KU LEUVEN

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The reappearance of logic in Flemish secondary mathematics education

Assessing the curriculum, available teaching material, and providing extra support

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Preface

I have many people to thank for the completion of this thesis but will attempt to list some of them here.
First and foremost, I am grateful for my promotor Professor Deprez. His guiding advice, interesting sources and most of all his flexibility are what made this thesis possible. This mentorship is especially exceptional, knowing how many other things need his attention.
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Summary

The mathematics curriculum for Flemish secondary education has changed. Logic has reappeared in the curriculum, and now the question of how to teach logic is on every mathematics teacher's mind. This thesis starts with an extensive literature review of both theoretical perspectives and practical pedagogical research on logic. For the theoretical perspectives, we discuss the mathematical, philosophical, psychological and cultural psychological view on logic. In the Section on pedagogical research we summarize research by mathematics educators and by psychologists.

This extensive review is used in the assessment of the curriculum and the available teaching material, providing a critical look on the current Flemish context. The curriculum is analyzed using all the central themes of the literature review. The available teaching material is concisely summarized with some extra remarks. This Chapter can be used by curriculum developers to reflect on choices in including logic, and by mathematics teachers as a guiding tool for selecting the right textbooks to teach logic.

Based on the weak spots of the curriculum and on what's less present in the available teaching material, extra educational resources were designed to support the teaching of logic in the second stage of Flemish secondary mathematics education. Some of these resources can be straightforwardly used by mathematics teachers to aid their logic instruction, and some, more innovative ones, can serve as a starting point for further development.

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AHOVOKS

Agentschap voor Hoger Onderwijs, Volwassenenonderwijs, Kwalificaties en Studietoelagen

GNP

Gross National Product

IT

Information and communications technology

MM

Mathématique Moderne

NA

Not Available

OECD

The Organisation for Economic Co-operation and Development

PISA

Programme for International Student Assessment

QED

Quod Erat Demonstrandum

STEM

Science, Technology, Engineering, and Mathematics

TICTTL

Tools for Teaching Logic - Third International Congress

TIMSS

Trends in International Mathematics and Science Study

VBTL

Van Basis Tot Limiet

VVWL

Flemish union for mathematics teachers

ZFC

Zermelo–Fraenkel set theory + Axiom of Choice

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'Contrariwise,' continued Tweedledee,
'if it was so, it might be;
and if it were so, it would be;
but as it isn't, it ain't.
That's logic.'

— Carroll, Lewis

Introduction

Logic has entered the stage in the Flemish mathematics curriculum yet again. "again", because logic once already was part of the curriculum. From the 60s till the 80s the New Mathematics movement had a strong grip on mathematics education, and also in Flanders. This pedagogical approach used the deductive structure and abstract nature of mathematics to draw up a model for mathematics education. This also meant the intense use of logic. After the initial enthusiasm and optimism in the beginning, the "downfall" (labeled this way by (De Bock and Vanpaemel, 2019)) of the New Mathematics movement started in the 80s. Will we be able to avoid the "downfall" of logic in the mathematics curriculum?

This thesis investigates the question on how to teach logic in the given context of Flemish secondary mathematics education. In Chapter 0 we start with some background in logic. We continue with Chapter 1 where we give an overview of theoretical perspectives and pedagogical research on logic. This literature review is then used in Chapter 2 to assess the curriculum and the available teaching material. Based on the findings in Chapter 2 extra educational resources were designed to support logic teaching in the second stage. The designed teaching material and its designing process is described in Chapter 3. The material itself can be found in Appendix B.

The axiom of choice is obviously true,
the well-ordering principle obviously false,
and who can tell about Zorn's lemma?

— Bona, Jerry

0
Background in Logic

This Chapter summarizes some interesting concepts in logic, that are handy to keep in mind for some of the arguments in later Chapters. I will mainly talk from the perspective of formal logic (rather than informal) and assume the reader has a master of Mathematics and thus has some prior knowledge. I will refrain from explicitly, formally defining terms, to make the material easier to grasp. The ideas of this Chapter should still be understandable for secondary mathematics teachers who have some background in logic.

0.1 Syntax versus Semantics

We start by talking about two important aspects in formal logic, but logic in general.[1] This explanation is based on de Pater and Vergauwen (1993) and Patton Burgess (2009).

First off, we have the syntax of a logic. The syntax is about the grammar of the logic, it decides what formulas are well-formed and what formulas aren't. This component doesn't rely on the meaning or truth of the formulas.

Example 1. In the example of propositional logic we have the alphabet[2]

$$A = \{(,), \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, p, q, r, s, \ldots\}$$

These are all the allowed characters in the language of propositional logic. Parentheses are technically also a character, so we will include them as well. You can see we also included

[1]Not every logic has the two aspects underneath, but they are very prevalent. In the next Section we will see that there are **many** logics.

[2]**Important footnote for logicians:** In Flemish mathematics education and in mathematical practice in general, the symbol $\Rightarrow$ is used to denote any kind of implication. When propositional and predicate logic are discussed, they keep using $\Rightarrow$, representing the material implication in this context. However, remain flexible in interpreting this notation when it is used in this thesis: assume initially the material implication is meant, if the sentence involved does not make sense, understand the $\Rightarrow$ to mean a general implication.

the $p, q, r, s, \ldots$ representing atomic statements. Instead of using separate variable names, it's also common to enumerate them and use $p_1, p_2, p_3, \ldots$ for them. Then we can define our alphabet explicitly as $A = \{(,), \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\} \cup \{p_i | i \in \mathbb{N}\}$.

Next, the syntax then also tells which formula is well-formed, such as $p \Rightarrow p$, and which formula isn't, such as $p(\neg q)$. Some examples of syntactical rules are underneath. Capital letters are used to denote any formula, both simple and complex.

- All the p_i are well-formed formulas.
- B and C are well-formed if and only if $(B \wedge C)$ is a well-formed formula.
- B is well-formed if and only if $(\neg B)$ is well-formed.

Such rules inductively construct the language of propositional logic.

We also have the semantics of a logic. We already considered the syntactical component, but so far, this is just meaningless manipulation of symbols. Logic is also concerned with truth and meaning, and this is what the semantical component does.

Example 2. Let's consider again the example of propositional logic. The only 'meaning' we might give to formulas is whether they are true or false, denoted respectively by 1 and 0. The semantics here is defined inductively with rules. You can find examples of such rules below.

- The well-formed formula B is true if and only if $(\neg B)$ is false.
- $(B \wedge C)$ is mapped to 1 if and only if B is mapped to 1 and C is mapped to 1.

Notice, these semantical rules are coordinated with the syntactical rules. They define the truth value for every propositional formula, based on the truth values of the atomic statements.

We discussed the two main components of a formal logical system with the example of propositional logic. The difference is important to understand for example the theories in Subsection 1.1.3.

0.2 Different logics

The reader is assumed to be familiar with propositional and predicate logic. Possibly new information for the reader is that there exist many logics, infinitely many to say the least. Let's now consider some examples of different logics than the ones we know already. We will discuss two. These logics are discussed in detail in (Patton Burgess, 2009).

Example 1. Temporal logic is an extension of propositional logic and introduces time operators apart from the classical operators like $\neg$ and $\wedge$. If we have the atomic statement p, then $\mathbf{F}p$ is another statement meaning "it will be the case that p". We can similarly define $\mathbf{P}p$ as "it was the case that p". This way you can extend the syntax of propositional logic with the symbols $\mathbf{F}$ and $\mathbf{P}$. For the semantics we cannot simply define it inductively based on the truth values of the atomic statement. The truth value of a statement may now depend on time, so this needs to be taken into account.

Formally, one considers a tuple $(S, <)$ where S is a set of times and the binary relation $<$ gives what counts as the future or past from any given point. This tuple is a formal model of time. We can consider the standard $(\mathbb{R}, <)$ for this model which is how, I would say, most people reason about time, but for example also $([0, 1], <)$ is possible where at some point there's no future anymore.

When assigning a truth value to an atomic statement p, you also need information about the time $t \in S$ at which p is stated. I already hinted at truth of a formula being some kind of mapping, in the example of propositional logic in the previous Section. You can thus understand the process of assigning truth values to the atomic statements in temporal logic as one function $f : S \times \{p, q, r, \ldots\} \to \{0, 1\}$. Then $\mathbf{F}p$ is true at time t if and only if p is true at some point $t_1 \in S$ with $t < t_1$. In a more intuitive way: "it will be the case that p" is true now if there's a point in the future where p is true. This truth assignment function for atomic statements can be extended with similar semantic rules as in propositional logic to all formulas in the system.

Example 2. Instead of formalizing temporal statements, the next example of a logical system tries to capture the modality of a sentence. This can again be done as an extension of propositional logic, called modal logic. Examples of a modality are "it is necessary that ..." and "it is possible that ...", which can then be formalized as another syntactical operator $\square$ and $\triangle$. That's the syntactical component.

The semantical component can be defined similarly to temporal logic. Formally we still have a tuple $(S, <)$ and some kind of function $f : S \times \{p, q, r, \ldots\} \to \{0, 1\}$ for assigning truth values. However, the interpretation of these objects is quite different. The set S doesn't contain times, but so-called 'worlds' and the binary relation $<$ tells what worlds are accessible from what world.

The function f tells for each 'world' for every atomic statement if it's true or not. Defining the semantics of one of the new operators goes as follows: $\triangle p$ is true in w if and only if p is true in some world $w_1 \in S$ with $w < w_1$.

These 'worlds' can be very different things, but you should think of them as different possible realities. For example, if we're studying statements about the law, it's common to imagine a different world where different laws apply. In this concrete example S would be the set of all possible constitutions, and the relation $<$ could be entire $S \times S$. This would mean that in any reality with a certain constitution, you can imagine or think of or "access" any other reality with a different constitution. In this concrete logic you would be able to analyses discussions about including or excluding certain laws from the constitution.

These two examples of extensions of propositional logic are mostly important to realize that there exist several logical systems. We will come back to this point in Subsection 1.1.2.

0.3 Formal versus informal arguments

A very common use of formal logic is to analyze if informal arguments in natural language are 'sensible'. This Section purposely contains only a simple introduction. Let's consider an example of an informal argument, inspired by Heylen (2019), and perform this analysis with propositional logic. The reasoning is written from the perspective of a teenager sitting in his room, smoking and he does not want his mother to find out about it.

Teen: I just saw my mother coming upstairs. This means she's going to bed in her bedroom or coming to my bedroom to check on me.

...

Teen: Wait no, she's not going to bed, because she still needs to work! Quick, I have to ventilate my room!

The analysis of informal arguments is somewhat of an art, as it is not the case that the logical structure in an argument is always clear. Often times there are even different sensible formalizations of an argument possible. An elaborate discussion of how to formalize arguments is given in Chapter 2 of (de Pater and Vergauwen, 1993). I present one formalization of the argument here.

First we split the argument into individual statements and use propositional letters to denote them. Let p="The mother is coming upstairs.", q ="The mother is going to bed in her bedroom.", r ="The mother is going to the teenager's bedroom.", s ="The mother still needs to work." and t ="I have to ventilate my room!". The reasoning of the teenager then went as follows.[3]

Teen: p

Teen: $p \Rightarrow (q \vee r)$

Teen: So, $(q \vee r)$

...

Teen: $s \Rightarrow \neg q$

Teen: s

Teen: $r \Rightarrow t$

Teen: So, t

Now we have a more formal version of the informal argument. Notice I have made choices in what logical system to use, what propositional letters to define, ... to construct this. Even more, I had to fill in extra, implicit assumptions the teenager made. For example, he does not state that $p \Rightarrow (q \vee r)$ is true, but he is definitely assuming it to be true for his reasoning. Even more, $p \Rightarrow (q \vee r)$ in general is probably not a true assumption. A situation is easily imaginable where the mother comes upstairs, and does not go to either her bedroom or the teenager's bedroom. Another implicit assumption is for example $r \Rightarrow t$, which is quite important. He is assuming he needs to hide his smoking from his mother, so he needs to ventilate the room. If he didn't make the assumption of $r \Rightarrow t$, he might have concluded to confess about his smoking to his mother.

[3] My formalization assumes that 'seeing' the mother coming upstairs is the same as the mother actually coming upstairs. This is not necessarily true, as he might have mistaken her for someone else.

Now we have simply written down a more formal way of his reasoning. We can see there are two moments where he makes a conclusion, preceded by "So,". We have not yet checked if the reasoning of the teenager is sensible.[4] This brings me to the concept of validity. An argument is valid if and only if, **if** all the assumptions are true, then the conclusion is true. The assumptions do not need to be true in reality for an argument to be valid. This concept of validity formalizes what we would call reasoning 'sensibly'. That might still be potentially reasoning about clearly false assumptions, but the conclusions necessarily follow from the assumptions.

An important question for formal logic is then: "How do you check the validity of a formal argument?". We only consider propositional logic to keep it simple. One way of doing this is semantically checking the formal argument. Make a truth table with all the combinations of the truth values of the assumptions, and check if the conclusion is always true when the assumptions are true. However, this reasoning yields $2^5 = 32$ possible combinations[5] for the truth values, which is impractical.

A better way is to break down the formal argument into steps. If each of these steps is allowed, also known as valid, then the whole argument is valid. For this purpose, propositional logic has several argumentation forms (sometimes called inference rules) of which it is established that they are valid. The first argument effectively has the following structure, where A and B denote a simple or complex propositional formula.

Teen: A

Teen: $A \Rightarrow B$

Teen: So, B

This argumentation form is known as Modus Ponens, and is valid. There's also a valid argumentation form called Modus Tollens which derives $\neg A$ from $A \Rightarrow B$ and $\neg B$.

His second argument builds up from the first one, so now he uses the first conclusion $q \vee r$ as an assumption in the second. It goes as follows.

Teen: $q \vee r$	Assumption
Teen: $s \Rightarrow \neg q$	Assumption
Teen: s	Assumption
Teen: $r \Rightarrow t$	Assumption
Teen: $\neg q$	Modus Ponens with s and $s \Rightarrow \neg q$
Teen: r	Disjunctive Syllogism with $q \vee r$ and $\neg q$
Teen: t	Modus Ponens with r and $r \Rightarrow t$

We used yet another valid argumentation form, namely Disjunctive Syllogism which derives B from $A \vee B$ and $\neg A$. To conclude this example, the reasoning of the teenager

[4]In my formalization I already filled in his assumptions to make his reasoning 'sensible'.

[5]because $q \vee r$ is used as an assumption in the second argument

is valid!

Notice that, the approach I presented is a syntactic way of checking validity. We could also make a (big) truth table to check it. This can be understood as the semantic way of checking validity. This semantic approach can effectively be understood as checking that a long propositional formula is always true, which is also called a 'logical law' or 'tautology'. For example, the first argument being valid is the same as the formula

$$(p \wedge (p \Rightarrow (q \vee r))) \Rightarrow (q \vee r)$$

being always true.

The fact that all Mathematics is Symbolic Logic
is one of the greatest discoveries of our age;
and when this fact has been established,
the remainder of the principles of mathematics
consists in the analysis of Symbolic Logic itself.
— Russell, Bertrand

1 Literature review

Now that we have some background in logic from Chapter 0, we start this Chapter with a broad overview of different perspectives on logic. We consider mathematical, philosophical, psychological and anthropological perspectives on logic, all of which bring a nuance to the table about the nature of logic. These perspectives have been selected because they were assumed to bring new/interesting insights to the reader who is assumed to be an 'average' mathematician with a teaching interest. That is not to say other perspectives, like the perspective of logic on logic, are not interesting; a choice had to be made. In the second Section we focus on the pedagogical side of logic, discussing factors influencing logic performance and related advice for teaching practices.

1.1 Perspectives on Logic

In the second "International Congress on Tools for Teaching Logic" (Manzano et al., 2006), Morado wrote about different disciplines being related to the didactics of logic. He posed several questions for each domain that need to be answered in order to teach logic properly. Inspired by his paper, I discuss four domains in this Section, namely mathematics, philosophy, psychology and cultural psychology on a theoretical level. This is in addition to Milbou (2013) who did her Master thesis with Professor Deprez some years ago on the topic of logic in secondary mathematics and mainly focused on available pedagogical research, which is in the next Section 1.2. In Subsection 1.1.1 I go over the recent history of mathematicians' attitude towards logic and its inclusion in mathematics education. In the philosophical part in Subsection 1.1.2 I discuss the normative status of logic and how it relates to different ways of teaching mathematics. Then we have a look at what psychology has to say about deductive reasoning and how exactly people think logically in Subsection 1.1.3. Finally, we will conclude with an anthropological point of view on the universality of logic in Subsection 1.1.4.

1.1.1 Mathematics: is logic any different?

In this Section we discuss several things related to logic from a mathematics perspective. To start, I will give a historical overview of the 20th century of the different philosophies of mathematics and how they relate to logic. Next, we follow the chronological order of events and thus focus more on mathematics education in the 60s-80s with the New Mathematics Movement. Then we will circle back to contemporary philosophies of mathematics, and how they relate to logic.

