchings in every step of the algorithm. For every step, the first column represents the school to which the corresponding student is assigned and the second column represent the position of that school in his/her preference list.

student	**step 1**		**step 2**	
c_1	s_1	1	s_1	1
c_2	s_3	1	s_3	1
c_3	0	0	s_2	2
c_4	0	0	0	0

Appendix B

Data and results

B.1 Distribution capacities of Antwerp and Ghent

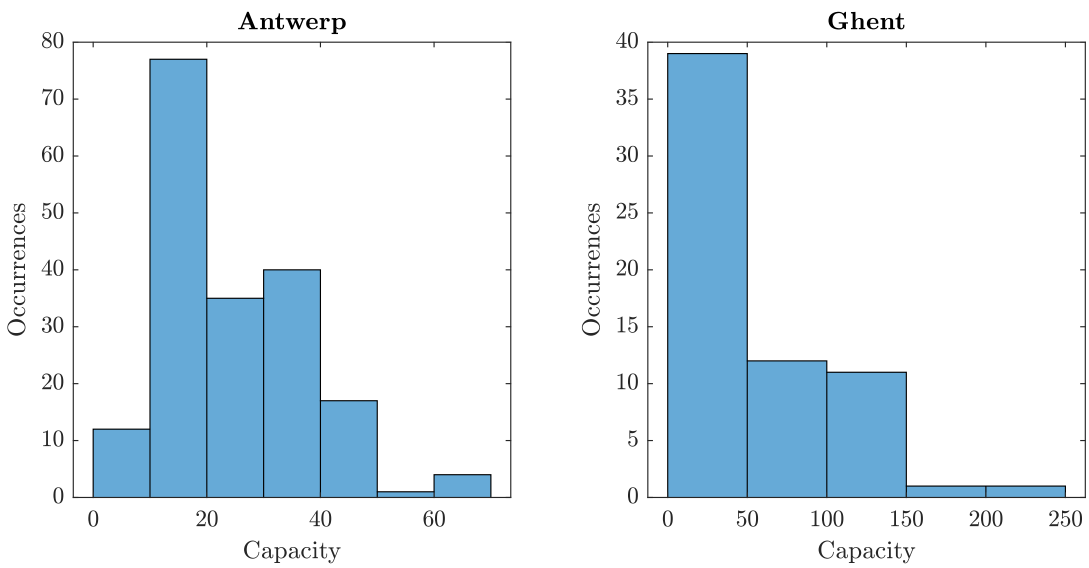


Figure B.1: Distribution of the capacities in Antwerp and Ghent

B.2 Distribution number of submitted preferences of Antwerp and Ghent

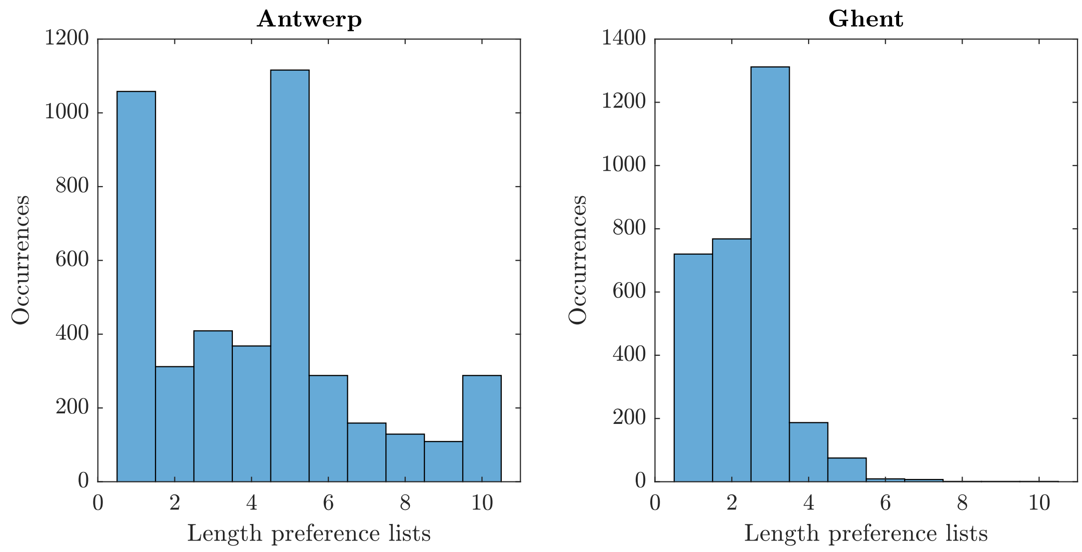


Figure B.2: Distribution of the number of submitted preferences per student in Antwerp and Ghent

B.3 Shortage of seats

Table B.1: Shortage of seats to assign all students to their school of first choice

	Antwerp	**Ghent**
Number of schools with shortage	66 (35.5%)	17 (26.6%)
Average shortage compared to capacity	41.5%	42.5%
Total shortage	508 (10.9%)	366 (9.9%)