This history is largely based on the lectures from Vanpaemel (2019) in the course about the history of mathematics. Before we jump into the brief overview of the beginning of the 20th century, we have to set the scene in the end of the 19th century. In the end of the 1800s there's a growing discomfort amongst mathematicians. Exotic mathematical objects have been discovered, such as non-Euclidean geometry, pathological functions in analysis, ... Mathematicians want to find certainty that aligns with their intuitions on mathematics, rather than being surprised every 50 years when yet another foundation is breaking. However, there's also a general optimism in this time. Yes, 'unholy' things have been studied in mathematics, but the diverse and large subject area also makes mathematicians believe they can tackle this issue. The initial attempts at formally grounding all of mathematics are focused on numbers. The idea is that if we can properly, axiomatically define numbers, everything else in mathematics is secure and has a solid basis. For example in 1872 Dedekind publishes his construction of the irrational numbers out of the integers (Dedekind, 1872) and in 1884, Frege publishes his theory of constructing the natural numbers using only logical laws and rules (Frege, 1884). It's also in this time that the Peano axioms for natural numbers were published (Peano, 1889). Also in 1873-1895 Cantor is manipulating transfinite numbers, which is especially unsettling (Cantor, 1874). So in the end of the 1800s, there's a growing discomfort with the fragile basis of mathematical truth but also optimism to secure it, since mathematics has been able to solve every problem so far.

In 1900 Hilbert gives his famous speech, where he proposes his list of problems and challenges the mathematical community to find the solution. The second problem asks to "prove that the axioms of arithmetic are consistent." In his speech there's a determination that mathematicians will find the solution to all these problems, including the foundational ones.[1] Despite the general optimism there were also different philosophies of mathematics that tried to secure the foundation of mathematics. These different philosophies are also in discussed in summary by Baker (2017).

The first one is logicism. The idea of this philosophy was to secure mathematics by building it up from logic. Note that "logic" wasn't defined back then as I have defined it in this thesis; there was more of a vague idea of what logic was, namely the uniform study of correct reasoning. There weren't really several logical systems studied (in the beginning at least). The main proponents were amongst others Frege and Russell. The idea is for example clear from Frege's work where he defined the natural numbers based on predicate logic. Also Russell published the Principia Mathematica around 1910 where

[1] A famous example of such a foundational problem, that was discovered around 1900, is Russell's Paradox.

he tried to define arithmetic from logic, since a contradiction had been found in Frege's theory (Russell and Whitehead, 1910).

Criticisms of this philosophy include the following: Logicists had technical difficulties to actually build mathematics from logic because they sometimes found contradictions in their own theories. They were criticized for using controversial axiom(s), showing that it's not entirely clear what logic is. More precisely, the idea was to secure mathematics by building it on the "secure" laws of reasoning, but if those are also up for debate, there's no security gained by building it on logic. Thirdly, if mathematics is purely built on logic, then what's the point of mathematics? Why study this as a separate domain? Also, this foundational approach had problems with explaining the applicability of mathematics (how is mathematics so useful in the physical world if it's just reasoning laws?) and with explaining or including how mathematicians actually perform research in practice (since mathematicians are not logic executing machines).

A second approach was formalism. Hilbert himself was a formalist. In formalism one believes that there's only a syntactical aspect to mathematics (and to logic). Mathematics is purely manipulating of meaningless symbols. You can define your axiomatic system however you want, the most important thing is that your chosen system is consistent. The formalists were mostly searching for consistency proofs of different axiomatic systems, including the one for arithmetic.

This approach was still problematic for some mathematicians. They didn't have to argue for or against certain logical laws, but had trouble answering again the question of the applicability of mathematics and of mathematical research practice. Also, according to some, it couldn't be that mathematics was purely formal and without meaning.

A third one was intuitionism. The main proponent was the Dutch mathematician Brouwer. In this philosophy a lot of importance is put on mathematical practice, how actual humans reason mathematically, rather than focusing on abstract formal systems. According to intuitionism mathematics is a mental process, that revolves around mathematical intuition, that has nothing to do with logic. This philosophy resisted the use of logic in securing mathematics, since mathematics is full of meaning and logic is not, and wanted to secure mathematics by regarding it as a human activity. However, intuitionists didn't stop there, but also introduced extra demands for mathematical proofs. Intuitionists only wanted to accept constructive proofs, meaning the axiom of choice is out of the question, the law of excluded middle is out of the question. Everything you use in a proof, needs to be constructed and not summoned into existence with an axiom.

This last point was the major criticism against this theory, since most mathematicians heavily rely on non-constructive proofs.

However, in the 30-40s these foundational quests came to an abrupt halt. Two events were important in this regard.[2] In 1931, Gödel published two incompleteness theorems that were a large blow for the formalists and the general enthusiasm of mathematicians. The first incompleteness theorem states that any consistent formal system, strong enough

[2]The development of "Zermelo–Fraenkel set theory, with the axiom of Choice" (abbreviated as ZFC) also undoubtedly calmed the nerves of mathematicians at the time. ZFC is a formal theory in predicate logic that solves Russell's paradox and is an, as far as we know, consistent foundation for most of mathematics.

to perform arithmetic in, is incomplete. The belief in the power of mathematics was mathematically disproven. Since all these philosophies of mathematics (and often intimately linked an attitude towards logic) had strong criticisms, including the final blow of Gödel, the foundational quest of mathematicians was never fully answered in this time. And even more, with the Second World War erupting, there wasn't any room to go on an internal search for foundation. Mathematicians had to justify what they were doing, and research useful areas. This was the quite abrupt ending of the foundational crisis and research in the beginning of the 20th century.

We have discussed some philosophies of mathematics and their views on logic. Let's follow the chronological order of events and focus more on mathematics education in the decades following the Second World War. This part is still based on the lectures from Vanpaemel (2019). There was also a recently published book by Vanpaemel and De Bock (De Bock and Vanpaemel, 2019) of several hundred pages, which provides a detailed overview of this period. However, I used more concise summaries/reviews of the book, namely (Bjarnadóttir, 2020) and (Van Bendegem, 2021).

After the Second World War the Cold War ensued and in the US the general feeling was that they were running behind on the USSR in terms of science (and mathematics) achievements. They needed to improve education in order to catch up, and rationality (and thus mathematics) should form the basis of the education of the workforce. With the extra funding there was a drastic change in mathematics education and the New Math Movement was born. This would not only influence the US, but also Western Europe. In Western Europe the pressure of the Cold War race were less present, but the New Math Movement's was also active here. A more important fear in Western Europe was that the textbooks at the time were getting outdated, while students should learn the most advanced, modern mathematics topics (thus originated the name "Moderne Wiskunde" in Dutch). There was namely a strong influence of 'Bourbaki', a group of mostly French mathematicians who hoped to write down all of mathematics in a series of textbooks. According to Vanpaemel (2019), this was partly in response to the German achievements in mathematics and they hoped in this way to also upgrade the mathematics achievements of France. Over the years they published several books in their series 'Elements of Mathematics'. Each textbook provided a rigorous, axiomatic overview of a domain. Examples of topics include set theory, abstract algebra, analysis, ... They wanted to build up their textbooks from the basics, and thus started from axiomatic set theory where logic clearly is very important. These textbooks were composed as an ideal for the science of mathematics, Bourbaki didn't intend the textbook series to be used as an ideal for mathematics education (De Bock et al., 2004). However, their publication had a large effect on the mathematical world, and in particular on Georges Papy. The Belgian mathematician Georges Papy with his own textbook series 'Mathématique Moderne' was very influential. De Bock and Vanpaemel (2019) devote a Chapter to his textbook series on page 117. To specify the relation between Papy and Bourbaki, De Bock and Vanpaemel (2019) write on page 125:"From a mathematical point of view (but certainly not from a didactical point of view), Papy's approach was quite similar to that of Bourbaki (1939), to which he explicitly referred in the Preface to MM3 [third volume of 'Mathématique Moderne']. Papy's method can be seen as an application of the model that Bourbaki developed for the *science* of mathematics as a model for mathematics *education*. The New Math Movement thus had a strong international influence on secondary mathematics education around 60-70s. Their didactical approach can be characterized as very formal and deductive. There was a strong focus on sets with the accompanying Venn diagrams, the natural numbers were based on set theory, and logic was thus very important. There was no attention to other sciences or to the students intuition. To make this more clear, let's sketch the contents of the first volume of 'Mathématique Moderne' by Georges Papy (De Bock and Vanpaemel, 2019). This first volume was intended for students aged 12, so pupils of grade 7. The volume had 24 Chapters with subjects (1–5) Algebra of sets; (6) First elements of geometry; (7–13) Relations, properties, composition; (14–15) Transformations of the plane; (16–18) Natural numbers (cardinal numbers), operations; (19) The binary numeral system; (20) integers; (21–23) Equipollence, translations, vectors, central symmetries and (24) Groups. Bear in mind, this is intended for twelve-year-olds!

With its quick implementation also comes its downfall shortly after. The deductive approach only worked for the already mathematically strongest students, while for most students it was just meaningless symbols. The approach was also criticized by society saying mathematicians don't understand their societal purpose. Students finishing a mathematics program based on New Math Movement, were not prepared for engineering or using mathematics in any real world application. One can ask how educators did not anticipate this. Did they not test this new pedagogical approach, before implementing it nationwide? Bjarnadóttir (2020) has the following to say about this.

> In his [Georges Papy] case and his collaborators, the research leading to the results was based on the personal experiences of the mathematicians themselves. There was a lack of a systematic and independent evaluation. There was never a thorough, comparative evaluation of the extent to which specific educational goals were met. Experiments were always considered successful by those in charge. The proposals remained mere hypotheses as long as they were not supported by objective verification. Eventually, it led to criticism: Nothing was objectively measured.

More about the experiments of Georges Papy can be found in Chapter 5 of (De Bock and Vanpaemel, 2019).

In the 80s a different mathematics education Movement, Realistic Mathematics Education, originating in the Netherlands with Hans Freudenthal as main proponent became very influential in Belgium. This approach focuses on meaningful, rich, realistic contexts as a basis for students constructing their own mathematical knowledge. Over the years both approaches merged into a mixture of both, and elements of the New Math Movement gradually disappeared from the Flemish curriculum (Milbou, 2013; De Bock et al., 2004). For example in 1982 logic disappeared as a separate subject, and can only be found in the Section on sets. And ultimately in 1997 logic disappeared from the curriculum completely.

Bjarnadóttir (2020) distinguishes several general characteristics of the New Math history, including the following. Firstly, she writes: "There was a widely held ideology on the absolute value of mathematics for future citizenship. This ideology was shared by all parties involved, including people outside of the mathematics community." Secondly, some individual events had a large influence. She discusses them in more detail, but for example an international seminar in France in 1959 about 'New Thinking in School

Mathematics' was very influential. Thirdly, individual people also determined a lot of the development. Bjarnadóttir (2020) writes: "One notices also that the great reform projects are often carried out by charismatic persons who manage to fascinate groups of people and convince them to work with them on great projects." Lastly, there's also a good effect of the New Math Movement, at least because of the wave of criticism after its implementation. This actually afforded the emergence of mathematical pedagogy as a distinct academic field.

We have sketched the common views on logic of mathematicians in the early 20th century and touched upon the New Math Movement (in general and in Belgium) in the second half of the 20th century. Let's circle back to current philosophy of mathematics, and see how it's related to logic. It is not easy constructing a brief but approximately exhaustive list of current philosophical positions on the matter. Since the 20th century, mathematics has diversified a lot and frankly sometimes isn't clearly distinguishable from theoretical physics, theoretical computer science, ... Also logic has grown a lot as a field on its own, but is also used in mathematics, computer science, philosophy, ... As we will see below, the questions asked have changed, and sometimes the philosophical theories don't immediately give an answer to the question "What does logic have to do with mathematics?". Underneath I give an overview based on Baker (2017). A more detailed overview can be found in (Shapiro, 2000).

In the beginning of the 20th century the main concern of philosophy of mathematics was establishing a foundation for mathematical truth. Baker (2017) formulates this as the question: "Are the central claims of our core mathematical theories true? If so, what makes them true?". Later the main focus shifted to two other questions, namely the question of knowledge ("How do we come to know the truth of the central claims of our core mathematical theories?") and of applicability ("What explains the usefulness of mathematics in science, and its applicability more generally to the world?"). Be aware that the question of knowledge is not the same as the one of foundation. In the question about knowledge, humans play an essential role. It asks how humans interact with (truthful statement about) mathematical objects. You will see in the philosophical views below, that this is translated to the question of whether or not numbers (and other mathematical objects) exist or in what sense they exist. The question of applicability became more important, as we saw before, since the Second World War.

The first candidate is neologicism. As the name suggests, this is the direct successor of logicism with Frege and Russell. In this theory they continue to try building up mathematics from logic. Proponents of this theory try to fix the paradoxes found in the previous century, and make reasonable choices for what logical laws they do or do not include in the one foundational logical system. Logic clearly is very important in this view and even quite literally foundational.

The second participant is structuralism. This theory mainly focuses on the second question of knowledge, and stands completely opposite of the belief in the human-independent existence of mathematical objects. They share with intuitionism the denial of separate objects. This theory claims mathematics is about structures, not about individual objects. There's no independent object for the number 17, anything that can serve as a 17 in the structure of natural numbers can be considered 17. In (Shapiro, 2000) and (Baker, 2017) I didn't immediately find a clear view on logic and how it relates to mathematics. However, I did find some papers that relate the structures, that are vital in structuralism, to different logical systems. For example, Feferman (2014) analyzes the structure of natural numbers and the structure of sets to determine the correct associated logical system. This effectively flips the idea of neologicism around, and builds (the correct) logic on top of mathematical structures.

The third option is fictionalism, which is a variant of formalism. As the name already suggests, the people from this camp believe that mathematicians make up fictional stories about imagined objects. They set out fictional scenarios and then explore their consequences. A rather extreme surprising stance in this theory is that all the mathematical statements (like '2 is a prime number') are false since they make statement about non-existing objects. Again, there wasn't a clear stance on logic for this view in (Shapiro, 2000) and (Baker, 2017). However, logic is still used by the discussed authors in the Chapter on fictionalism in (Shapiro, 2000) as a tool to describe their theories.

These are just some examples of philosophies of mathematics, and you can see the role of logic also varies amongst them. The philosophical debate on these three questions about the nature of mathematics is far from settled, and new theories are still being proposed. However, this doesn't seem to affect mathematicians. For example Baker (2017) ends his paper with the quote: "It is ironic, therefore, that these developments [in philosophy of mathematics] have occurred just as philosophy of mathematics has fallen off most mathematicians' radar.". Also in the preface of (Friend, 2014), a recent book where yet another theory of the nature of mathematics is presented, the author notes: "I observed that, for the most part, mathematicians and logicians did not behave as though they adhered to a philosophy of mathematics. In particular, with some exceptions, they did not seem to show adherence to one foundation of mathematics in a philosophical way.". Despite not influencing most mathematicians, it is still interesting to introduce different (historical) possible positions, since they also influence pedagogical and curricular choices.

1.1.2 Philosophy: is logic normative?

This Section is primarily based on the Stanford Encyclopedia entry about the normative status of logic (Steinberger, 2020). Other sources include the entry on psychologism (Kusch, 2020), a book about Philosophical Logic from Patton Burgess (2009) and a textbook for logic (de Pater and Vergauwen, 1993).

Most humans consider it a bad thing to be inconsistent. We call someone a hypocrite if he does not act on the principles that his other beliefs logically imply. This gives the intuitive idea of the normative status of logic. The question is whether (a) logic gives us norms, obligatory reasoning patterns, and if so, how these norms relate to the formal system that is a logic. We have defined a logic in the previous Section in a quite formal way, we continue using this definition (though we are discarding some logical systems this way).

We consider different nuances to this question. First off, we compare normative monism and pluralism. Proponents of monism argue for a singular, normative logic.

They believe that one logical system, such as propositional logic or syllogism theory or a more complicated one, prescribe the correct way of thinking. In pluralism they believe that there are several logical systems equally worthy of being normatively correct. Monists also acknowledge the usefulness of other logics in specific situations, but stick with one logic.

This distinction is related to the history of logic. For example for quite some time after Aristotle, his work on logic was by many considered to be the definitive treatise of logic. In the 19th-20th century the propositional and predicate logic gained more popularity for monists. However, this was also the time where different logics flourished in research, so pluralism began to be more prevalent. This debate still continues up to this day. The typical move of a monist will be to develop a large logical system that works normatively for many different situations (like Fu (2016) does, which we will see in Subsection 1.1.4) and to show ways to embed different logical systems into his chosen one.[3] The typical move of a pluralist will be to point out irreconcilable differences between different logics and show the correctness of both sides, and to devise new logics for concrete situations.

I provided some examples of different logics in Chapter 0 so that now the readers realizes pluralism is actually an option. If you only use propositional logic and predicate logic (which nicely contains propositional logic), and only see the enormous power of predicate logic in mathematics, it's not surprising to be an implicit monist for predicate logic. For example the (neo)logicists, but especially the first logicists were also monists. We will now give some examples for the mathematical reader of possible normative arguments related to modal logic (from Chapter 0) and intuitionism (from Subsection 1.1.1).

The implication in predicate logic only looks at truth values of its components; so it also comes with some paradoxes. For example the formula $p \Rightarrow (q \Rightarrow p)$ is a tautology. The interpretation of this formula is that any true statement is implied by any other statement. This feels clearly wrong, but to label it as clearly wrong we need to derive our norms from a different logical system than predicate logic. C.I. Lewis noticed this and other paradoxes with the implication in predicate logic, and developed a new kind of implication to better grasp the implicit norms we mathematicians have (Goldblatt, 2003). His implication $p \multimap q$ is defined in terms of the modal logic of Chapter 0 as $\Box(p \Rightarrow q)$. This last formula is read as "It is necessarily true that when p is true, then q is true.". This alternative is an attempt to better formalize the implicit norms associated to the implication in mathematical practice.

In Subsection 1.1.1 we talked about intuitionism from the mathematician Brouwer as a philosophy of mathematics. This philosophy was never really popular, until it was rejuvenated in 1970s by philosopher Michael Dummett. Funnily enough, the idea of intuitionism, with only accepting constructive arguments and rejecting the use of formal logic as a basis for mathematics, was then formalized into a logic. So this logic, called intuitionistic logic, is a formalization of Brouwer's ideas. Intuitively speaking, the truth value of a sentence isn't an independent thing, but lies in the verification of a sentence. This mirrors mathematical practice; we only accept the truth of a statement if there's a

[3]Despite this being a monist move, Fu (2016) still admits the cultural influence on choice of logic and only wants to claim his logic as uniquely normative for Chinese culture. In this regard he still allows some form of logical pluralism. More details in Subsection 1.1.4.

proof. In this logic non-constructive proofs are not allowed, and this causes for example the law of excluded middle ($p \vee \neg p$) to not be a tautology. When you want to prove something like $p \vee q$, you should pick one side of the disjunction and prove either p or q directly. For example, the typical non-constructive proof that there exist irrational numbers a and b such that a^b is rational goes as follows. Define $A = \sqrt{2}^{\sqrt{2}}$. Using the law of excluded middle, A is either rational or irrational. If it's rational, then we are done with $a = b = \sqrt{2}$. If it's irrational, then take $a = A$ and $b = \sqrt{2}$. Then

$$a^b = A^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

, which is rational! This proof is not allowed in intuitionistic logic and thus (tries to) impose a norm on mathematicians. There's also a more contemporary example of the norms of intuitionistic logic. Some mathematical theorems have only been proven by computers with theorem provers like Coq. Coq actually uses some elements of intuitionistic logic internally.

With these two examples I have given some examples of various logics and their influence on mathematical norms, in order to exemplify the debate amongst philosophers of logical monism versus pluralism. After reformulating the question "Is logic normative?" into "What logic is normative?" we can also reformulate it as "What is logic normative for?". So another nuance is what the, if any, norms from logic apply to? In the entry Steinberger (2020) specifies three ways of answering this new subquestion.

The first possibility is that logic is normative for (individual) reasoning. Steinberger (2020) distinguishes between 'reasoning' and 'thinking'. Reasoning is "presumably a connected, usually goal-directed, process by which we form, reinstate or revise doxastic attitudes [beliefs] (and perhaps other types of states) through inference." while he defines thinking as "which might consist merely of disconnected sequences of conceptual activity". Reasoning is a part of thinking according to this definition. In this possibility logic is not only normative in academic disciplines such as mathematics, but also in everyday problem solving. There are however still conceptual activities that are exempt from the logical norms, but remain unnamed in the entry.

The second, scarier, possibility is that logic is normative for thinking as a whole. This includes judging, believing, inferring, . . . ; any mental activity that involves concepts. This position was historically taken by for example Kant and Frege. Some philosophers have even gone so far as to only call a conceptual activity 'thinking' if the agent acknowledges the norms imposed on his activity by logic. The author compares thinking to playing chess. You can't simultaneously claim to be playing chess, while knowingly not following the rules of chess. The rules of chess are what make 'chess' chess. This view is a scary option, because you can think about chess without playing chess but unlike chess, you can't think about thinking without thinking. In this view there's no escape from logic in all conceptual activity.

A third option is to look externally instead of internally. In this view one claims that logic is normative for public practices, for rational dialogue, and for example for the community of mathematicians researching together. According to this view, you're technically allowed to have illogical thoughts on your own, but when you want to make a public statement or research paper, you have to follow the logical norms. It's also explained that through a process of interiorization (internalizing the social norms of thought), the normative impact of logic can also be felt on an internal level.

I have now highlighted two questions from the entry, namely “What logic is normative?” and “What is logic normative for?”. The text goes on by investigating the gap between the formal logical system and its normative role, if it has a normative role at all. We’re not choosing any stance on the normative status of logic here, but building up terminology and perspectives for analyzing the upcoming teaching material and curriculum. Rounding off this Subsection, I provide some interesting links to other domains.

Before the first World War, there was a war going on between philosophers and psychologists. In the period between 1890 and 1914 they were fighting over the land of logic, and wanted to claim it as part of their discipline. You can already see in the previous part that some views on logic’s normativity are closer/farther from psychology. In this time the term ‘psychologism’ was used between philosophers and depending on the speaker, as an insult. You can call something ‘psychologistic’ if it’s wrongly labeling non-psychological entities as psychological, where examples of those entities can be logical laws. The psychologists claiming logic considered logical laws to be only descriptive of psychological patterns, while the philosophers considered them to be normative and thus excluding them from the scope of natural science that is psychology. This term ‘descriptive’ is often used as an antonym of ‘normative’; logic isn’t prescribing rules, it’s describing them. Technically speaking these terms aren’t a priori exclusive, and not necessarily related to the so-called ‘Psychologismus-Streit’ between 1890 and 1914. Logical laws can be norms you have to abide by and simultaneously describing common deductive patterns. They might also be neither normative nor describing anything, and just be a formal game (like formalism from Subsection 1.1.1 might say). This debate was put to a halt with the first World War, but still lingers on in research on logic in these two disciplines. In Subsection 1.1.3 on the psychological perspective we go over a debate amongst psychologists about the mental nature of logic, which shows that psychologists haven’t given up on logic as a research area.

We can also already hint to an upcoming Subsection 1.1.4, namely the one from the standpoint of cultural psychology. There we will investigate the influence of culture on cognition (and thus logic). In the current Subsection you can see the possibility for cultural influence in the viewpoint of logical pluralism or in logic being normative for public practice (embedded in a culture). However, in Subsection 1.1.4 we will circle back to the main question “Is logic normative?” instead of the subquestions we saw here.

To finish this Subsection, let’s provide a concrete link to logic education in the mathematics classroom to show the relevance of this discussion for this thesis. We might find more examples of connections later, when analyzing the curriculum and available teaching material. I will formulate the links as questions where possible answers are related to the nuances to the normative status of logic discussed above. When (if ever) are students allowed to make logical mistakes? Possible situations are: when thinking on their own, when solving a problem not related to logic/mathematics in homework, when solving a mathematics problem on their own, when discussing a mathematical problem/solution in the classroom, . . . Is there only one logic discussed or are there multiple (pairwise exclusive) logics? Are assumptions of logical systems posed as a choice or as an unavoidable observation? Are logical fallacies discussed? Are the influence of context/situation discussed when talking about fallacies? Apart from these concrete questions, the reader can also understand the relevance of the question at hand for logic education, by looking at language courses. In teaching English/Dutch/French one has decide whether the grammar rules are prescribing correct language usage or describing common usage. This is similar to whether logic is normative and/or descriptive.

1.1.3 Psychology: how does logic work mentally?

In this Subsection we analyze two psychological papers on how humans think logically. They represent opposing theories, and both try to debunk each other. The papers considered for this Subsection speak in a very aggressive tone. This means that it's hard to find an objective explanation for their differences, because every author tends to clearly prefer one theory over the other. However, this part of the literature review is largely based on Bonatti in (Bonatti, 1994), a clear proponent of the so-called 'mental logic' theory, and on Johnson-Laird, Byrne and Schaeken in (Johnson-Laird et al., 1994) written by clear proponents of the so-called 'mental models' theory.

Let's start by defining the individual theories in arbitrary order. Recall the difference between semantics and syntax from Chapter 0. The main idea of the mental models theory is that people reason deductively by imagining several 'models of reality' (Johnson-Laird et al., 1994). In other words, people reason semantically in every situation, even in a very abstract one. If you show someone a deductive argument, they will agree or disagree with it not because some formal rule was followed/violated, but because the models of reality they considered either agreed or disagreed with the argument.

This contrasts with the mental logic theory, where the focus lies on the syntactical aspect of logic. The theory postulates that every human has his own internal logical system (not necessarily propositional logic), which he uses to judge deductive arguments (Bonatti, 1994). One can imagine for example Modus Ponens being one of the accepted rules in that logical system. A person will agree with a given argument, if every transition agrees with his own mental logic or disagree if it conflicts with his internal logic.

One should be aware that both of these theories are very flexible. In the mental models theory it is hard to define what a "model" is. It is unclear if it could best be understood as an image representation of reality, an abstract neural connection, a simulation of reality, ... Apart from the vagueness in the main concept, it's also up to debate within proponents of this theory how the models are constructed and manipulated. Both of these aspects make it hard to devise a psychological experiment that would debunk the theory. A similar problem also holds for the mental logic theory. It is unclear whether this mental logic is something you are born with or not, whether it depends on the person or not, whether it can be an inconsistent logical system, ... This also makes it hard to design a possibly refuting experiment. In fact, in both papers considered, the authors go so far as to accuse each other of working in an unfalsifiable theory. In both papers, results of a psychological experiment are used as an argument against the other theory, but then in the other paper it is shown that there was a flaw in the experiment, or their theory is modified ad hoc.

I would like remark that both of the theories still need both a syntactical and semantical component. The mental models theory still needs to work with syntax because you need to understand the syntax of a sentence in order to construct a corresponding model. “Sophie kills Ben” and “Ben kills Sophie” are both sentences that can be used to deductively reason, mean different things, but in order to understand the difference one needs to understand the grammatical structure of the sentences a.k.a the syntax. The mental logic theory also needs semantical support at some point in the reasoning process. Consider the premises “I hate bats, I’m too afraid of them.” and “Bats are some kind of sticks, used in baseball.”. No sensible person would conclude “I hate some kind of sticks that are used in baseball.”, though purely syntactically speaking, there does not seem to be any reason to prevent this conclusion.

These theories take a very different stance on deductive reasoning, and also have an effect on how the teaching of deductive reasoning should be understood. If we follow the mental models theory to the extreme, we should only focus on the semantical aspect of deductive reasoning. We would need to help students creating mental models that capture all different possibilities so they can better judge validity of an argument. When there’s for example a deductive argument about “all parallelograms”, you should focus on all the different possibilities to imagine when you hear “parallelogram” which includes squares, rectangles, rhombuses and ‘real’ parallelograms. It wouldn’t help to focus on teaching students formal inference rules such as contraposition of meaningless statements, because no one’s brain works like this.

On the other hand, if we follow the mental logic theory to the extreme, it would only be confusing to talk about different semantical examples to teach deductive reasoning. The different contexts wouldn’t have an effect on the purely syntactical reasoning systems of the students. According to this theory the mathematics/logic teacher should focus on expanding and polishing the internal logic of every individual student. This could for example be done by giving gibberish but valid arguments. An example is “if ktont, then proska, and ktont. So proska” to show students a correct inference rule. The focus lies completely on the syntactical aspect of reasoning.

Since the 21st century, the mental models theory has overwhelming support amongst psychologists, Bonatti (1994) admits, despite his active attempt to debunk the mental models. This can be seen in (Schaeken et al., 2000) which is an elaborate book, edited by Schaeken, De Vooght, Vandierendonck and d’Ydewalle, Gérythat, that tries to summarize the psychological research on deductive reasoning. The majority of the authors of this book are proponents of the mental models theory, although the mental logic theory still receives some attention in the book. Historically speaking, mental logic theory was very important, but was mostly used by logicians and philosophers, rather than psychologists. This is in contrast to the relatively new theory of mental models. It’s only since the 1960s where both theories really could compete with each other, and where the mental models theory outcompeted mental logic.

There’s however also a downside for the pedagogy of logic, if we follow the mental models theory. Some proponents of the mental models theory namely use their theoretical interpretations of experiments (for example the Wason Selection Task which is discussed in Subsection 1.2.1) to conclude that logic has no place in education. According to Durand-Guerrier, Boero, Douek, Epp and Tanguay in (Durand-Guerrier et al., 2012) this stems from an inaccurate understanding of “logic”. Logic is wrongly understood as purely syntactical by some psychologists. Durand-Guerrier et al. (2012) argue for not discarding logic as irrelevant for deductive reasoning in the mathematics classroom. This is firstly because logic isn’t purely syntactical. In (Durand-Guerrier, 2008) she uses “Tarski’s theory of truth” (a historically important analysis of the semantics of a logic which is similar to how I illustrated semantics in Chapter 0) to illustrate this point. Also in (Durand-Guerrier, 2020) she uses more recent theories of semantics to argue the same point. Secondly, it is not because there would be some focus on syntactical rules that all semantical activities in the mathematics classroom would be abandoned.

To conclude, it is important to be aware that this debate was important in the psychological community and is now mostly settled in the psychologists community. This can help us to later analyze available teaching material. In Subsection 1.2.2 the tension between psychologists and advocates for teaching logic in mathematics lessons will come back to the surface.

1.1.4 Cultural Psychology: is logic universal?

In this Section we focus on the question of whether logic is universal with insights from cultural psychology. This Section is largely based on the “Handbook of Cultural Psychology” from 2007, more specifically on Chapter 23 “Perception and Cognition” by Norenzayan, Choi and Peng (Norenzayan et al., 2007). We focus more on the cognitive aspects, though these two are of course intertwined. In this Chapter the authors mostly contrast “The West” and “The East” and don’t really consider other cultures. This is because most of the currently available research focuses on these two cultural groups.

In the Chapter we get two reasons why the universality of cognition has been a long standing assumption in psychology. Psychologists more concretely assumed that what people thought and believed was vastly different, but that the way of thinking was uniform across humanity. Firstly, since psychology historically speaking originated in biology, psychology inherited the way of looking at humans as one species, which is common in the theory of evolution. In the rudimentary theory of evolution all the traits of a species are ascribed to a shared, species-wide genome, thus also including cognitive traits. This leaves out culture, and implies the universality of cognition amongst human kind. Secondly, psychology was also influenced by other fields such as (the start of) computer science. This influence is particularly visible in the comparison of human brain and computer. The idea was that every human had the same “processing device” in their head. Differences in beliefs and cultures were explained by inserting different inputs in this processing device we call “brain”. You can still see this computer science like terminology in psychology papers, for example that treat the mental logic theory from Subsection 1.1.3.

Since the beginning of the 20th century, there were also voices rising to include the influence of culture in psychological theories. The first to raise this concern was Wilhelm Wundt in 1916, who is commonly regarded as the founder of experimental psychology. Also in the Russian School of Lev Vygotsky the impact of the cultural context on cognition was proclaimed. Thirdly, and most influentially, was the linguistic relativity hypothesis by Whorf in 1956. This hypothesis states in its simplest form that the language people speak *influences* their thought processes. Meaning that you're more or less likely to think in a certain way, but there's no strictly impossible thought processes. The hypothesis also has a strong form called linguistic determinism, which states that the language people speak *determines* their thought processes. According to linguistic determinism your spoken language makes certain thought processes strictly inaccessible for you.

Now that we saw several theoretical reasons to (not) assume the universal character of cognition, we can review some empirical research papers. In short, we can already quote the Chapter to set the scene: "The cross-cultural evidence reveals marked cultural differences in the cognitive strategies, or habits of thought, recruited to solve a given cognitive problem." (Norenzayan et al., 2007).- In order to summarize the literature structuredly, the authors try to dissect the cultures involved into separate variables such as language, level of industrialization, ... It's important to realize that this dissection isn't easy, because different aspects of a culture can be intertwined. First we will talk about the influence of the language, a.k.a. the Whorfian hypothesis. Secondly we will consider the influence of industrialization. Lastly we will consider two cultural groups in a more general way, namely the West (USA) and the East (China, Japan, Korea), that have a similar level of industrialization (but unfortunately a different language). This last point illustrates the previous warning: in a world ideal for cultural psychology research, for every cultural aspect you would have two nearly identical cultures that only differ on the aspect you want to investigate. But we don't live in such a world.

Let's consider first the linguistic relativity hypothesis. The literature provides some support for the conjecture although the precise implications of these studies continue to be debated. In particular, a study regarding the influence of the counting system in different languages is mentioned in the Chapter. More concretely, in English the base-10 structure for numbers between 10 and 20 is quite hard to recognize with names as "eleven", while this is clearly present in the Chinese names for these numbers. This structural difference does influence the way children learn to count. Another study comparing Chinese and English was done about hypothetical reasoning. In English the subjunctive mode of a verb makes clear that we're talking hypothetically (for example "If mathematics were beautiful, the Borwein Integrals wouldn't exist."). This mode doesn't exist in Chinese, where the hypothetical nature of a statement should be made clear from context. In this study it was found that Chinese speakers indeed performed more poorly than English speakers in hypothetical reasoning (though this study has also been criticized about the accuracy of the Chinese translations, used in the test).

These are two examples that offer some support for the Whorfian hypothesis, specifically related to mathematics and logic. So let it be clear that this really is relevant for the logic/mathematics teacher. Further research is needed to determine the exact influence of language on cognition. However, researchers already agree that linguistic determinism is definitely not true.