B.4 Distribution popularity ratios of Antwerp and Ghent

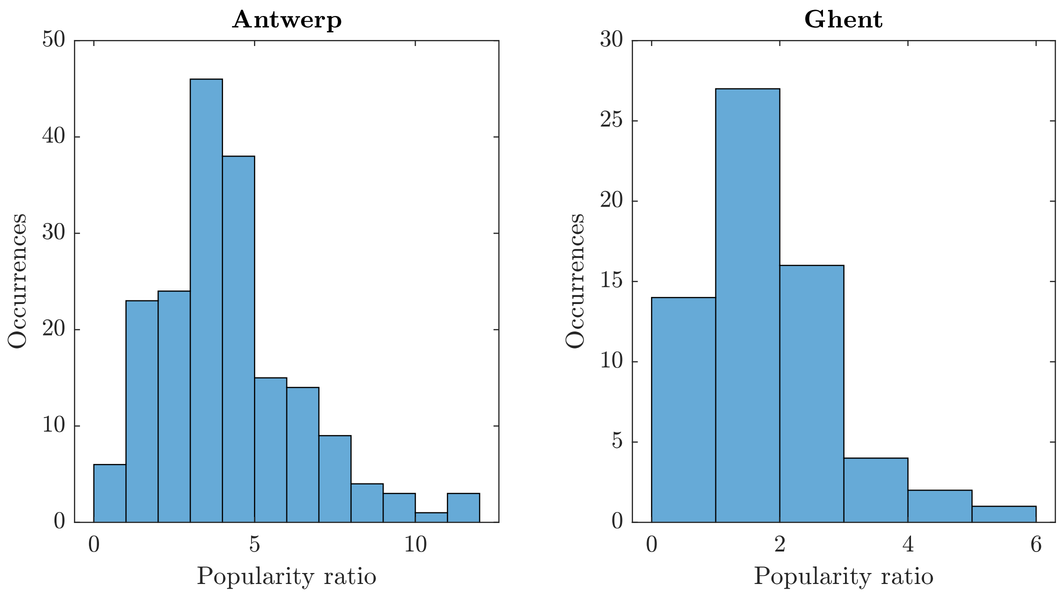


Figure B.3: Distribution of the popularity ratios in Antwerp and Ghent

B.5 Data generation: parameter benchmarks

Table B.2: Values of the parameters in the data sets of Antwerp and Ghent (for popularity % = 0.10)

Parameter	**Antwerp**	**Ghent**
Nstudents	4236	3081
Nschools	186	64
Capacity ratio	1.10	1.20
ρ_{cp}	-0.33	0.21
μ_{pref}	4.18	2.42
σ_{pref}	2.66	1.05
CV_c	0.49	0.80
CV_p	0.52	0.60
Δ_1	5.2%	14.0%
Δ_2	-5.6%	0.9%

B.6 Data generation: Cholesky factorization

Consider the matrix $S_0 \in \mathbb{R}^{m\times 2}$ with randomly generated values from the standard normal distribution. Imagine we want to obtain two variables $C, R \in \mathbb{R}^m$ that have a certain correlation ρ_{cr}, for example $\rho_{cr} = -0.5$. These variables can represent, for instance, the capacities and the priority ratios of the schools. Denote the desired covariance matrix by $K_{CR} \in \mathbb{R}^{2\times 2}$. As K_{CR} is a symmetric positive definite matrix, there exists a unique lower triangular $G \in \mathbb{R}^{m\times m}$ with positive diagonal entries such that $K_{CR} = GG^T$ (Cholesky factorization) (Golub and Van Loan, 1996, p. 143):

$$K_{CR} = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -0.5 & 0.8660 \end{pmatrix} \begin{pmatrix} 1 & -0.5 \\ 0 & 0.8660 \end{pmatrix} = GG^T$$

Now define the matrix $S_1 \in \mathbb{R}^{m\times 2} : S_1 = S_0 G^T$. If we set C equal to the first column of S_1 and R to the second column, then the variables C and R will have an expected correlation of -0.5. This approach can be generalized for a larger number of variables.

B.7 Data generation: effect of re-sampling on standard deviations

Consider a setting in which, for both the capacity and the popularity ratio, the parameters of the coefficients of variation (CV_c and CV_p) are initially set to 0.5. Figure B.4 displays the size of both parameters after re-sampling for different numbers of students and schools, in which each data point is the average ratio in 5000 generated data sets.

For the capacities, the observed coefficient of variation is about 0.05 smaller than the desired value of 0.5, but the observed value seems to be approximately constant with respect to the number of students. When the average number of students per school is very low, the deviation from the desired value is slightly higher.

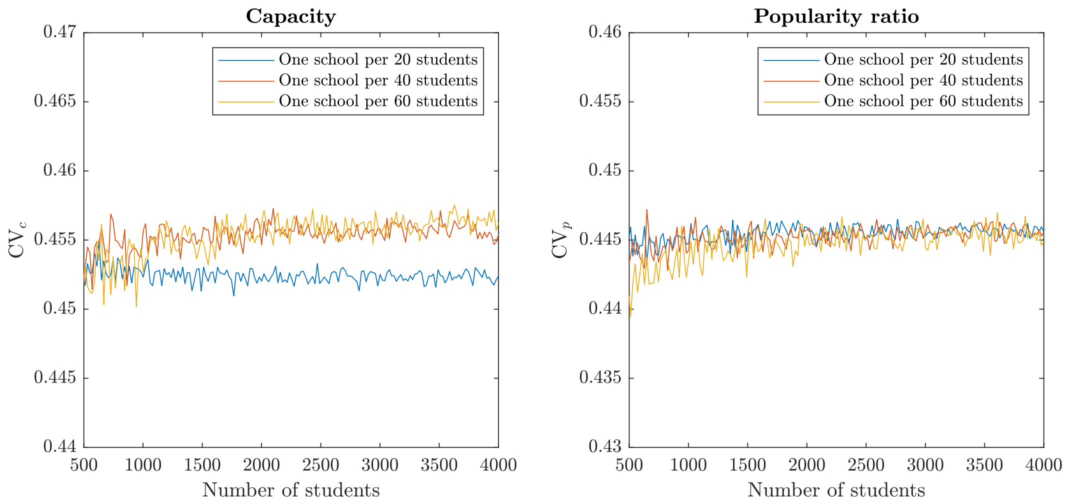


Figure B.4: Monte-Carlo simulation ($N = 5{,}000$) to obtain the average observed coefficients of variation CV_c and CV_p after re-sampling for different numbers of students and schools