Let's now consider the influence of industrialization on deductive reasoning. One of the first studies in this area was done by Luria in the 1930s (Luria, 1931). She went on an expedition to remote Central Asia where she presented syllogisms to people with different levels of modernization. The main results of this study are concisely summarized by Norenzayan et al. (2007) and are paraphrased here in the following three points. Firstly, Western-style schooling seems to facilitate logical reasoning. Increases in performance are detected in people with 2-3 years of schooling. Secondly, people from different cultural backgrounds scored similarly if controlled for age and schooling. Thirdly, the most important generalization is that in not modernized societies there's a preference for concrete thinking based on direct personal knowledge. They simply didn't deem the logical structure of an argument as relevant, and were more focused on the concrete subject. This was especially clear with syllogisms about contents unfamiliar to the participants. In those cases one could derive a correct conclusion using formal logical rules, without knowledge about the specific subject. Some participants however refused to answer the question themselves, and referred the researcher to someone else who might know more about the specific subject. However, this doesn't mean that the logical reasoning abilities were absent in these people. We can conclude that industrialization influenced people in their deductive strategies. There's no evidence that people from a less industrialized society score lower (taking schooling into account). They do prefer different reasoning strategies, and usually don't rely on formal logic.

There was also research in comparing two approximately equally industrialized cultural groups, namely the West (USA, Western Europe) and the East (China, Japan, Korea) Also in this case systematic differences were found, and researchers even went so far to categorize it as two different ways of reasoning. The *holistic* way of reasoning involves an orientation to the context as a whole, including attention to relationships between a focal object and the context, and a preference for explaining and predicting events on the basis of such relation- ships. Holistic approaches rely on experience based knowledge rather than abstract logic and are dialectical, meaning that there is an emphasis on change, a recognition of contradiction and the need for multiple perspectives, and a search for the "middle way" between opposing propositions. The *analytic* way of reasoning involves detachment of the object from its context, a tendency to focus on attributes of the object to assign it to categories, and a preference for using rules about the categories to explain and predict the object's behavior. Inferences rest in part on decontextualization of structure from content, use of formal logic, and avoidance of contradiction. People from the West seem to prefer the analytic way, while people from the East prefer the holistic approach. Again, there wasn't any difference in the capacity for applying formal logical rules if the subject was purely formal (and thus without context). A blunt example to understand the difference between those ways of reasoning is the following. Imagine a situation where you're with a group of friends. One of them says it's raining, but the other one says it's not raining. The typical way of an analytic reasoner is to be confused, and think one of them is (accidentally) saying something untrue. The typical way of a holistic reasoner is to find the middle way by concluding it's drizzling. This cultural difference is obviously relevant for (the teaching of) logic, in for example teaching proof by contradiction. It also has an influence on the normative status of logic, which we hinted at in Section 1.1.2. Holistic reasoners ascribe less importance, and thus also less or no normative status, to logic in general.

Several theories have been proposed for the observed differences, but we won't go into them in detail. One possibly explanation is the different usage of internal speech in the two cultural groups. In the West open debate and discussion are historically important, so reasoning is associated with talking. In the East talking is discouraged and silence is associated with reflection and thinking, in for example silent meditation. However, more research is needed to determine if there is a link to the inference strategies. A more general explanation is to attribute this difference to the different most important philosophies of the two cultures. For example, the principle of avoiding contradiction is and was used in the West to discard falsified scientific theories, while the principle of accepting contradiction is and was an important part of the Eastern yin and yang. Regardless of the cause for this difference in reasoning, I would like to provide a more detailed example in contemporary logic research that shows there's a real conflict between the two perspectives.

In the paper of Fu (2016) with title "Is Universal Logic 'Universal'?" we find a quite interesting conflict. In 2007 there was a congress and school on "Universal Logic" in China. This was an initiative from the Western Universal Logic Project, which is an initiative originating in Switzerland that aims to find general properties of all logical systems. It's already clear that this is a very analytic approach. One of their proud achievements is for example reducing some many-valued logics (where there are more than two truth values) to binary logics (with two truth values). More details of this can be found in (Fu, 2016). However, at this congress, Hucan He was presenting his own universal logic, named "C-UniLog", short for universal logic in China. His team also wanted to study logic in general, but proposed an encompassing, holistic approach based on the Eastern view on logic.[4] They for example accepted contradiction as part of their system. It's quite ironic, and really exemplary for the cultural difference discussed before, that different research groups claim to be working on "Universal" logic. Fu (2016) then goes further by analyzing the concrete differences between the two projects.

To summarize the findings so far: there's some support for the weak linguistic relativity hypothesis and there's evidence for some cultural difference in reasoning. There's however no difference in performance, if age and schooling are taken into account. Cultural psychologists therefore characterize the cognitive apparatus of people as a toolbox with different levels of universality. Some tools might be universally available in the toolbox, and equally used by everyone around the globe. Some tools might be available in everyone's toolbox, but used differently for the same problem depending on the culture. And there might even be some tools that are simply not universal, and not available in some people's toolboxes.

Why is this cross-cultural research relevant for the Flemish context? I wanted to include this Section in response to mathematicians/teachers who consider logic to be purely objective and universal. This Section is a reminder that including (certain parts of) logic in the curriculum is a cultural choice, which we will see in action later. One lesson we can already take from this Chapter though, is that using a cultural deficit model for logic (assuming that some cultures are better at reasoning logically) isn't supported by the empirical research, and should (also for other reasons) be avoided in teaching logic.

[4] This C-UniLog is not quite a logic, but more a 'metatheory' according to (Fu, 2016)

1.2 Pedagogical Research

After the theoretical perspectives in the previous Section I will now turn to a different kind of academic research. In this Section I discuss what we have learned from empirical research for teaching logic. Firstly, I give some factors influencing logic performance, both from psychological and pedagogical research. Then, there's a brief Subsection on the place of logic in mathematics education, followed by some pedagogical advice for teaching logic. In the end, I finish with a concrete list of elements of propositional/predicate logic that students have difficulties with. As I said in the introduction, some points were already highlighted by Milbou (2013) and they will also be treated within the previously described structure of this Section.

1.2.1 Factors influencing logic performance

A potential factor in logic performance is age. Milbou (2013) summarized two papers that conclude there's no positive effect on conditional reasoning (= reasoning with material implications) based on age. However, schooling does have a positive effect, and it would be possible to already teach logic at a young age. The positive effect of schooling was also touched upon in Subsection 1.1.4. In the papers discussed by Bronkhorst (2006) he found a positive age effect on logic performance, but there was no control on schooling, while this was present in the papers of Milbou (2013).

In the paper (Markovits, 1993) we get a seemingly disappointing overview of the psychological research on logic performance. Markovits (1993) provides several psychological studies that contradict each other in the influence of age on logic. There are studies, like the one in (Milbou, 2013), that conclude that very young children can already reason logically. There's however also a study cited that only adolescents and adults can do so. In the most extreme case, we have the Wason test and its derived experiments, that seemed to conclude that not even educated adults can reason logically on a consistent basis. The Wason test is a famous experiment from 1968 that tested the deductive reasoning skills of participants with the following test. This explanation comes from Bronkhorst (2006). Participants were shown four cards as in Figure 1.1. Each card had a letter on one side and a number on the other. Only the front sides of the four cards are visible. Participants were informed of the rule that "every card with a D on one side, has a 3 on the other side.". Then, the participant was asked what cards they would need to turn around, to know for sure that the rule is followed.

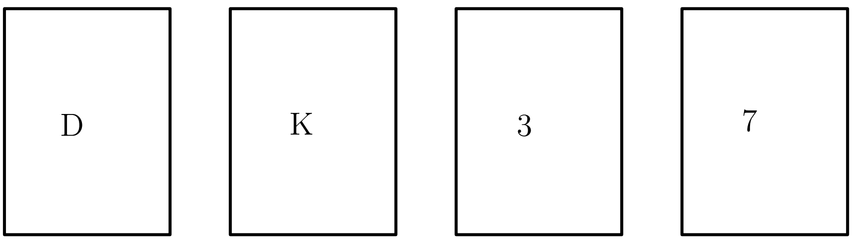

Figure 1.1: Four cards of which only the front is visible in the Wason test.

The correct answer, according to propositional logic, is the card with D and the card with 7 on it. It's clear from the given rule that the card with D should be switched around, but if we contrapose the rule, we can see that also the other card with a 7 needs to be turned around. Performance on this test was very bad. It was also specifically tested on mathematicians, who scored higher than people with different backgrounds, but still gave the correct answer only 47 % of the time. Markovits (1993) goes on to lay out a reformulation of the mental models theory we saw in Subsection 1.1.3 that better explains the differences found in empirical research. In his theory he takes a developmental component into account, which allows influence from schooling.

As you can see from the papers and studies discussed above, empirical research didn't conclude in unison that age does (not) matter and schooling does (not) matter. In fact, the different views on logic in human cognition and education has an influence here. The significantly bad results on the Wason test, regardless of age and schooling, have been historically used by proponents of the mental models theory (see Subsection 1.1.3) to conclude that humans as a whole do not reason logically and their reasoning should be better understood by mental models without formal logic. They have been criticized both for having a too limited (syntactical) view on logic as we saw in Subsection 1.1.3 They have also been criticized for using inappropriate logical systems for the analysis of experiments. We namely already saw in Subsection 1.1.2 that not everyone agrees propositional logic is the appropriate normative logical system for every situation imaginable. For example, Stenning and van Lambalgen (2004) aim to extend the logic used in psychological empirical research. However, mathematics educators use the empirical results to argue for age not being a factor, that even very young children are able to reason logically. They often conclude that schooling does have a positive effect on logical reasoning. This optimism can be seen in (Bronkhorst, 2006; Milbou, 2013; Durand-Guerrier et al., 2012; Bakó, 2002), where educational contexts are designed that provoke and improve logical reasoning.

With the inconsistency, and influence of underlying theory, laid out in an honest way, I want to conclude that age does not matter, but schooling does for logic performance based on the criticisms and extensions of older psychological research in deductive performance discussed above. The next two themes weren't discussed yet by Milbou (2013).

The next point is the potential influence of (natural) language on logic performance. In Subsection 1.1.4 we already discussed the linguistic relativity hypothesis, and saw that language can have an influence on performance in deductive reasoning. That is, we saw that using different natural languages can influence the performance. Apart from comparing natural languages, also the difference between formal language and natural language is a difficult point that can decrease logic performance. In the psychological paper (Markovits, 1993) this is also listed as a factor. Also papers from mathematics education researchers elucidate this, as in (Polé, 2016), based on their teaching of mathematics courses. We provide some examples of differences that have been shown to negatively impact performance (more sources are listed by Durand-Guerrier et al. (2012)). The difference in meaning between the always inclusive $\vee$ and sometimes inclusive, sometimes exclusive "or" (Dawkins, 2017). The logical $\wedge$ is symmetric, while the "and" in natural language often has a temporal meaning like "and then ... ", as made clear in the teaching material of (Milbou, 2013). The implication "if ... , then ... " is for most people not the same as $\Rightarrow$, especially not if the antecedent is false (see Subsection 1.2.4). Durand-Guerrier et al. (2012) also give the examples underneath with quantifiers where the difference with natural language can cause problems. The proverb “There is a lid for every pot” is more formally stated as “For every pot, there is a fitting lid.” So the order of quantifiers, which is important in mathematical theorems, is not so strict in natural language. Another example with quantifiers, which is very common in the mathematics classroom, is implicit quantification of a variable (Durand-Guerrier, 2003). More concretely, the statement $A =$ “if x is a square, then the diagonals of x have the same length” technically does not have a truth value if we translate it into predicate logic as $P(x) \Rightarrow Q(x)$ where P denotes the property of being a square and Q denotes the property of having same-length diagonals. Even mathematicians will find this surprising, since we actually mean $\forall x : P(x) \Rightarrow Q(x)$ with A, which does have a truth value. Sentences with open variables (so variables that are not bound to a quantifier) do not have a truth value. Reading A as $P(x) \Rightarrow Q(x)$ in the mathematics classroom, won’t cause problems because $\forall x : P(x) \Rightarrow Q(x)$ is actually true. So anything you put in x will make $P(x) \Rightarrow Q(x)$ true. There’s however a problem when a statement B like A is read as the open sentence $P(x) \Rightarrow Q(x)$ and it is contingent (meaning sometimes false, and sometimes true depending on x). Take for example $B =$“if x is divisible by 2, then x is divisible by 6.”. Teachers will argue for B being false, because they implicitly quantify B into $\forall x : P(x) \Rightarrow Q(x)$ which is indeed false if one counterexample is found. However, some students (and this has been empirically shown by Durand-Guerrier (2003)) will not accept this, and will call B neither true nor false, but “can’t tell”. This is because indeed the truth value of the $P(x) \Rightarrow Q(x)$ interpretation of B depends on x.

The last theme is the influence of context on deductive reasoning. We already saw in Subsection 1.1.4 that the general cultural context can be important. Not so much in actual performance (if controlled for other variables), but in attitude towards logic. There we also saw that if the contents of a logical problem are unfamiliar some people even refuse to answer questions about the problem, even if the answer could be deduced using logic. The importance of context is also clear in general psychological research as discussed for example by Bronkhorst (2006). It goes in both directions, if the context starts becoming more unfamiliar, people score lower, and if the context is more familiar, people score higher. For example, in a variant of the Wason test described by Bronkhorst (2006), people score way higher while they are logically equivalent. Instead of cards with abstract letters and numbers, it now uses cards that represent people in a known situation. There are again four cards as in Figure 1.2. Each card has the age of the person on one side and the name of his/her drink on the other side. Only the front sides of the four cards are visible. The rule becomes “If someone drinks alcohol, then they have to be at least 16 years old.” and the question is still what cards need to be turned around to completely verify the rule. The correct answer here, according to propositional logic, is the card with “beer” and the card with 15 on it.

Durand-Guerrier et al. (2012) also note the influence of context specifically about mathematics education. They write: “Knowledge of the principles of logical reasoning becomes most important when familiarity with the mathematical subject matter does not suffice by itself to ascertain truth or falsity in a given situation.” Also Dawkins and Cook (2017) write: “Students’ initial reasoning heavily reflected content-specific and pragmatic factors in ways inconsistent with the norms and conventions of mathematical logic.”

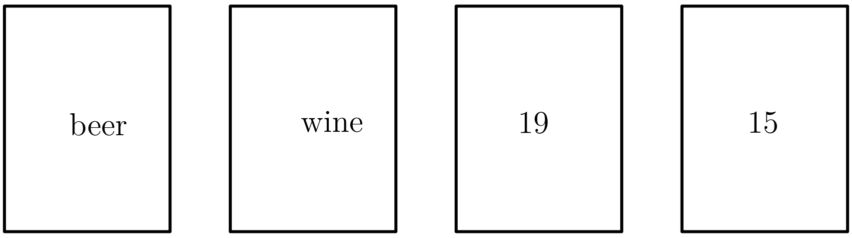

Figure 1.2: Four cards of which only the front is visible in the altered Wason test with a more familiar context.

1.2.2 Logic and mathematics education

Milbou (2013) goes on to discuss the relation between performance in logic and mathematics. She discussed an empirical paper with qualitative interviews where logic performance predicted mathematical performance. Also an other paper/essay was considered that argued for the inclusion of logic in mathematics education.

As touched upon before in the Subsection above 1.2.1, this is in fact the overall sentiment amongst researchers who investigate the influence of logic on mathematics performance. For example in papers (Durand-Guerrier, 2020) and (Durand-Guerrier et al., 2012) based in French high school and university education, in an Indian paper (Guha, 2014), in a Hungarian paper (Bakó, 2002), in a paper from the Netherlands (Bronkhorst, 2006) and in a thesis from Belgium (Milbou, 2013), it is argued for including logic in mathematics education. Even Freudenthal, the main proponent of Realistic Mathematics Education that started as a reaction to the heavily logic based New Maths Movement, wrote a paper (Freudenthal, 1985) to advocate for making reasoning moves explicit and for students to reflect on mathematical thinking. The main argument provided, and this is especially made clear by Durand-Guerrier (2020), is that studying logic in mathematics helps to differentiate valid from invalid forms of deductive reasoning. If students only ever reason in natural language, it often leads to invalid logical moves when students reason on themselves (as more concretely listed in Subsection 1.2.4). It is not enough to reason and prove mathematically with implicit logical moves, and hope that students will find the logical structure on their own. The different themes above in Subsection 1.2.1 can and do prevent this. Underneath we list pedagogical advice for teaching logic in the mathematics classroom.

1.2.3 Pedagogical advice for teaching logic

In this Subsection I will thematically list some points of advice, based on the available literature.

We noted the factor of (natural) language has an influence on logic performance. One might think that in order to avoid the associated problems, you need to avoid the use of natural language. For example, educators might try to visualize the mathematical objects that are discussed in the problem. However, this kind of visualization negatively impacts logic performance in one paper that Milbou (2013) covered. In another paper on the matter that she summarized, visualization didn't have an effect. These findings were

also confirmed in a later experiment by Ott, Brünken, Vogel and Malone that compared the performance if different representations are used in the problems (Ott et al., 2018). In their paper they conclude that the textual description of the logic problem received the most attention (compared to the formula and the visual representation) and was the reference representation in all combinations of the three representations. In conclusion, despite the possible difficulties with natural language, empirical research shows that this is the most accessible representation for logic. Also in papers from mathematics educators like (Bakó, 2002) and (Guha, 2014) natural language is used as a start for students, for example in the form of logical riddles. Durand-Guerrier et al. (2012) state very clearly that the difference with natural language should be explicit in lessons and textbooks. According to Durand-Guerrier et al. (2012), it is even possible to use simple sentences in the beginning of learning logic ("Miranda is tall" as a predicate $T(m)$ where m stands for Miranda and T for being tall), even if it sounds artificial. So the pedagogical advice is two-fold: you should not avoid natural language and it's a good idea to use natural language with riddles/puzzles as a start for teaching logic.
That is not to say other starting points aren't possible as we will see later in this Section.