B.8 Data generation: re-sampling frequency

In the same setting as in Appendix B.7, also the proportion of the schools for which the capacity or the popularity ratio has to be re-sampled can be considered. These proportions are displayed in Figure B.5. It can be seen that the proportion of the schools for which the capacity or the popularity ratio have to be re-sampled is slightly higher when the average number of students per school is lower (i.e. when there are relatively many schools). However, the overall frequencies are rather small.

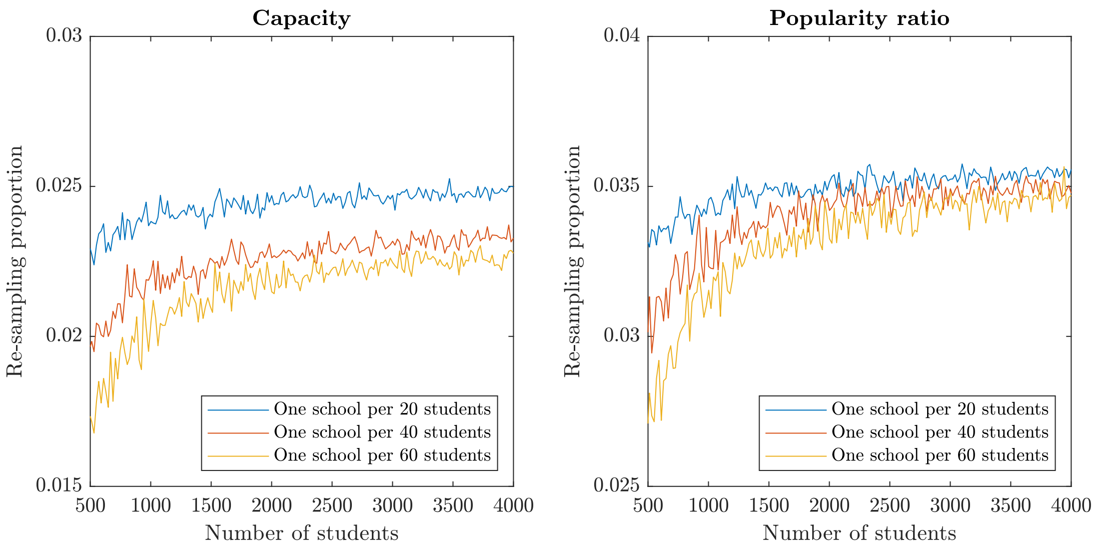


Figure B.5: Monte-Carlo simulation ($N = 5{,}000$) to obtain the proportion of the schools for which the capacity/popularity ratio has to be re-sampled

B.9 Profile of Antwerp

Table B.3: Number of students (proportion) that is assigned to their school of p-th choice in Antwerp (average of 2,000 tie-breaking rules)

Preference	DA(MTB)	School-prop. DA(MTB)	RSD	PS	BOSTON
1	**3333** (78.70%)	**3329** (78.58%)	**3595** (84.87%)	**3598** (84.95%)	**3728** (88.00%)
2	**527.3** (12.45%)	**531.0** (12.54%)	**275.1** (6.49%)	**278.1** (6.56%)	**160.9** (3.80%)
3	**134.8** (3.18%)	**135.8** (3.21%)	**91.1** (2.15%)	**90.2** (2.13%)	**67.6** (1.59%)
4	**36.4** (0.86%)	**36.8** (0.87%)	**46.6** (1.10%)	**46.1** (1.09%)	**42.1** (1.00%)
5	**8.62** (0.20%)	**8.78** (0.21%)	**20.6** (0.49%)	**19.6** (0.46%)	**17.6** (0.42%)
6	**1.42** (0.03%)	**1.45** (0.03%)	**7.83** (0.18%)	**7.59** (0.18%)	**11.5** (0.27%)
7	**0.27** (0.01%)	**0.27** (0.01%)	**2.82** (0.07%)	**2.66** (0.06%)	**4.05** (0.10%)
8	**0.07** (0.00%)	**0.06** (0.00%)	**2.04** (0.05%)	**2.03** (0.05%)	**4.95** (0.12%)
9	**0.01** (0.00%)	**0.01** (0.00%)	**0.76** (0.02%)	**0.67** (0.02%)	**1.12** (0.03%)
10	**0.00** (0.00%)	**0.01** (0.00%)	**0.41** (0.01%)	**0.40** (0.01%)	**1.30** (0.03%)
UNASSIGNED	**193.5** (4.57%)	**193.2** (4.56%)	**193.7** (4.57%)	**190.3** (4.49%)	**197.0** (4.65%)

B.10 Profile of Ghent

Table B.4: Number of students (proportion) that is assigned to their school of p-th choice in Ghent (average of 2,000 tie-breaking rules)

Preference	DA(MTB)	School-prop. DA(MTB)	RSD	PS	BOSTON
1	**2405** (78.05%)	**2403** (78.00%)	**2645** (85.86%)	**2646** (85.89%)	**2715** (88.12%)
2	**417.2** (13.54%)	**418.1** (13.57%)	**175.9** (5.71%)	**176.7** (5.73%)	**124.6** (4.05%)
3	**79.3** (2.57%)	**79.8** (2.59%)	**54.6** (1.77%)	**54.3** (1.76%)	**30.6** (0.99%)
4	**3.88** (0.13%)	**3.93** (0.13%)	**5.67** (0.18%)	**5.72** (0.19%)	**3.28** (0.11%)
5	**0.30** (0.01%)	**0.31** (0.01%)	**0.94** (0.03%)	**0.92** (0.03%)	**0.56** (0.02%)
6	**0.01** (0.00%)	**0.01** (0.00%)	**0.06** (0.00%)	**0.07** (0.00%)	**0.11** (0.00%)
7	**0.00** (0.00%)	**0.00** (0.00%)	**0.09** (0.00%)	**0.08** (0.00%)	**0.09** (0.00%)
8	**0.00** (0.00%)	**0.00** (0.00%)	**0.01** (0.00%)	**0.01** (0.00%)	**0.06** (0.00%)
UNASSIGNED	**175.63** (5.70%)	**175.63** (5.70%)	**198.4** (6.44%)	**196.9** (6.39%)	**206.7** (6.71%)

B.11 Effects of Δ_1 and ρ_{cp} on waste reduction possibilities

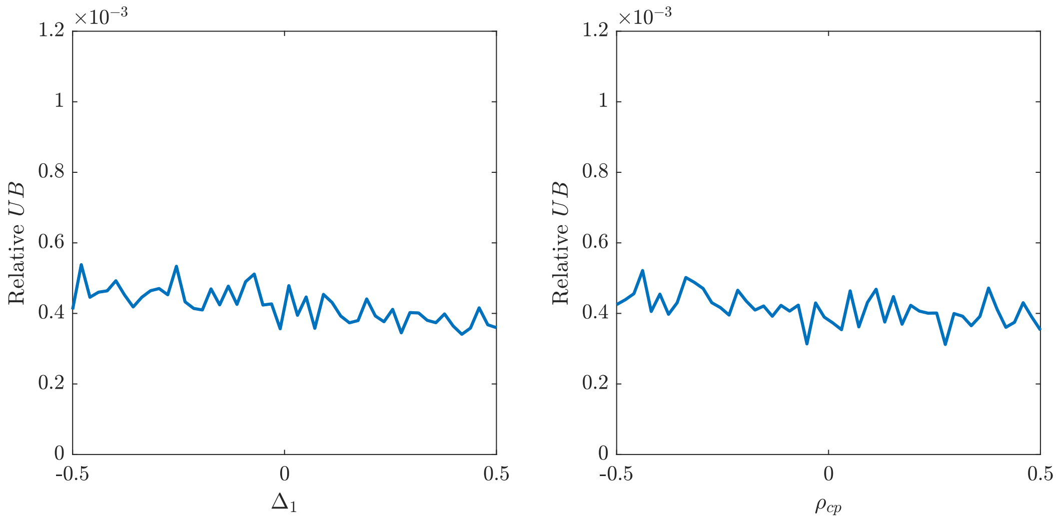


Figure B.6: Average relative waste reduction possibilities with respect to Δ_1 and ρ_{cp} (average over 200 data sets with 500 students, each with 200 considered tie-breaking rules)

B.12 Relative difference between the minimum and the maximum number of assigned students

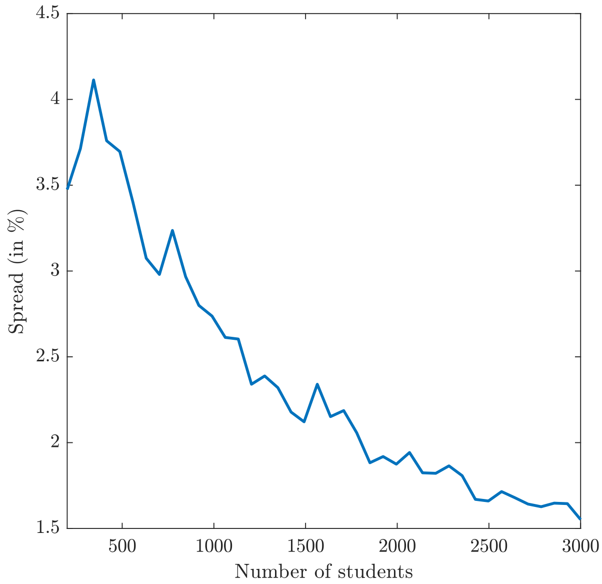


Figure B.7: Relative difference between the maximum and the minimum number of assigned students (*spread*) over all matchings, with respect to the number of students (average over 20 data sets, each with 1,000 considered tie-breaking rules)