There's an important remark however about "visualization". It can come initially as a surprise that visualizing does not help, since this often helps in other mathematical subjects. The three original papers in the paragraph that conclude visualization does not work, use pictures to show the objects involved in the logical problems. They visualize the semantics of the problem, so to speak, with concrete examples. If you would study logical problems about squares, you would see a picture of a few squares. However, a different kind of visualization is also often used in logic textbooks. Instead of visualizing the objects involved, you can visualize the logical structure of a problem/argument. I give a few examples of visualizations used. In a paper by Bronkhorst, Roorda, Suhre and Goedhart, it's advised to use visualizations of the logical structure of an argument (Bronkhorst et al., 2020). This is sometimes called 'argument mapping'. He gives the example underneath in Figure 1.3 of an argument about factors in life span. The main takeaway is that students had to read an article about factors that (can) influence your life span and represent the structure of the reasoning visually.

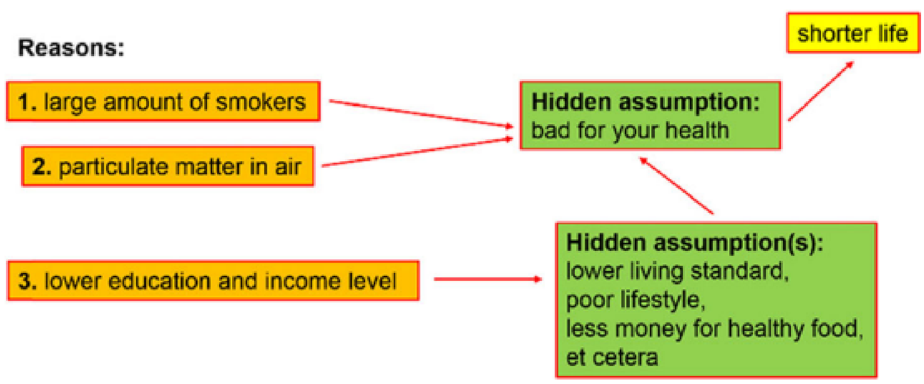

Fig. 6 Formal scheme for the everyday reasoning task

Figure 1.3: Example of visualization of logical structure in (Bronkhorst et al., 2020)

A second example is a visual procedure for determining if a formula in propositional logic is a tautology or not. The method called 'truth-trees' is described in (de Pater and

Vergauwen, 1993). In short, the method works by deconstructing the given formula into a tree where one searches for ways to make the formula false. If no way is found (in the terminology of this procedure: every branch of the tree is 'closed'), then the formula is a tautology. An example is given in Figure 1.4.

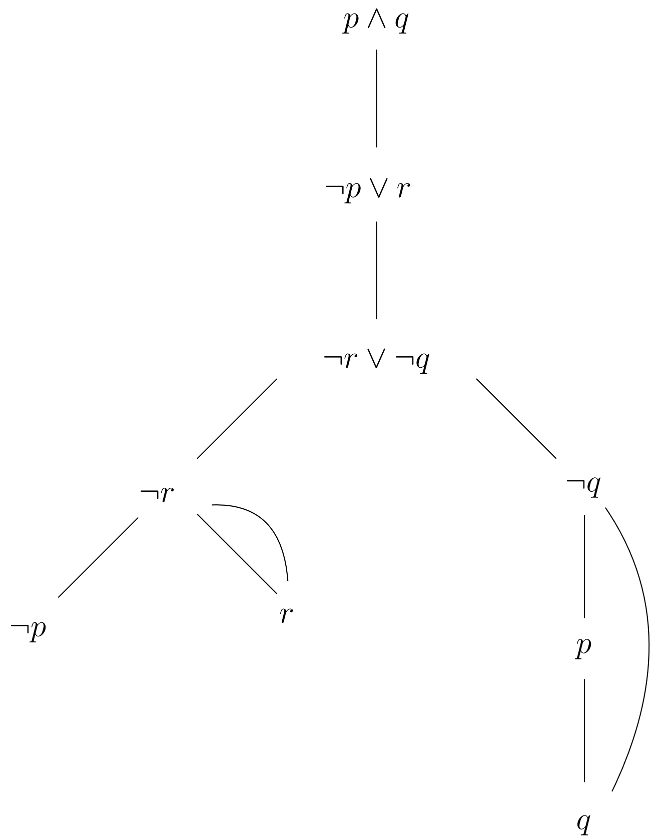

Figure 1.4: Example of (the start of) a truth-tree for the formula $(p \wedge q) \wedge (\neg p \vee r) \wedge (\neg r \vee \neg q)$, based on de Pater and Vergauwen (1993).

A third example is using electrical circuits to illustrate a logical formula. An extensive guide, and mathematical analysis, can be found in (Allwein and Barwise, 1996). Underneath is an example in Figure 1.5. To construct a logical circuit of a given propositional formula the atomic letters are the inputs of the circuit. Every logical negation is replaced with a NOT-gate, every logical conjunction with an AND-gate, and so on. Other connectives can be constructed from the basic connectives.

Even more examples of visualization exist, namely Venn diagrams can represent logical connectives and in syllogisms visual schematics are also used (de Pater and Vergauwen, 1993). These examples were simply to illustrate a different way of 'visualizing'. That is not to say these visualizations solve all problems. There are some sources that suggest a positive effect of this understanding of visualization. Bronkhorst (2006) used his method of visualization amongst other techniques (such as classroom discourse) and found a positive effect of his teaching (material) on the students' logic performance. Cullen et al. (2018) talk about a similar visualization method as Bronkhorst (2006) used and experimentally also found a positive effect.

In terms of teaching logic with or without context, we can be very clear. It is not

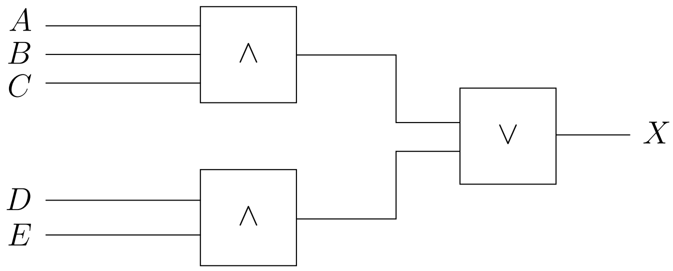

Figure 1.5: Electrical circuit representation of the formula $(A \wedge B \wedge C) \vee (D \wedge E)$ based on (Eggermont et al., 2021). Capital letters are used for the subformulas since $A, B, C, \ldots$ can be complex formulas themselves.

recommended to only teach logic without context, meaning purely formal where the semantics are literally limited to zeros and ones. Durand-Guerrier et al. (2012) clearly state this, and advise to use both new, unfamiliar contexts to foster the need for formal logic and familiar contexts to train logic. In the extensive paper several possibilities are listed. The idea is that if you start teaching logic in a completely known context, the need for formal logic won't arise, but if you start in a completely unknown or completely formal context, logic will be considered meaningless. The question is to find a balance that invites logical reflection, which can be done by using accessible but unknown contexts of which Durand-Guerrier et al. (2012) give some examples. Also Bakó (2002) says: "...we need to teach logic, but not as a separated part in mathematics.". We can also see ourselves from the unfortunate history of the New Maths Movement in Subsection 1.1.1 that no context is not a good idea.

An extra point that wasn't fully covered yet by Milbou (2013), is the extensive use of virtual tools and software in teaching logic. She already explained in detail "Tarski's World", which is a world where virtual objects are displayed that can be manipulated. The users of the software practice their logic skills by manipulating the objects on screen, in order to make a given logic formula true. The possible manipulations include enlarging or shrinking and dragging objects around in 3D. The formula can include predicates such as "... is smaller than ... " or "... is on the left of ... ". According to two sources listed in (Durand-Guerrier et al., 2012), one of which also discussed in (Milbou, 2013), this software was successful in aiding teaching of first order logic. It was mainly the manipulation aspect and the instructions that guided student to known difficult situations that determined its success. However, we have gone a long way in terms of technology since the 2000s, when Tarski's World came out. There have been several international congresses, one in 2006 (Manzano et al., 2006) and one in 2011 (Blackburn et al., 2011), about "Tools for Teaching Logic". In these congresses there are often a lot of different virtual tools presented that help students in the university level with learning logic (for example in the program for computer science or philosophy). Not only that, there have even been studies that use the popular video game Minecraft to teach students logic (Prayaga et al., 2016; Duncan, 2011). In Minecraft, there's a 3D world in which you can walk around in first person perspective. You can manipulate and change your environment, and the game has a game mechanic similar to real life electricity. Students learn propositional logic, based on riddles with 'electrical' circuits in the game.

1.2.4 Concrete difficulties in propositional and predicate logic

After some concise pedagogical advice, I want to include a very concrete list of difficulties found in the papers that I gathered. A portion of this list was already composed by Milbou (2013) based on her literature study. This list will be based on propositional and predicate logic, since this is most commonly used in mathematics education in high school and in the papers of psychologists and mathematics educators. This is stated here clearly, because some psychological experiments have been criticized for rashly using propositional logic (see Subsection 1.2.1).

- If you would have to order the connectives in propositional logic from most to least difficult, you would have implication and equivalence in number one, then negation and then disjunction and conjunction in approximately the same place (Durand-Guerrier et al., 2012; Dawkins, 2017). Note however that conditional reasoning has been more researched than using for example disjunctions.
- Confusing implication and equivalence (Milbou, 2013; Polé, 2016).
- Confusing contraposition (going from $p \Rightarrow q$ to $\neg q \Rightarrow \neg p$) and inverse of implication (going from $p \Rightarrow q$ to $p \Leftarrow q$)
- Negation of implication.
- There are also a lot of problems with quantifiers.
  - Negation of (multiple) quantifiers (which is sometimes necessary for a proof by contradiction, which is also difficult) (Durand-Guerrier et al., 2012) (Milbou, 2013).
  - Switching $\exists$ and $\forall$ (Durand-Guerrier et al., 2012; Bakó, 2002).
  - Variable management in proofs when dealing with quantifiers (when you want to prove $\forall x : P(x)$, the proof usually starts by taking an arbitrary x and showing that P is true for that x). Durand-Guerrier et al. (2012) explicitly mention the inference rules of universal/existential instantation/generalization, which are further explained in (de Pater and Vergauwen, 1993). These inference rules are what mathematicians implicitly apply when proving a theorem with quantifiers. Also Guha (2014) mentions this variable management. The difficulty comes from several things. Firstly, sometimes you need to introduce an arbitrary y for a theorem $\forall x : P(x)$ because an x is already defined somewhere else in the proof Secondly, the different meaning of variables in proofs is confusing. Sometimes an x comes from a $\forall x$ and is a placeholder for an arbitrary value, sometimes x is a constructed value in the proof to conclude $\exists x$ in the end. Thirdly, when working with multiple quantifiers at once, the introduced variables depend on each other.
  - Implicitly quantifying an implication with an open variable (Durand-Guerrier et al., 2012).

Van zodra de leerkracht zich hult in een soort neutraliteit,
en de voorgeschreven kennis en vaardigheden enkel overdraagt,
zal hij eigenlijk gewoon handlanger worden van bepaalde groepen
die hun stempel proberen drukken op het curriculum.

— Agirdag et al. (2019)

2 Assessment of curriculum and teaching material

After the exposition part of this thesis, we can now turn towards the more creative areas. With the perspectives and research of Chapter 1, we can namely assess the state of the new (logic) mathematics curriculum and the available teaching material. We go into much detail for the curriculum, and provide a succinct yet comprehensive list of the available material.

2.1 The curriculum

We start this Section by summarizing the general structure of the Flemish curriculum in Subsection 2.1.1. Then we focus on the mathematics curriculum, and finally even more specifically on the learning trajectory for reasoning and abstracting within the mathematics curriculum. This will be a quite technical explanation. We follow up with Subsection 2.1.2 where we use all the theoretical perspectives of Section 1.1 to better understand the idea of the logic curriculum. We then turn our attention in Subsection 2.1.3 towards the pedagogical research of Section 1.2. In the end of this Section in Subsection 2.1.4, I give some final, more personal remarks, reflecting on this analysis and the role of the logic curriculum.

2.1.1 Exposition of the curriculum

Let's start with a (concise) structure of Flemish secondary education.[1] Secondary education in Flanders consists of 6 grades, from grade 7 till (including) grade 12. We are using American terminology (like "grade") to make this structure easily understandable for the non-Flemish reader. Pupils start around the age of 12 years old and finish high school at

[1]This structure is concise in the sense that we will strongly simplify our explanation, and only discuss the necessary parts for this thesis.

the age of 18. Pairs of grades are grouped into so-called "stages". The first stage consists of grade 7 and 8, the second stage contains grade 9 and 10 and finally the third stage is composed of grade 11 and 12. This is important, because the curriculum is split into separate parts for each stage.

There was also a recent reform of Flemish secondary education. We now divide students based on their orientation on what they will do after secondary education.[2] We have three options for the orientation: 'orientation to higher education' for study programs where one is expected to go to higher education, 'vocational orientation' where one is expected to start working after high school and 'combined orientation' for study programs where one can choose between higher education or starting to work. **From now on, we will only talk about students with the 'orientation to higher education', since it's only these people who have to learn logic in their mathematics class.**[3] A more detailed explanation of the structure of secondary education can be found here.

Let's focus more on the structure of the curriculum. A full explanation can be found here (in Dutch). In the first stage there is the core curriculum and electives. The learning objectives in the core curriculum are divided based on 16 key competences such as 'Foreign Languages', 'Physical and Mental Health' and also 'STEM'. This means that the learning objectives are no longer bound to specific courses, but to these competences. The mathematics curriculum is part of the wider competence for STEM. In the second and third stage the core curriculum is again divided based on the same 16 key competences, including the one for STEM. Everyone in one stage gets the same core curriculum. Apart from the core curriculum, there is also a specialization curriculum for the second and third stage. The learning objectives in the specialization curriculum are called 'specific learning objectives'. These specific learning objectives are divided based on 16 science domains, that are not the same as the competences. There is a domain 'Mathematics', which is separated from 'STEM', and also 'Economics', 'Physics', . . . Per domain they are also grouped in packages, that are assigned to pupils based on their individual study program.

Every learning objective follows a standardized form. The full explanation of the form can be found here (in Dutch). A learning objective can contain the following parts. The obligatory elements are colored red.

- **X.XX (the number of the learning objective): A single sentence expressing the objective, using an operationalized verb**

- Including knowledge:
  The knowledge needed to realize the objective is listed, divided into separate categories. Not every category needs to be present. The possible categories are below.

  In the knowledge elements there is often a "such as" construction after the general idea, giving lots of different examples of the knowledge element. These are just suggestions; to realize the learning objective and the knowledge element it is allowed to

[2] In contrast to the old system that was based on a historical division between subject areas.

[3] That is: students in the first stage in the A-stream and students in the second and third stage with 'orientation to higher education'.

learn completely different things than the items on the list after the "such as", as long as the general idea of the knowledge element is followed. For example to realize a knowledge element like "Fallacies such as wrongly inverting an implication, confusing an implication with an equivalence" can be realized by studying the fallacies of False dilemma and False authority.

  - Factual knowledge:
    Factual knowledge contains the terms, definitions, symbols that pupils need to actively employ and understand. In mathematics this knowledge category is focused on terminology.
  - Conceptual knowledge:
    Conceptual knowledge contains a list of concepts, principles, theories, models that students need to comprehend. The pupils do not need to be able to define these concepts (unless they are also listed in factual knowledge).
  - Procedural knowledge:
    Procedural knowledge consists of the techniques, methods and algorithms that pupils use. These are things that pupils *do*. Note that there can be very creative, non-algorithmic items listed here, such as "composing a proof for a new theorem".
  - Metacognitive knowledge:
    Metacognitive knowledge consists of knowledge about the learning process, heuristics, . . . This is never used in the learning objectives on logic.

- Including context:
  In this optional part it is specified whether a context should be used for the learning objective. In the learning objectives for mathematics, if no context is specified, one can choose to realize the learning objective with or without a context.

- Including dimensions of the final learning objective:
  This specifies what competence level needs to be achieved to realize the entire learning objective, based on the revised taxonomy of Bloom (Anderson and Krathwohl, 2001). I only discuss the cognitive dimension (ignoring the psychomotorical and affective dimension), since this is the only one used in the logic learning objectives. The possible competence levels are listed below. Note that these do not have a hierarchical order.

  - "know"
    The pupil remembers/recognizes the content in the same way it was presented.
  - "comprehend"
    The pupil comprehends the content, and can provide evidence for this comprehension by summarizing, organizing, performing small operations on the content.
  - "apply"
    The pupil applies the knowledge by solving exercises or problems.
  - "analyze"
    The pupil can examine the information, break it into smaller parts and identify how they relate to each other.

- "evaluate"
  The pupil can make judgments and present their opinions, and provide support for them based on the taught criteria and standards.
- "create"
  The pupil comes up with new or alternative approaches to a problem or expresses themselves in an unique way.

Now that we know the general structure of Flemish secondary education, the structure of the curriculum and the standard form of a learning objective, one last note on overarching influences for the mathematics curriculum needs to be made.

For the core curriculum, the mathematics curriculum is embedded in the 'STEM' competence. There are overarching STEM learning objectives, that also influence mathematics. For example, "6.41: The students design a solution for a problem by using concepts and practices from different STEM subject areas in an integrated way." is a learning objective in the core curriculum of the third stage. Students also need to learn the idea of the scientific method for example, which is also applicable to mathematics.

There is also a separate competence influential for logic in the core curriculum. The "Digital skills" competence refers to all kinds of knowledge and skills students can acquire related to IT. Students are required to use IT consciously, be critical of information online, ... They also learn to use some basic software, like Microsoft Word or Microsoft Powerpoint. One important building block of "Digital skills" is titled "Computational thinking and acting". Students learn to think with algorithms, see basic structures used in programming, learn basic hardware concepts, ... The link to logic is evidently present.

Let's focus now on the mathematics curriculum. There are mathematical learning objectives present in both the core curriculum and the electives/specialization curriculum. In the core curriculum for mathematics, the framework of PISA (OECD, 2021) was used as an inspiration to distinguish separate building blocks. There are 6 building blocks, and they are the following. The last two building blocks are more overarching, rather than individual pieces of mathematical content. This structure applies to all stages.

- Numbers and quantities
  This building block contains number theory and performing calculations with numbers.

- Space and shape
  This building block contains geometry and geometrical reasoning.

- Change and relationships
  Here, you can find algebra, calculus and discrete mathematics.

- Data and uncertainty
  This building block is for statistics and probability.

- Reasoning and abstracting
  This is the building block we have been waiting for, since this building block contains the learning objectives related to logic. The official title is "Reasoning and abstracting taking into account the coherence and structure of mathematics", and we will go into detail how this building block is constructed over the different stages.

- Modeling and problem solving
  This building block contains more general learning objectives about translating a given realistic context into a mathematical model, using heuristics in problem solving, ...

Let us focus on the building block of reasoning and abstracting in the core curriculum of mathematics. This contains a total of 6 learning objectives. Every stage there are two learning objectives, one of which is logic inspired and the other one is more mathematical and uses logic in evaluating mathematical arguments, statements, proofs. The competence level for the more mathematical learning objective is always "evaluate" on the cognitive dimension, since students need to judge mathematical reasonings. The competence level for the logic learning objective is usually "apply" but turns into "analyze" in the third stage.

In the first stage students learn very basic set theory, and understand the difference between implication and equivalence using the correct symbols. This logical knowledge is applied conceptually when studying congruence proofs, properties of geometrical transformations and other mathematical content of the first stage. In this stage there's not really a procedural component yet, so the focus is on the conceptual part.

In the second stage the logical repertoire is extended with conjunction, disjunction and negation from propositional logic. Students learn the truth tables of the logical operators, and translate a statement in everyday language into logical language. Logic gates are proposed as a stepping stone to introduce the operators. This logical knowledge is applied, yet again, in the mathematical content of that stage such as the proof of the irrationality of $\sqrt{2}$. However, the procedural component is now extended to include many options. Possible procedural knowledge elements are translating a statement in words to a statement in symbols, exemplifying a statement, verifying the correctness of a mathematical statement, reconstructing proofs in a slightly altered situation, ... In the principles of the building block of this stage it is specified that teachers can attune to different study programs by putting more or less weight on certain procedural elements.

In the third stage, the logical contents are extended with analyzing derivations and arguments.[4] Instead of only dealing with individual statements, students analyze logical derivations and judge their validity. They learn different logical laws and inference rules. Common fallacies are also treated. This logical knowledge is applied again in the mathematical content of that stage such as properties of sequences or derivatives of standard functions. The procedural component is exactly the same as the one in the second stage, and again differentiation based on study program is possible.

In the electives of the first stage and the specialization curriculum of the second stage, there are no extra learning objectives for logic. The specialization curriculum of the third stage does contain extra learning objectives for reasoning and abstracting. Three 'packages' of specific learning objectives contain extra relevant learning objectives.

- Extended mathematics in service of science
  This contains only one learning objective that merely links the logical knowledge to extra mathematical content these students see, outside of the core curriculum.

[4]We will investigate later if the curriculum is talking about informal or formal arguments, as explained in Chapter 0.

- Extended mathematics in service of economics
  This contains only one learning objective that merely links the logical knowledge to extra mathematical content these students see, outside of the core curriculum.

- Advanced mathematics
  There are two extra learning objectives, one for logic and one more mathematical. Predicate logic is on the menu: students learn the meaning of 'argument', 'predicate', ... They learn to take the negation of a quantified logical statement and when reversing the order of quantifiers is allowed. This logical knowledge is used in the mathematical learning objective that links the logical knowledge to extra mathematical content these students see, outside of the core curriculum. However, this learning objective does not fall under the competence level of "evaluate", but has the competence level "create". Pupils have to construct their own proofs!

This gives us a total of ten learning objectives that are part of the abstraction building block. In the curriculum there are also several Sections with 'Principles' where the assumptions are stated and choices in the curriculum are explained. The principles relevant for the abstraction building block are the principles of the mathematics curriculum, of the abstraction building block specifically, of the first stage learning objectives in the abstraction building block, of the second stage learning objectives in the abstraction building block, of the third stage core curriculum learning objectives in the abstraction building block and of the third stage specific learning objectives about reasoning and abstracting. I will occasionally use the term "logic curriculum" for referring to these learning objectives and their principles. We can thus summarize the logic curriculum in the Table underneath.[5]

Overview	Stage 1	Stage 2	Stage 3	Stage 3: Science	Stage 3: Economics	Stage 3: Advanced
Logic	basic set theory	truth tables statements, $\neg, \wedge, \vee$	Analyze arguments	/	/	predicate logic
Maths	application above, $\Rightarrow$ and $\Leftrightarrow$	application above, $\forall$ and $\exists$	application above	trivial extension core curriculum	trivial extension core curriculum	proving themselves

Table 2.1: This Table displays an overview of the abstraction building block in the mathematics curriculum. "/" means there is no learning objective for that cell. "application above" means the logic content is applied by evaluating, providing foundation for mathematical statements and arguments. I don't list all symbols, just the new ones per stage.

A full translation of the logic curriculum[6], made by myself but checked by several people involved in the development of the logic curriculum, can be found in Appendix A. The original Dutch version can be found online (AHOVOKS, 2021).

[5]This structure is not perfect. For example, the "logical" learning objective in first stage is purely set theory, which might not be considered "logic".

[6]The principles of the third stage specific learning objectives about reasoning and abstracting were not included, because they did not contain anything extra information compared to the learning objectives.

2.1.2 Analysis based on Theoretical Perspectives

In this Subsection, we analyze the curriculum using the theoretical perspectives of Section 1.1. A warning in advance is appropriate: the curriculum does not clearly state its standpoint on the matters we have seen. Despite the 'Principles' being the place where one would expect to find the theoretical assumptions, these texts mostly contain a description of the content already listed in the learning objectives. We can still perform an analysis, though carefully.

Is logic any different?

Firstly, we can look at the change of the mathematics curriculum and compare it to the New Math Movement in the 60s, as discussed in Subsection 1.1.1. It should be no surprise that there are some New Math elements in the curriculum, since the Movement merged with other pedagogical approaches after the 80s in Flanders. The New Math Movement was kickstarted, at least in the USA, because of the perceived decline of mathematics compared to the USSR. An analogy can be drawn with the Flemish context. Similarly, there's a fear of decline in mathematics education, because of the results of (inter)national rankings/studies. There have been several news articles in Flanders reporting the decline of Flemish mathematics education based on results of PISA (Rooms, 2019), TIMSS (Torbeyns, 2020) and in the 'peilingsonderzoek' (the recurring assessment investigating the achievement rate for the learning objectives) (Haeck, 2019).

Not only newspapers, but also individual, important figures publicly share the same sentiment. In an article (Vangelder, 2021) about the decline of Flemish mathematics education where the rector of KU Leuven was interviewed, he stated (translated in English) the following.

> We feel the decline in literacy and mathematical skills of the youth.
> This translates to an increased number of students dropping out.

Also the Minister of Education strengthened the perception of this decline in an online show (Weyts et al., 2020a) presented to mathematics teachers about the new curriculum for the second stage where he said (translated in English) the following.

> [. . .]
> Because I find it essential that we focus more on Dutch and on mathematics. Simply because those are the two subjects, that make other subjects possible. They determine the context, the framework in which children and teenagers can acquire all other knowledge. And that's why they are essential.
> [. . .] [intermission about primary education]
> Also in secondary education we raise the bar. And that brings me satisfaction. I also hope that this can maybe, in an indirect way, help take on the shortage of (mathematics) teachers. Finally, some recent scientific news. I have heard of a study in the Netherlands which showed that 30% of the GNP there is linked to mathematical skills. Knowing the parsimony of the Dutch, this 30% will be much higher in Flanders. But that's a question for mathematicians.
> I wish you good luck in this session, that you learn something from it. But mainly that we give the message: 'We raise the bar. We have experienced some decline in quality of education. We have to catch up, and Dutch and mathematics are essential for that.'

So, not only has the news repeatedly reported on the decline of mathematics education, but also influential people reinforce the fear. This atmosphere of fear of decline in mathematics (education) is similar to the start of the New Mathematics Movement.

However, not only the fear of decline is similar. There are also similarities as to what the status mathematics gets within society. The Minister of Education finds mathematics essential for **all** other knowledge in his quote. He even explicitly hints at mathematics education strongly influencing the economy. This sounds very similar to the characteristic of the New Math Movement written down by Bjarnadóttir (2020): "There was a widely held ideology on the absolute value of mathematics for future citizenship.".

A last similarity is the optimistic attitude of "taking on the danger", and "raising the bar", as the Minister puts it. In the beginning of the New Mathematics Movement there also was an optimism for this new pedagogical approach. Another speech in a show (De Winne et al., 2018) presented to mathematics teachers about the new curriculum for the **first** stage shows the analogy in the present. The show has been put online here. Ivan De Winne, president of the Flemish union for mathematics teachers, started the show by saying the following (translated to English)[7].

> Good afternoon!
> Welcome to this info session of the Flemish union for mathematics teachers, with topic: the new mathematics curriculum for the first stage which will take effect on September 1, 2019, together with the anticipated reform of secondary education. But let me first take you 50 years back in time. September 1, 1968 was a very special day for me, because then I started in the first year of secondary education. Also for mathematics teachers, this was a memorable school year because of the introduction of New Math, a radical reshape of mathematics education, inspired by the axiomatic, deductive-logical structure of mathematics. A slight panic took hold of the classically trained mathematics teachers, when exploring amongst other things Venn diagrams and mathematical structures like groups, rings, fields and vector spaces. The introduction on September 1, 2019 of the new curriculum and course specific curriculum for mathematics in the first stage is most likely somewhat less far-reaching, but necessary. The curriculum namely refers to what we as a society expect as a minimum from all pupils in education in terms of knowledge and skills. Since the introduction of the last curriculum in 1991 society has rapidly changed. An upgrade of the curriculum could not wait any longer. As representative for mathematics teachers in the curricular development committee VVWL [Flemish union for mathematics teachers] has played an important role.
> [...]

This quote explicitly compares the new curriculum with the New Math Movement, and labels the new curriculum as possibly less radical, but necessary. There seems to be a mostly positive attitude towards the New Math Movement, calling it a "memorable year". The sentiment of this speech falls within the "Yes, fear the decline of mathematics education, but we can make it better!" I described earlier.

This is not the only thing in this show that further confirms the similarities with start of the New Math Movement. After the speech of Ivan De Winne, there was a speech by

[7] The composition of the development committee is explained later.

Prof. Ellen Vandervieren who mentions the same waning scores from PISA and other studies, and further confirms the need for an upgrade. The best example I have come across of the optimistic attitude in all sources, comes later in the show. The parts of the new mathematics curriculum are being described, and at 1h53m30s the presenters get to the abstraction building block. A small video of Jean Paul Van Bendegem, mathematician and philosopher associated to KU Leuven, is displayed, who gives a short introduction to logic. He talks about the difference between natural language and logical language, more specifically about how the implication in propositional logic is remarkably different from everyday conditional reasoning. He gives several examples of non-sensible statements with implications that are tautologies in propositional logic. He does not say anything about logic being useful for mathematics teaching; he even gives examples that might make you be skeptical of teaching logic. Yet, the show goes on with the presenter talking about $\Rightarrow$ and $\Leftrightarrow$ being reintroduced in the mathematics curriculum. After that, the audience, comprising mathematics teachers, starts applauding.

However, there are also differences with the start of the New Mathematics Movement. Unlike the New Math Movement, there are no clear individual figures deciding on the mathematics curriculum. In Belgium, Georges Papy was **the** figure representing the new approach. For example now, the Minister of Education and the rector of KU Leuven do not have a direct influence on what is included in the curriculum. More specifically, the development process of the curriculum went as follows. Firstly, AHOVOKS, a governmental institution which concerns itself with the curricula amongst other things, compiled a proposal based on national and international curricula and frameworks. This proposal was then presented to a development committee where it was studied and modified based on discussion between academic experts, representatives of school networks like 'Katholiek Onderwijs Vlaanderen' and 'GO!', and representatives for teachers like members of 'VVWL' or other teacher unions.[8] AHOVOKS supervises this modification process. They do not have any political ties, and that includes the Cabinet of the Minister of Education. There was also a validation committee which regularly checked the coherence and testability of the learning objectives. After this revision and modification process, the proposed curriculum was presented to the Flemish Government which has accepted it. There may have been influential voices within the development committees[9] of the mathematics curriculum (and specifically for the inclusion of logic), but the final version that was sent to the Flemish government, was approved by the committee as a whole through democratic vote.[10]

There are other differences with the New Math Movement. The abstract, deductive flavor that was essential to every mathematical topic in the New Math Movement is surely not present in every part of the current curriculum. Logic may be labeled as being close to the topics studied during the 60s, but nowadays we also have statistics, modeling

[8]Member lists are not publicly available

[9]There were several development committees that proposed things for the mathematics curriculum. There was a committee for 'STEM' competence, core curriculum, first stage; one for 'STEM' competence, core curriculum, second and third stage; and one for the specific learning objectives of the 'Mathematics' domain in third stage.

[10] Though an open letter was published in De Morgen (Loobuyck et al., 2020) by members of development committees who had discovered that the proposals of the development committees had been meddled with before being handed to the Flemish government.

of physical phenomena, ... This is precisely because the current pedagogical approach to mathematics can be described as a mixture of Realistic Mathematics Education and New Math. To say it bluntly, if this abstraction building block of the curriculum "fails", the other building blocks won't necessarily also fall. Also the general principles of the mathematics curriculum start with a paragraph that is clear about the role of mathematics to society.

> The idea that mathematics is abstract and formal and that it is actually separate from reality, is correct to some extent. Mathematics education revolves around the meaningful development and construction of mathematical knowledge and reasoning. This requires a pedagogical approach that pays sufficient attention to finding meaning in the more abstract mathematical concepts. This is why, when interpreting the learning objectives, one needs to not only take mathematics as a subject into account but also the pupil and the society in which that pupil will function. The mathematical contents should therefore preferably be applied in various situations. A provision relating to the context was included in almost every learning objective. [...]

In short, the current situation has similarities with (the start of) the New Math Movement, but also differences. In time we will find out if these similarities extend to the learning performance of pupils when there's another 'peilingsonderzoek'.

Another topic in Subsection 1.1.1 was the philosophical point of view of mathematicians on logic. We can ask a similar question of the curriculum: "What philosophical stance does the logic curriculum take on logic?". Just like we struggled in Subsection 1.1.1 to find contemporary clear views on logic from mathematicians, the curriculum doesn't reveal much about its position. The overall structure of the curriculum, with in every stage a logic learning objective one connected to one for mathematics, seems to suggest that logic is a **tool** used for understanding mathematical reasoning.

Let's look closer at the stages in the logic curriculum. The attitude towards logic seems to vary between the first stage core curriculum, the second and third stage core curriculum, and the specific learning objectives of the third stage.[11] The idea of the first stage is mainly "logic as a handy tool to support mathematical thinking". Specific logic concepts get minimal attention, only $\Rightarrow$ and $\Leftrightarrow$ are seen. Logic has a purely supportive role. In the second and third stage core curriculum, logic is more independent than in the first stage. There is also a mechanical view on logic, for example to calculate truth values. Here, logic would be best described as "logic as a calculation device to support mathematical thinking". In the specific learning objectives of the third stage, there's another nuance. The learning objectives in packages 'extended math in service of economics' and 'extended math in service of science' do not reveal much. The logic learning objective in 'advanced mathematics' still has some mechanical elements, but focuses more on mathematical language explicitly than the rest of the logic learning objectives. It also studies (predicate) logic as an independent subject by for example listing obligatory theorems about predicate logic, rather than just some concepts. Here, logic would be best described as "logic as a subject, useful for understanding mathematical language".

[11]This is possibly explainable because of the different development committees working separately.