B.13 Effects of Δ_1 and ρ_{cp} on worst-case difference

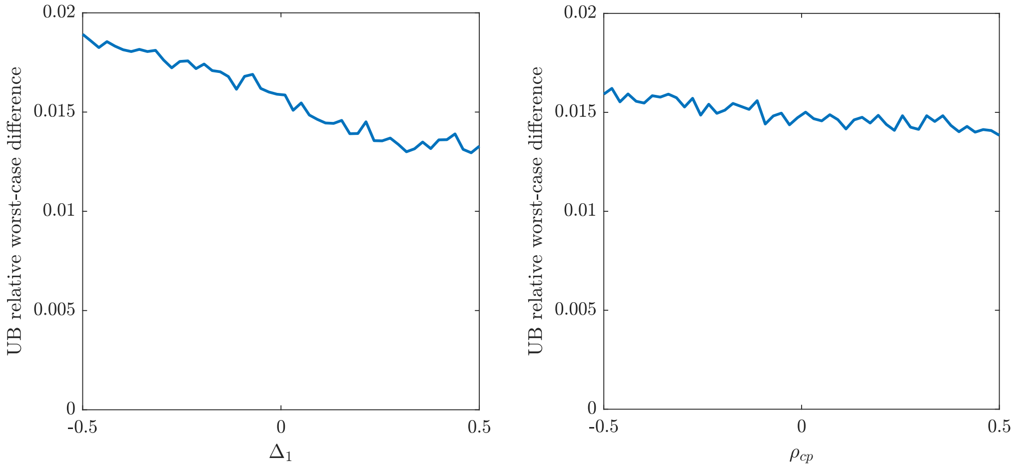


Figure B.8: Average relative worst-case difference with respect to Δ_1 (left panel) and ρ_{cp} (right panel) (average over 200 data sets with 500 students, each with 200 considered tie-breaking rules)

B.14 Illustration stochastic dominance PS and RSD

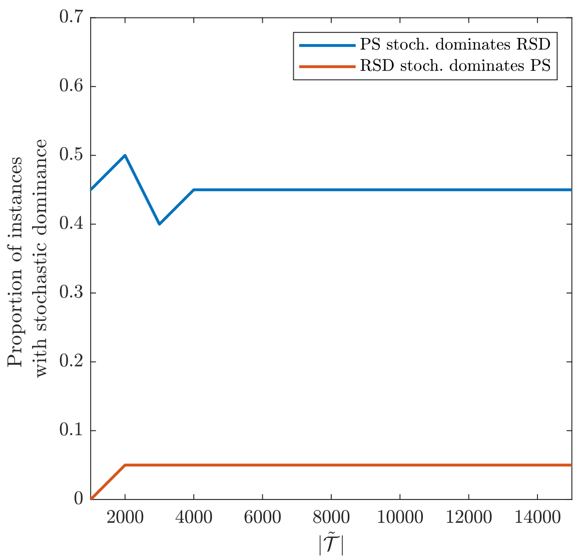


Figure B.9: Proportion of the data sets in which PS stochastically dominates RSD, and vice versa, with respect to the sample size (20 data sets with 500 students)

B.15 Feasible decomposition PS with randomly sample of tie-breaking rules $\tilde{\mathcal{T}}$

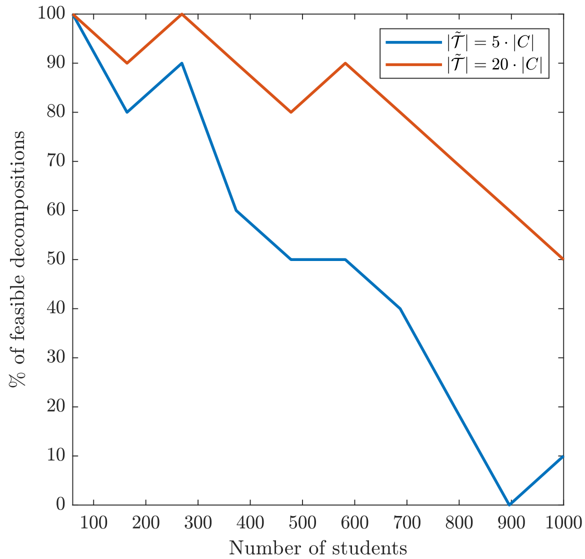


Figure B.10: Proportion of the data sets for which a feasible decomposition of the PS assignment exists, with respect to the number of students and the sample size $|\tilde{\mathcal{T}}|$ (average over 10 data sets)

B.16 r-partial strategy-proofness

B.16.1 Axioms and r-partial strategy-proofness

Mennle and Seuken (2014) stated that a mechanism is strategy-proof if it meets the following three axioms: *swap monotonicity*, *upper invariance* and *lower invariance*. Define the *neighborhood* $N_{>_{c_i}}$ of a preference list $>_{c_i}$ as the set of all possible preference lists $>'_{c_i}$ that can be obtained by swapping the order of two adjacent schools (or the outside option) in $>_{c_i}$. Imagine, for example, three preference lists $>_a$: $s_1 > s_2 > s_3$, $>_b$: $s_1 > s_3 > s_2$ and $>_c$: $s_2 > s_3 > s_1$, then only $>_b$ is part of the neighborhood $N_{>_a}$ of preference list $>_a$, and not $>_c$.

First of all, a mechanism is *swap monotonic* if submitting a preference list $>'_{c_i}$ that is part of the neighborhood $N_{>_{c_i}}$ of preference list $>_{c_i}$, with $s_j >_{c_i} s_k$ and $s_k >'_{c_i} s_j$ for $s_j, s_k \in S \cup \{0\}$, will either leave the allocation probabilities of both schools unchanged, or will cause both a strict increase in the allocation probability of school s_k and a strict decrease in the allocation probability of s_j. To illustrate the intuition behind this axiom, consider a simplified example with one student c_1 and two schools s_1 and s_2. A mechanism that would assign student c_1 to school s_2 for the preference list $s_1 >_{c_1} s_2$ and to s_1 for preference list $s_2 >_{c_1} s_1$ is not swap monotonic and not strategy-proof (Mennle and Seuken, 2014).