Every time the logic content is applied in the mathematical learning objectives. So in general the stance seems to be “logic as a tool to support mathematical thinking”, though it varies quite a bit between the stages.

We have ignored the principles of the abstraction building block so far. The principles (translated in English) say the following.

> Mathematical insight is developed and problem solving skills are promoted by mathematical reasoning through formulation of a conjecture, argumentation, explanation, proof, generalization, structuring, ordering, working analogously and/or synthesizing. It is about more than just writing down a proof from rote memory. In this way the coherence can be overseen, apart from the fragmentary.
> A number of mathematical thinking methods must be acquired. It is important that students are obliged to reflect on the thinking process.

These principles talk about the scientific method as a mathematician and reflection on the thinking process. The first sentence almost could have been written by intuitionist mathematician Brouwer, who was strongly against the use of logic in describing mathematical practices. There seems to be no explained transition from “It is important that students are obliged to reflect on the thinking process.” in the principles of the abstraction building block to a quote from the principles of the first stage: “It is more important here that the students can use the relations[$\Rightarrow$ and $\Leftrightarrow$] than that they can explain in words what they mean.”.

In conclusion, the general idea seemed to be “logic as a handy tool to support mathematical thinking”, though there is a remarkable variation within the logic curriculum. This can possibly be explained by different development committees designing different parts.

Is logic normative?

Now that we have analyzed the curriculum based on Subsection 1.1.1, we can transition to the philosophical questions of Subsection 1.1.2. There is a slight monist, normative hint in the curriculum. Some examples can be found in the core curriculum of the second and third stage.

The most clear example is the phrasing of the logic learning objective in the core curriculum of the second stage. It reads: “6.20: The students determine the truth value of logical statements.”. The monist hint is because of “**the** truth value”; if there’s only on possible truth value, it’s most likely because only one logical system is considered. The normative hint is because of the use of “**logical** statement”. A statement being “logical” has a positive connotation attached to it, so this terminology seems to suggest that there are norms of logic and following them is good. Similarly in the third stage core curriculum one talks about “logical derivations and reasonings”. The term “proposition” explicitly is not used in the learning objectives, while it could have been done. For example, the learning objective about logic for advanced mathematics reads: “6.4.15: The students examine mathematical statements using predicate logic.”. To be clear, it is clarified in the principles of the second stage that the intended logical system is propositional logic. Seeing the choice in wording and the presence of fallacies on the menu in the third stage, there's a slight normative presence in the logic curriculum. Though as said before, it is not so clear.

The phrasing of some learning objectives and the inclusion of some fallacies is normative. However, the inclusion of "Meaning of negation, conjunction, disjunction, implication and equivalence in logic, including differences with the meaning of 'or' and 'if ... then ...' in everyday language" seems to be more on the descriptive side. The use of "if ... then ..." in everyday language is labeled as a difference, not as a mistake. The suggested fallacies in the third stage are "wrongly inverting an implication, confusing an implication with an equivalence". Now, "if ... then ..." in everyday language is used as an example of a fallacy, so a mistake, which falls under a normative view. It's not clear where the line between descriptive and normative view on logic lays in the curriculum. For example the statement "if you have good grades, you will get a present." is potentially on this line. If I understood this implication as an equivalence, since this is most likely meant be the speaker, is this labeled as wrong logical reasoning or as simply a difference with everyday language?

How does logic work mentally?

We continue our investigation with Subsection 1.1.3 that introduced the mental logic and the mental models theory about deductive reasoning. Here we try answering the question: "According to the curriculum, how does logic work mentally?".

According to the general principles of the mathematics curriculum, the curriculum is inspired by the PISA mathematics framework (OECD, 2021). Indeed, this framework contains a component called 'Mathematics as a system based on abstraction and symbolic representation'. That sounds very similar to the building block in the Flemish curriculum with title 'Reasoning and abstracting taking into account the coherence and structure of mathematics'. Also, the principles about the abstraction building block as a whole are clearly inspired by PISA. You can find the PISA framework online and the principles of the abstraction building block in Appendix A. They both talk about the process of mathematics research, the human side of mathematics and the need for coherence and predictability. They both very explicitly mention mathematics as a coherent system. How is this relevant to the question of how logic works mentally?

Well, there is one aspect in the abstraction component of PISA that is not included in the principles of the abstraction building block. In PISA, there is a big focus on symbolic representation. They write for example: "Abstraction involves deliberately and selectively attending to structural similarities between objects and constructing relationships between those objects based on these similarities. [...] Students use representations – whether symbolic, graphical, numerical or geometric – to organize and communicate their mathematical thinking.". In a way, the 'abstraction' is understood as semantically abstracting: students still focus on meaning, but in a more abstract way with graphs, diagrams, ... PISA goes even all the way by explicitly mentioning "mental models".[12] Even more, logic is not mentioned at all in their abstraction building block, not on the website

[12]Remember that this is different from using mathematics as modeling of reality. Both the PISA framework and the mathematics curriculum have a separate component for modeling.

and not in their written out article (OECD, 2021). That is not very surprising, as we saw earlier in Subsection 1.1.3; the mental models theory has historically even been used to argue against the teaching of logic. This then poses the question how it is possible that the mathematics curriculum is inspired by the PISA framework, which is strongly silent about logic and leaning towards mental models theory, while the current curriculum does include logic. This is an explanation you would expect to find in the principles, and could have been based on the responses of mathematics educators I discussed in Subsection 1.1.3.

Apart from the above apparent contradiction, it's unclear what psychological stance is taken in the curriculum. It seems to lean closer towards mental logic theory since formal logic is taken to be an appropriate choice for understanding deductive reasoning, but cannot be surely determined.

Is logic universal?

Lastly, let's discuss the cultural question in Subsection 1.1.4. The choice for including formal logic is not explained in the principles (as we saw in the previous part about the psychological perspective). Nor is the choice for propositional and predicate logic explained.

There is however an argumentation possible as to why these two choices make sense. In Subsection 1.1.4 we saw that Western cultures, which also includes Flanders, give more value to formal logic and analytical reasoning. There is also an extensive history of logic in the West (think Aristotle, Frege, Russell, ...). For mathematics in particular, propositional and predicate logic are very powerful and used all around.[13] So including formal logic in the mathematics curriculum in Flanders, and focusing on propositional and predicate logic, makes sense.

2.1.3 Analysis based on Pedagogical Research

After our analysis of the principles of the curriculum using the theoretical perspectives of Section 1.1, we will now focus on the learning objectives themselves and use the pedagogical research, described in Section 1.2. We will try to stick to a similar structure of Section 1.2.

In the pedagogical research of mathematics educators there is a clear consensus that teaching logic can support mathematical thinking. The curriculum is aligned with this sentiment, as it includes logic. A more extensive argumentation for this inclusion possibly should have been included in the principles of the curriculum, but the sources of Section 1.2 already provide support.

Natural language

The link and difference with natural language is included in the curriculum. More specifically, in the core curriculum of the second stage the logic learning objective 6.20 mentions: "Meaning of negation, conjunction, disjunction, implication and equivalence in logic, including differences with the meaning of 'or' and 'if ... then ...' in everyday language"

[13]Think about group axioms, topology axioms, ZFC axioms, ...

in the conceptual knowledge Section. This is good news and is in accordance with the experimentally confirmed difficulties students have in mathematics.

Some improvements could be made to the natural language component. Firstly, it is odd that there is no "such as" construction in learning objective 6.20 of the second stage. There are many more differences with natural language than only 'or' and 'if ... then ...'. So the learning objective could have been phrased as "[...], including differences with natural language such as in the meaning of 'or' and 'if ... then ...'.

Secondly, the differences with natural language component should have been extended to come up in the entire logic learning trajectory, since this is such a difficult aspect for students in learning logic. In the current curriculum these differences are only included in the core curriculum of the second stage. Already in the first stage students see $\Rightarrow$ and $\Leftrightarrow$, so the difference with natural language is also possible there. Also in the third stage, when analyzing arguments in natural language, the difference between logic and natural language should be on the curriculum, since it will be so apparent when translating arguments in natural language into formal ones.[14]

Thirdly, the curriculum should be more clear, explicit and coherent when it is talking about logical language or natural language. In the core curriculum of the second and third stage the terms "logical statement" and "logical derivation" are used. This does not make clear if the learning objectives talk about statements/derivations in natural language or in logical language. The learning objective of the second stage makes this more clear by listing as procedural knowledge "Translating a statement in words to a statement in symbols". We will talk later about the third stage. In the logic learning objective for the advanced mathematics students, suddenly no "logical statements" are analyzed but only "mathematical statements". The attention towards natural language there disappears completely, while the study of quantifiers is an ideal situation to talk about differences between logical and natural language.

Visualization

Visualization is not strongly included in the curriculum. It is not explicitly mentioned in the principles. However, students do learn basic set theory in the first stage where Venn diagrams offer visual support and in the second stage, logic gates/circuits are discussed. These can be used as visual aid. You would expect the basic set theory of the first stage (with symbols $\cap$ and $\cup$) to be linked to symbols in propositional logic like $\wedge$ and $\vee$, but this is not included.[15] Venn diagrams and basic set theory could also be used to introduce quantifiers, but this is not suggested nor demanded. Even more, quantifiers as a concept are not even obligatory! This brings me to my next paragraph on the logic concept included in the curriculum.

[14] We will investigate later if the curriculum is talking about informal or formal arguments, as explained in Chapter 0.

[15] Technically, the link between these symbols should be done with predicate logic, since you need sets to range over with variables. In didactical practice this is sometimes ignored by using implicit variables. One can for example study "the pupils in the class" as a set and define sentences like $p =$"This pupil has blue eyes.".

Logic content

Let's also discuss the specific logic content elements in the curriculum. I will try to make connections to the list in Subsection 1.2.4 as much as possible. As we saw on that list, the implication (and in association the equivalence) in propositional logic are definitely the hardest operators for students. So, it is a good choice to start early on in the first stage with these operators, and then extend this in the second stage with other operators.

One thing that is quite weird in the curriculum if we compare with the list of difficulties in Subsection 1.2.4 is how the quantifiers are treated. Quantifiers are listed in the factual knowledge Section of the mathematics learning objectives of the core curriculum in the second and third stage. This means pupils should only know the symbol $\forall$ and $\exists$, but since quantifiers are never listed in the conceptual knowledge Section in the core curriculum, pupils do not have to learn the concepts that govern the symbols $\forall$ and $\exists$. This is quite odd, since many problems have been observed with quantifiers in experimental research, as is listed in Subsection 1.2.4. Also, many of the theorems and properties in secondary education use quantifiers. With this research in mind, it might have been a good idea to include basic concepts of predicate logic for everyone, not just for pupils who have an advanced mathematics component. In the package 'Advanced Mathematics' predicate logic is then in contrast studied in great depth. For example, pupils have to learn theorems about predicate logic when quantifier switching is allowed. The intended theorems are listed underneath.

$$\forall x : \forall y : P(x, y) \Leftrightarrow \forall y : \forall x : P(x, y)$$
$$\exists x : \exists y : P(x, y) \Leftrightarrow \exists y : \exists x : P(x, y)$$
$$\exists x : \forall y : P(x, y) \Rightarrow \forall y : \exists x : P(x, y)$$

The reader can judge for themselves if this level of predicate logic study is useful for mathematics reasoning.

Another element in the logic content that needs to be talked about, is the subject of the logic learning objective in the core curriculum of the third stage. In this learning objective "6.14: The students analyze logical derivations and reasonings.". There were several footnotes in the text so far that we need to discuss if formal or informal derivations are meant, as we saw in Chapter 0. Well, frankly, it is not super clear.

The learning objective needs to be realized both with and without a context, so you could conclude both formal and informal arguments need to be studied. Also in a show about the curriculum(Weyts et al., 2020b) an informal argument was described, then formalized, and then analyzed for validity. However, the principles and terminology in the logic curriculum never shows this. Even more, the procedural knowledge does not list anything about translating arguments in natural language to formal arguments, while the analogue for the second stage does. No differences between natural language and logical language are listed, in contrast to the analogue for the second stage. From these observations I conclude both formal and informal arguments should be studied, and that the curriculum should be more clear in this aspect.

Something else that is quite weird in this learning objective and the analogue in the second stage, is the division of concepts and competence levels between the two. On a surface level the transition from the second to the third stage makes sense. Students start by determining the truth value of individual statements in the second stage, and in the third stage they analyze arguments which consist of several individual statements. The first thing that is surprising is the shift in competence level. In the second stage students have to reach competence level "apply", which reflects the more mechanical view on logic there. In the third stage students have to reach competence level "analyze", which seems to suggest a less mechanical, more insightful role for logic. However, validity is never on the conceptual menu. This is however a fundamental concept in logic, and is essential for analyzing arguments in natural language. The term "validity" is only used in the procedural knowledge Section of the logic learning objective in the third stage, where it reads: "Determining the validity of a logical reasoning". If no conceptual knowledge of "validity" is needed to determine the validity of an argument, we can only assume there is a rather mechanical approach to analyzing arguments. I argued earlier both formal and informal arguments are meant. So it seems that analyzing an informal argument is quite mechanical, according to this learning objective. All the difficulties listed in Subsection 1.2.4 and mental models theory being the prevalent theory amongst psychologists for deductive reasoning (see Subsection 1.1.3) should make clear that judging the validity of an argument in natural language is not mechanical. My own example in Chapter 0 also shows this. Apart from that, validity as a concept would be helpful for seeing the "the coherence and structure of mathematics" as in the title of the abstraction building block.

Another point for these two learning objectives is that they are not consistent in the difference between semantically or syntactically analyzing validity, and the difference between an individual statement and an argument. For example, "Tautology, contradiction" is listed in the conceptual knowledge of the third stage. This is weird, because being a tautology or contradiction in propositional logic is a semantical property of an individual statement and is something you calculate with a truth table. Truth tables are the main focus in the second stage, so this concept would be more appropriate in the second stage. It would also offer a better transition from mostly calculating truth values in the second stage to non-mechanically analyzing complex natural language arguments in the third stage. Also, in the third stage pupils have to learn "Logical laws, inference rules, valid argumentation forms". The same confusion between statement and argument comes up again, since being a logical law is a semantical property of individual statements, while inference rules and valid argumentation forms are about arguments and more syntactical. It would make the curriculum smoother and more coherent by moving some of the suggested logical laws to the second stage.

The last point for this logic learning objective is that the listed inference rules could be improved according to the pedagogical literature. Including "Contraposition of an implication, proof by contradiction" was definitely a good choice, based on Subsection 1.2.4. It is quite odd to list Modus Tollens separately if Contraposition and Modus Ponens is already included, because these two inference rules immediately combine into Modus Tollens. Durand-Guerrier (2008) lists other inference rules that are especially important in mathematics, and relate to quantifiers. It's about inference rules that deal with the introduction/elimination of quantifiers when making a mathematical argument.

In conclusion, the logic core curriculum for the second and third stage shows there is a difference between a statement and an argument, between natural language and logical language and between semantics and syntax, but does not drive these differences all the way home. In my personal opinion, based on the arguments I gave above, including some basic predicate logic instead of the not so nicely implemented analysis of arguments, might have been more in line with the idea of the abstraction building block titled "Reasoning and abstracting taking into account the coherence and structure of mathematics" and with the idea of using logic as a supporting tool for mathematical reasoning.

Context

Let's focus now on the context. In Subsection 1.2.3 we saw that it is important to teach logic with a context, which was commonly based around natural language (riddles, puzzles, examples from literature). The general principles of the mathematics curriculum state: "A provision relating to the context was included in almost every learning objective. In learning objectives where the (use of) context is not specified, one can choose to realize them with or without context.". The principles of the first stage for the mathematical learning objective list some mathematical contexts where pupils can study mathematical reasoning and statements. Also in the second and third stage, the principles specifically for the mathematical learning objectives talk about the context by stating: "It is important to not cover this [mathematical] learning objective in isolation, but to address it in an integrated way with the other learning objectives of this stage, whenever those other learning objectives provide interesting properties, theorems, statements, . . . ". So in the principles the link with a (mathematical) context is clear. However, in the principles of the logical learning objectives the context is usually not mentioned.

Let us look in more detail at the context in the learning objectives of the abstraction building block. Table 2.2 underneath gives a clear overview. Keep the structure of Table 2.1 with the general content per learning objective in mind. I made a distinction between a (certain) context being **demanded** and being **suggested** in the curriculum.[16] The first two rows are the demanded part, the last two rows are the suggested part.

This Table tells quite a different story than the principles. We noticed that the principles said the mathematical learning objectives should be seen with a mathematical context, not in isolation, while they were silent about the logic learning objectives. However, in the actual learning objectives it is almost always demanded to realize the logic learning objectives with and without a context. Only for predicate logic for pupils with an advanced mathematics component it is not stated. There are also many different contexts suggested for the logic learning objectives.

For the mathematics learning objectives, the demanded context varies quite a bit. From the second stage onward either nothing is specified about the context or the context specified is simply the logic learning objectives. There are **many** suggested theorems and topics, but technically when realizing the learning objectives, you are allowed to only discuss one or two mathematical theorems per stage. In the second and third stage, it is technically allowed to teach pupils an abstract, formal course in logic, and finishing with discussing just one or two mathematical theorems or properties. Even more, the demanded context in the specialized Section of the third stage seems to point even stronger at the idea of a logic course with some mathematics in the end. The mathematical learning

[16] When a specific context is demanded, I understand it as the context also being suggested.