Furthermore, a mechanism is *upper invariant* if a student cannot influence the allocation probabilities of one of his/her most preferred schools by swapping the order of two less preferred schools in the preference list. More formally, a mechanism is upper invariant if for preference lists $>_{c_i}$ and $>'_{c_i} \in N_{>_{c_i}}$, with $s_j >_{c_i} s_k$ and $s_k >'_{c_i} s_j$ for $s_j, s_k \in S \cup \{0\}$, the allocation probabilities for all schools $s_l \in S$, with $s_l >_{c_i} s_j$, do not change by submitting $>'_{c_i}$ instead of $>_{c_i}$.

Lastly, the concept of lower invariance is very similar to upper invariance. A mechanism is *lower invariant* if a student cannot influence the allocation probabilities of one of his/her least preferred schools by swapping the order of two more preferred schools in the preference list, i.e. if for preference lists $>_{c_i}$ and $>'_{c_i} \in N_{>_{c_i}}$, with $s_j >_{c_i} s_k$ and $s_k >'_{c_i} s_j$ for $s_j, s_k \in S \cup \{0\}$, the allocation probabilities for all schools $s_l \in S$, with $s_k >_{c_i} s_l$, do not change by submitting $>'_{c_i}$ instead of $>_{c_i}$.

As a strategy-proof mechanism satisfies all three axioms, swapping two adjacent schools in the preference list can only cause a change in the allocation probabilities of both schools and not in the allocation probabilities of other schools. According to Mennle and Seuken (2014), lower invariance is the least intuitive of the three axioms and they call a mechanism that is only swap monotonic and upper invariant but not lower invariant *partially strategy-proof*. Clearly, partially strategy-proof mechanisms will not be strategy-proof, but Mennle and Seuken (2014) nevertheless show that partially strategy-proof mechanisms are still strategy-proof for a large subset of all utility functions.

Example B.16.1. To illustrate for which utility functions partially strategy-proof mechanisms will be strategy-proof, consider the PS mechanism in a setting[1] with three students and three schools with unit capacity, in which the true student preferences are:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$
s_1	s_2	s_2
s_2	s_1	s_3
s_3	s_3	s_1

Suppose student c_2 and c_3 report truthfully. If student c_1 reports truthfully as well, (s)he will be allocated to school (s_1, s_2, s_3) with probability $(\frac{3}{4}, 0, \frac{1}{4})$. If c_1 submits preference list $>'_{c_1}$: $s_2 > s_1 > s_3$, the allocation probabilities are $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. Suppose that the utility $u_i(s_3)$ of c_1 of being assigned to s_3 equals zero. In this case, whether or not reporting truthfully is a dominant strategy for c_1 depends on the relative value of $u_1(s_1)$ and $u_1(s_2)$. More specifically, reporting truthfully is a dominant strategy if $\frac{1}{2}u_1(s_1) + \frac{1}{3}u_1(s_2) \leq \frac{3}{4}u_1(s_1)$, i.e. if $u_1(s_2) \leq \frac{3}{4}u_1(s_1)$. This illustrates that the incentive for a student to misreport is larger when a student is closer to being indifferent between being assigned to one of both schools.

In correspondence to the observation in Example B.16.1, Mennle and Seuken (2014) introduced a new property for utility functions. A utility function u_i for student c_i satisfies *uniformly relatively bounded indifference* (URBI(r)) if c_i experiences a utility of being assigned to school $s_k \in S \cup \{0\}$ that is at least a factor $r \in [0, 1]$ smaller than the utility c_i experiences of being assigned to a more preferred school $s_j \in S$:

$$u(s_k) \leq r \cdot u(s_j), \text{ assuming } \min(u) = 0.$$

Denote by $\mathcal{U}(r)$ the set of all utility functions that satisfy URBI(r). Consider, for example, two utility functions $u = (1, \frac{1}{2}, \frac{1}{4}, 0)$ and $u' = (1, \frac{3}{4}, \frac{1}{4}, 0)$, in which the schools are ranked in order of decreasing utility. While both u and u' belong to the set $\mathcal{U}(\frac{3}{4})$, only u is an element of $\mathcal{U}(\frac{1}{2})$. In general, $\mathcal{U}(r_1) \subset \mathcal{U}(r_2)$ for any $r_1, r_2 \in [0, 1]$ with $r_1 < r_2$.

If, for a given problem instance, a mechanism is strategy-proof for all utility functions in $\mathcal{U}(r)$, this mechanism is called *r-partially strategy-proof* (r-PSP) for the given problem instance. An intuitive measure for the *degree of strategy-proofness* of a certain mechanism for a given problem instance could then be defined as the maximum value of $r \in [0, 1]$ for which the mechanism is r-partially strategy-proof. For an r-partially strategy-proof mechanism, the only honest advice that a mechanism designer can give to the students is that truthful reporting is a dominant strategy if the student's utility function lies within $\mathcal{U}(r)$.

[1]Example from Mennle and Seuken (2014)

B.16.2 Axioms and r-PSP for different mechanisms

First of all, Table B.5 shows that the PS mechanism is r-partially strategy-proof, whereas the Boston mechanism is not as it is not swap monotonic. The proofs for these results have been given by Mennle and Seuken (2014) and the intuition behind the proofs are shown in Appendix B.16.3. Furthermore, the WRLD procedure is not guaranteed to satisfy any of the axioms for an arbitrary problem instance. An illustration of a problem instance in which none of the three axioms is satisfied by the WRLD procedure is given in Appendix B.16.3. Lastly, the School-proposing DA with MTB is neither swap monotonic nor upper invariant, as shown by an example in Appendix B.16.3. No problem instance could be identified for which the School-proposing DA is not lower invariant, but a formal proof could not be found.

Table B.5: Comparison strategy-proofness axioms for different mechanism (Mennle and Seuken, 2014)

Mechanism	Swap monotonicity	Upper invariance	Lower invariance	Strategy-proofness	r-PSP
DA(MTB)	✓	✓	✓	✓	✓
RSD	✓	✓	✓	✓	✓
PS	✓	✓	✗	✗	✓
BOSTON	✗	✓	✗	✗	✗
WRLD	✗	✗	✗	✗	✗
School-prop. DA(MTB)	✗	✗	(✓)	✗	✗

B.16.3 Examples axioms and r-PSP for different mechanisms

Probabilistic Serial (PS) mechanism

Intuitively, the PS mechanism is upper invariant as a student's school of k-th choice will not influence the probability of being assigned to a school of higher choice, because that school will only be taken into consideration by the PS mechanism if a student has been rejected on all schools of higher choice.

Moreover, the proof of the swap monotonicity of the PS mechanism has been given by Mennle and Seuken (2014). To state the outline of the proof, consider the moments in time on which a school has been completely *eaten up*. Imagine that student c_i would swap two schools in his preference order, e.g. from $s_j >_{c_i} s_k$ to $s_k >'_{c_i} s_j$. If anything changes to the allocation probabilities of c_i at all, the student would spend strictly more time on consuming s_k. Hence, when s_k has been completely *eaten up*, there will be strictly less of s_j available or there will be strictly more students eating from s_j (in comparison to submitting $s_j >_{c_i} s_k$).

To illustrate that the PS mechanism is not lower invariant, consider the following example with three students and three schools with unit capacity in which the students preferences are given by:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$
s_1	s_1	s_2
0	s_2	s_1
0	s_3	s_3

Applying the PS mechanism to this problem instance results in the probabilistic assignment P_P.

$$P_P = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.50 & 0 & 0 \\ 0.50 & 0.25 & 0.25 \\ 0 & 0.75 & 0.25 \end{pmatrix} \end{array}$$

Imagine that student c_2 would submit an alternative preference list $>'_{c_2}$: $s_2 > s_1 > s_3$. The resulting probabilistic assignment by the PS mechanism is then equal to P'_P.

$$P'_P = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0.667 & 0 & 0 \\ 0.167 & 0.50 & 0.333 \\ \mathbf{0.167} & \mathbf{0.50} & \mathbf{0.333} \end{pmatrix} \end{array}$$

By swapping the order of schools s_1 and s_2 in c_s's preference list, student c_2 can increase the probability of being assigned to school s_3 by 0.083, which is in violation with the axiom of lower invariance.

Boston mechanism

Intuitively, the Boston mechanism is upper invariant as a student's school of k-th choice will not influence the probability of being assigned to a school of higher choice, because that school will only be taken into consideration by the Boston mechanism if a student has been rejected on all schools of higher choice. However, the Boston mechanism is neither swap monotonic nor lower invariant, as shown in the following example (Mennle and Seuken, 2014).

Consider a problem instance with four students and four schools with unit capacity in which the true preferences of the students are:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$	$>_{c_4}$
s_1	s_1	s_2	s_2
0	s_2	s_3	s_3
0	s_3	s_4	s_1
0	0	0	s_4

For this example, the allocation probabilities P_B of the Boston mechanism when all $4! = 24$ tie-breaking rules in $\mathcal{T}$ are considered, are equal to:

$$P_B = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 0.50 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 & 0 \end{pmatrix} \end{array}$$

If instead of truthful reporting, student c_4 would submit preference list $>'_{c_4}$: $s_2 > s_1 > s_3 > s_4$, the resulting probabilistic assignment would be P'_B:

$$P'_B = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 0.50 & 0 & 0 & 0 \\ 0.50 & 0 & 0.25 & 0 \\ 0 & 0.50 & 0.375 & 0.125 \\ 0 & 0.50 & \mathbf{0.375} & \mathbf{0.125} \end{pmatrix} \end{array}$$

This example is in violation with swap monotonicity as listing s_1 higher in the preference list does not increase the allocation probability to that school, whereas the allocation probability to s_3 decreased by 0.125. Moreover, lower invariance is violated as the allocation probability to school s_4 increases by 0.125 by misreporting $>'_{c_4}$ instead of $>_{c_4}$.

Waste-Reducing Lottery Design (WRLD) procedure

Consider an example with six students and four schools in which the student preferences $>_{c_i}$ and the school capacities Q are equal to:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$	$>_{c_4}$	$>_{c_5}$	$>_{c_6}$
s_4	s_4	s_4	s_4	s_3	s_3
0	s_2	s_2	s_1	s_2	s_4
0	0	s_1	s_3	s_4	s_1
0	0	s_3	s_2	0	0

$$Q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

Denote the probabilistic assignment from the RSD mechanism over all $6! = 720$ tie-breaking rules in $\mathcal{T}$ by P_R and the probabilistic assignment that is obtained by applying the WRLD procedure to P_R by P_W. Students c_3, c_5 and c_6 experience a waste reduction of 0.20 in total, which is equal to the upper bound on the waste that can be captured. This means that for every five random tie-breaking rules, on average one extra student will be assigned to a school.

$$P_W = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0.475 \\ 0 & 0.31 & 0 & 0.475 \\ 0.15 & 0.31 & 0 & 0.475 \\ 0.42 & 0.04 & 0 & 0.475 \\ 0 & 0.29 & 0.50 & 0 \\ 0.27 & 0 & 0.50 & 0.10 \end{pmatrix} \end{array} \qquad P_W = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0.475 \\ 0 & 0.31 & 0 & 0.475 \\ \mathbf{0.18} & 0.31 & 0 & 0.475 \\ 0.42 & 0.04 & 0 & 0.475 \\ 0 & \mathbf{0.33} & 0.50 & 0 \\ \mathbf{0.40} & 0 & 0.50 & 0.10 \end{pmatrix} \end{array}$$

For this example, the WRLD procedure does not satisfy any of the three strategy-proofness axioms that were introduced in Section B.16.

First of all, the WRLD procedure is not swap monotonic for this example. Imagine that student c_6 would submit a shortened preference list $>'_{c_6}$: $s_3 > s_1 > s_4 > 0$. In that case, the probability of being assigned to school s_4 would decrease by 0.10, which is intuitive, but the probability of being assigned to s_1 would also decrease by 0.03. By submitting s_1 as the school of second choice, c_6 would have a probability of 0.37 of being assigned to s_1 by the RSD mechanism. School s_1 would still be wasteful, but the WRLD procedure would now increase the allocation probabilities of c_3 and c_4 to s_1 each by 0.07 and c_6 would not benefit from the WRLD procedure. Therefore, overall, c_6 would have a smaller allocation probability to school s_1 by putting the school higher in his preference list.

Secondly, the WRLD procedure is not upper invariant for this example. Consider the case in which student c_3 submits preference list $>'_{c_3}$: $4 > 2 > 0$. This would increase the allocation probability of c_3 to school s_2 by 0.04, which is equal to the increase in allocation probability to s_2 for c_3 by the WRLD procedure when $>'_{c_3}$ is submitted.

Lastly, the WRLD procedure also violates lower invariance for student c_3 in this example. If student c_3 would submit preference list $>''_{c_3}$: $s_2 > s_4 > s_1 > s_3$, the allocation probability of c_3 to s_1 would decrease by 0.03, because (s)he would no longer benefit from the WRLD procedure.

School-proposing Deferred Acceptance mechanism

Consider an example with three students and three schools with unit capacity in which the true preferences of the students are:

$>_{c_1}$	$>_{c_2}$	$>_{c_3}$
s_1	s_2	s_3
0	s_3	s_2
0	0	s_1

For this example, the allocation probabilities P_D of the School-proposing DA with MTB when all $(3!)^3 = 216$ possible tie-breaking rules in $\mathcal{T}$ are considered, are equal to:

$$P_D = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.72 & 0.28 \\ 0 & 0.28 & 0.72 \end{pmatrix} \end{array}$$

The School-proposing DA with MTB is lower invariant, in this example, but not swap monotonic or upper invariant. Consider the situation in which student c_3 would submit preference list $>'_{c_3}$: $s_3 > s_1 > s_2$. The resulting probabilistic assignment P'_D is equal to:

$$P'_D = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.80 & 0.20 \\ 0 & \mathbf{0.20} & \mathbf{0.80} \end{pmatrix} \end{array}$$

First of all, this would harm the upper invariance of the mechanism as this would result in an increase of the allocation probability of c_3 to s_3 by 0.08. Secondly, this would harm the swap monotonicity of the mechanism as submitting a preference list in which s_1 and s_2 are swapped results in an assignment in which the allocation probability to s_1 remains equal to zero, whereas the allocation probability to s_2 decreases by 0.08. Moreover, both students c_2 and c_3 could increase the probability of being assigned to their school of first choice with 0.28 by only submitting that school, which implies a violation of upper invariance.

B.17 Survey utility function Ghent 2013-2014

Table B.6: Results survey on utility function in Ghent in 2013-2014 (D'haeseleer, 2016)

Preference allocated school	**Utility**
1	9.8
2	7.8
3	6.6
4	5.7
5	5.3
$>$ **5**	5.0
UNASSIGNED	N/A

B.18 Difference in allocation probabilities for twins

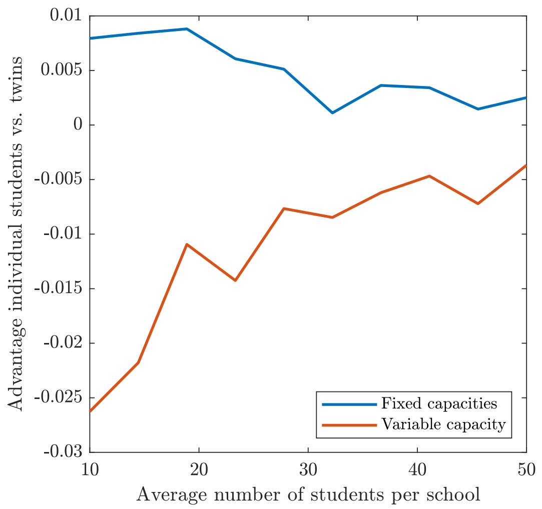


Figure B.11: Advantage in allocation probability to the school of first choice for individual students, in comparison to twins, with respect to the average number of students per school for the methods described in Section 4.2 (Average over 100 data sets with 500 students and $|\tilde{\mathcal{T}}| = 1{,}000$)

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