Context	Stage 1	Stage 2	Stage 3	Stage 3: Science	Stage 3: Economics	Stage 3: Advanced
Logic	with and without	with and without (and logic gates)	with and without	/	/	NA
Maths	with and without	NA	NA	with*	with*	with*
Logic	set theory	logic gates, natural language	rhetoric	/	/	NA
Maths	many suggested theorems and topics					

Table 2.2: This Table displays an overview of where context is demanded (first two rows) and suggested (last two rows) in the curriculum. Any specific contexts are also included. "/" means there is no learning objective for that cell. "NA" means there is a learning objective for that cell, but there is no suggestion/demand about the context present. According to the general principles of the mathematics curriculum, this means you can choose to realize this learning objective with or without a context. "with*" means that the context is all the logic learning objectives that apply to these pupils.

objective needs to be realized within the context of logic; it is not logic that needs to be realized in the context of mathematics.

This is very different from what the principles say and what the pedagogical literature written by mathematics educators recommends. There, the main idea is to see (some) logic while studying/learning mathematical reasoning. So to complete this idea, the demanded context of the logical learning objectives should be the mathematical learning objectives, and not the other way around.

IT

There are at least two uses IT can be used in teaching logic; it can be used as a didactical tool to support teaching and as a context where logic is applied. This second option clearly plays a role in the logic curriculum, since logic gates are mandatory as a context and since logic can play a part in the computational thinking aspect.

In Subsection 1.2.3 I briefly gave some examples of virtual tools that are currently being used, which are examples of the first option. Surprisingly, the option of IT as didactical support is not included in the logic curriculum, and even not allowed for realizing the learning objectives. In the general principles of the mathematics curriculum one can read the following.

> The use of technical aids offers added value and support in researching and/or solving problems. "Technical aids" include a set square, compass, ruler but also IT. "IT" is used as an all-encompassing name for digital tools such as a (graphing) calculator, a computer with a software package, a tablet, a smartphone,... It must always be decided in a thoughtful way when and which aid to use. The idea should be to use the aid to illustrate mathematical concepts and simplify calculations or other operations.

> In each element of procedural knowledge in the learning objectives, it is indicated whether or not it should be realized with IT. The following system is used for this:
>
> - No mention of IT use: this element must be realized without IT. However, the fact still holds that illustration with IT can provide added value in the pedagogical process.
> - Mentioning of "with and without IT": this element must be realized both with and without the use of IT.
> - Mentioning of "with functional use of IT": certain aspects of this element are easy to realize without IT, while other aspects cannot be efficiently calculated manually. The complexity of a specific test question may also play a role. IT should therefore consciously and effectively be used for every aspect of this element where IT is necessary.
>
> For calculations that are not explicitly mentioned in a procedural knowledge element, one is referred to learning objective 6.1 for IT use: feasible calculations are done without IT, complex calculations with IT.

Since there's no mention of IT use in any of the learning objectives of the abstraction building block, this means the learning objectives should be realized without IT. It is quite unfortunate that, because of the current label of IT use, all the various kinds of virtual tools for teaching logic listed in Subsection 1.2.3 are less attractive. However, even in the case of no IT mention, the principles note: "illustration with IT can provide added value in the pedagogical process." This still creates some odd situations, and a label of at least "with functional use of IT" in some places in the logic curriculum would have been more appropriate.

Consider the logic gates of the second stage for example. The curriculum allows illustration with IT, so the teacher can demonstrate logic gates with IT. They can use real-life electrical circuits or even Minecraft with virtual electrical circuits to do this. However, to realize the learning objective with its specified context of logic gates, it seems to not be allowed to let students interact themselves with these IT tools.

2.1.4 Final remarks

After this extensive analysis of the logic curriculum, based on the theoretical perspectives of Section 1.1 and the pedagogical research of Section 1.2, I would like to add some final remarks about the logic curriculum on a more personal note.

In the previous Subsections I have discussed several good aspects of the curriculum, but also many points of criticism. Often the principles are not very detailed or don't specify what theoretical stance is taken. Also the learning objectives themselves sometimes could have been improved, or should have been more supported by arguments. However, I want to put this criticism into perspective.

It is not surprising the logic curriculum is not super coherent or well documented. AHOVOKS submitted their proposals for the different parts of the mathematics curriculum to several development committees. These committees definitely had less than 3 years to analyze and modify the assigned proposal. Members of the committees were not fully appointed; they still had to work in their full-time job. The committees understandably did not have enough time to research every individual part (and thus also the logic curriculum) to the fullest detail. This thesis contains more documentation and research about the logic curriculum, which is not surprising as it was written in around 6 months by one full-time student.

Another possible reason for the slight incoherence of the logic curriculum is that there were separate committees, and each committee included a very varied set of people. This means the final product of their debates, which formed the current logic curriculum, is a jigsaw puzzle of all the different ideas and opinions represented. In conclusion, there are reasons for the shortcomings of the logic curriculum.

Despite the shortcomings of the curriculum being understandable, they can still have a big effect. Admittedly the curriculum contains **minimum** goals which technically does not limit teaching in classrooms. You are allowed to teach anything in the way you want it, so also content outside of the curriculum is possible. You are only required to realize the applicable learning objectives at some point.[17] However, this is just in theory. In practice the curriculum is very influential for the learning plans of school networks and for textbooks. The 'learning plans of school networks' are built on the curriculum. Stated in a simple way, the curriculum lists what needs to be learned, while the learning plans contain how different groups of schools are actually going to meet these objectives.

If we look at the learning plans for mathematics for second stage students with an orientation towards higher education, we can see how the logic core curriculum for the second stage was transformed into learning plans. The learning plans of private Catholic education (Katholiek Onderwijs Vlaanderen, 2021a,b,c,d,e) and the one of public education (GO!, 2021) for this subject and target group, will cover together more than half of the students (Agirdag et al., 2019). All these learning plans copied almost literally the learning objectives of the abstraction building block for the second stage. Since the learning plans are normally the go-to tool for designing lessons and textbooks, mathematics teachers and textbook authors will find very little extra pedagogical support in these documents for implementing logic. Not only is there little pedagogical support, by copy-pasting the curriculum the learning plans inherit the same problematic points the curriculum has. So despite the shortcomings of the curriculum being understandable and technically not binding teaching practices, it is quite unfortunate how they will in practice directly influence teaching.

2.2 The available teaching material

After the analysis of the curriculum in the previous Section 2.1, with a link to the most influential learning plans in Subsection 2.1.4, we will shift our focus to the available teaching material. We start with an overview and summary of the studied material in Subsection 2.2.1. This list is not exhaustive for all possible textbooks that mathematics teachers can use, but is an easily usable overview. We will also use the analysis of previous

[17]This is technically still a conditional and not absolute requirement. As a school you can choose to never realize the curriculum, but you won't receive governmental funding nor will you be able to award legally valid diplomas (Agirdag et al., 2019).

Chapters and Sections to assess the listed available teaching material, already partly in Subsection 2.2.1 and more thematically in Subsection 2.2.2. We won't go into too much detail, but will provide central points.

2.2.1 List and short analysis

For each source I provide a short summary, including some remarkable or interesting features. In this first list I include **all** kinds of teaching material I looked at, so for example also a website and a downloadable PDF count as teaching material. Later, I will mainly use the books from Flemish educational publishers, because they are more mainstream and most likely to be used next year by teachers (Goffin et al., 2016; Bellens et al., 2020).[18] These books will be summarized in a separate list and not included in the first list.

The process of finding appropriate teaching material for logic in the mathematics classroom followed its natural, chaotic path. There were recommendations from my promotor, previous Flemish master theses, an article+PDF in Uitwiskeling (a Flemish magazine for mathematics teachers), . . . One important criterion was that the teaching material is written in Dutch, that its target audience is pupils in secondary mathematics education, and that it is publicly available. This already limits the options quite strongly. Another thing to keep in mind, is that none of these sources are intentionally adjusted to the current curriculum, since they were all designed earlier than the approval of the curriculum. The order of the list is arbitrary.

- The first source (Milbou, 2013) is a textbook written in the context of a master thesis. This master thesis analyzed in 2013 if logic could be implemented in Flemish secondary mathematics education. For this purpose, the author designed her own textbook and performed an experiment to see if the conditional reasoning skills of the students improved. Her textbook (mostly studied by students on their own) achieved positive results, though the sample size was small. Also, the pupils stated that they didn't see the connection or use of logic in fields outside of mathematics after studying the textbook.

  The textbook is written for pupils with an advanced mathematics background, and serves as an independent logic textbook. It contains all the logic content a pupil with the 'Advanced Mathematics' package will ever need to see. It is in general written in a quite deductive way: "some examples in natural or mathematical language, definition and then (many) exercises at the end of the Chapter" is the general structure. Despite the deductive structure, there are interesting activating activities; for example when predicate logic is introduced by forcing the students to formalize a historical syllogism (a reasoning form categorized by Aristotle, which inherently uses quantification). However, there's an introductory Chapter that uses puzzles to introduce logic, and a Chapter that analyzes advanced semantics of a logical system with predicate logic with computer software. Logic gates are treated by using the graphing calculator. There's no link to set theory. Some historical notes are sprinkled around in the textbook where logicians are introduced or historical

[18] That is not to say they completely determine the teaching practice; a mathematics textbook/workbook can be used in many different ways.

problems are explained. Despite being developed before the current curriculum, this textbook can be a good starting point for designing your own textbook for logic in mathematics.

- Another source comes from another master thesis (Polé, 2016). The idea is similar in this master thesis: designing a textbook for logic and testing its merit. However, no experiment of deductive reasoning was performed, but the students filled in a survey and the secondary teachers involved were interviewed. This textbook was not super successful; it needed to be linked better with mathematical contexts and with everyday contexts. There was also a problem of the mathematics teacher involved not having the right technological background for using the textbook.

  The textbook was an adaptation of a programming course in higher education. In this course, and in this textbook, logic is understood in a very computational and technological way. Problems/puzzles are formalized in a logic-based programming language to solve them. Logic gates are not discussed. There's not really a link to set theory, natural language or history of logic. To achieve the best results, the target audience should be students with a computer science background.

- Yet another source comes from another master thesis (Bronkhorst, 2006). This master thesis was written in the Dutch context, where logic (in a rather different shape than now in Flanders) was introduced in the curriculum for students with a non-mathematical background. The master thesis contains a description of a teaching module with a written out workbook, with suggested exercises for homework. Both an experiment in logic performance and a survey was conducted. The teaching module (taught by the author with his own workbook) significantly improved the logic performance of students. The survey made clear that pupils liked this teaching module.

  The teaching module is intended for around 10 lessons and is built around exercises/games that are discussed with the whole class. There's a strong connection to natural language and philosophy (for example fallacies). Also historical anecdotes are interwoven with the text. It discusses the truth tables of $\Rightarrow$, $\wedge$, $\neg$ and $\vee$. There's no connection to IT or computational thinking; logic gates are not part of the text. Despite being developed in the Dutch context, this can be an interesting source for teaching logic to Flemish pupils who do not have a mathematical background.

- In 2011 a 26 page workbook for logic in mathematics appeared in Uitwiskeling (Roelens and Tytgat, 2011). There was no experiment conducted with this workbook.

  The target audience of this workbook is two-fold. The workbook consists of three Chapters: the first one explains the principles and theoretical assumptions, the second one introduces logical operators with their truth tables for a general audience and the third one studies mathematical theorems with a logical eye. The solutions to all the questions are written in italic in document, since this workbook is intended to be read by teachers. Throughout the entire second Chapter there is a strong link with natural language; almost every operator is discussed in contract with its natural language counterpart. The third Chapter is strongly linked to different mathematical contexts. In the workbook there's no link to set theory nor are there many historical anecdotes. There is no link to computational thinking and IT.

For the books by Flemish educational publishers, a more systematic approach was possible in deciding which ones to study. Several mainstream Flemish educational publishers have an agreement with the teachers program in KU Leuven to share their teaching material, so that future teachers can prepare for internships without spending money on textbooks and workbooks. The future teachers can access these materials online. I went through every educational publisher's website that had such an agreement, and checked if they published a new version of their mathematics books in the light of the new curriculum. There were only books available for grade 7-8-9 that were attuned to the new curriculum. Some educational publishers decided to allocate the logic content of the second stage to grade 10, but they didn't publish a new book yet. I will focus on grade 9 in the list underneath. In the end I was left with 7 mathematics books that contained logic, and which will be used from September onward in grade 9. Be aware that most of these books are only available in an experimental format, and will only be published in final form very close to the start of the school year. They also have not been experimentally validated. I will usually refer to the books with their title, and not with the name of the authors. More details about the books can be found online and I will also link to the correct websites.

When analyzing these books, I focused specifically on

- what elements of the logic curriculum are already contained in them.
  The curriculum for the second stage only specifies the learning objectives for the entire stage, and not in what grade these objectives have to addressed. Since only books for grade 9 were available, it is not possible to judge whether a textbook series will realize the learning objectives. However, I definitely argue against only dealing with logic in grade 10. The principles and structure of the curriculum clearly indicate that the abstraction building block should be intertwined with the other mathematical content of the stages.

- the link to mathematics.
  As said before, the logic content needs to be intertwined with the mathematical content per stage. Also one important point of advice in Subsection 1.2.3 was to use a context for teaching logic, and since the curriculum focuses on using logic as a tool for supporting mathematical reasoning, it makes sense to use mathematical contexts.

- the link to natural language.
  In Subsection 1.2.3, we saw that this point is very important when teaching logic in the mathematics classroom. The curriculum also, though in a rather limited way, demands to teach the difference between $\vee$ and "or", and $\Rightarrow$ and "if ... then ..." in the second stage. Possible questions for this point are: "Is there a link with natural language?", "Are more complex statements/arguments in natural language used?", "Is natural language treated before or after the logic definitions?", ...

- the link to set theory.
  We saw in Subsection 2.1.3, Venn diagrams can be used as a visualization tool, both for linking for example $\wedge$ and $\cap$, and for explaining quantifiers.

- the link to history.
  In the previous list of sources, it happened quite often that a historical setting was

used to introduce a certain logic concept, or that there were historical anecdotes in the text.

These points generated the following list of in total 7 books, published by Flemish educational publishers.

- "Matrix" is a mathematics workbook series by publisher Pelckmans. The new version of their workbooks for grade 9 is already online available. There's one book for pupils with 4-5 hours of mathematics per week (Tydtgat et al., 2021a) and one book for pupils with 5 hours of mathematics per week (Tydtgat et al., 2021b). In other words, they have one book for pupils with an orientation towards higher education who don't have a strong mathematical component in their study program, and one book for students who do have a strong mathematics focus. In each book logic takes up one Chapter in the beginning.

  In this Chapter all the different operators and truth tables are introduced in a rather deductive way. Every time, there are small, simple examples in natural language, and then the symbols are defined. Quantifiers are not introduced. The exercises stick to rather simple, artificial sentences. The implication gets the same amount of attention as other operators. There is not really a strong link with mathematics, set theory, history of logic or the differences with natural language. Only the obligatory differences with natural language, listed in the curriculum, are treated. After the initial Section on propositional logic, there's a Section that focuses purely on the technological aspect. Logic gates are discussed. Also complex electrical circuits and chains of logic gates are treated, in theory and with exercises. There's mainly a strong technological focus in this workbook.

- The advanced book of "Matrix" has the same two Sections as the basic book in the Chapter on logic, but there's an extra Section in the middle after introducing propositional logic. This middle Section links logic more intimately to mathematics and introduces the difference between an individual statement and an argument. The implication gets more attention by discussing necessary and sufficient conditions. This is rather odd since everyone has to see the difference between necessary and sufficient conditions, because it is included in the core curriculum. Some proof techniques are already introduced (though technically only required in the third stage), such as direct proof, proof by counterexample and proof by contradiction. Quantifiers are also not introduced in this Section. There are many exercises that treat simple mathematical statements. It is again quite odd that only pupils with a mathematical focus see this, because every pupil needs to see the connection between logic and mathematics in the second stage and needs to analyze arguments in the third stage. This book still has a strong technological focus, but also links with mathematical reasoning in the middle Section.

- "Van Basis Tot Limiet" (abbreviated as "VBTL") is a textbook series of publisher Die Keure. The current new version of their books can be found online. They plan on releasing a separate book with title "Logic and computational thinking", that contains at least all the logic content for the second stage (Bogaert et al., 2021). The target audience is not specified. There is a deductive structure in the book, apart from an introduction to logic based on syllogisms (a reasoning form categorized by

Aristotle). Each Section ends with several exercises. After the introduction, all the operators in propositional logic are defined after some small, simple examples. Then the focus shifts to logical laws, tautologies and contradictions and using them to prove logical statements are tautologies. After the more abstract Sections, there's a Section on logical riddles with many exercises where using a truth table can help solve a riddle, a Section on logic gates, and finally a Section on mathematical reasoning. In this last Section quantifiers are introduced and several proof techniques are listed, each time with one example of the proof technique. Necessary and sufficient conditions are in this last Section on mathematical reasoning. There's only a small link to set theory, in the Section with syllogisms. The obligatory differences with natural languages are treated, but apart from the riddles Section, natural language does not really play a role in this book. There are some historical anecdotes on logicians throughout the book. There is a very strong link to technology and computational thinking, since an entire Chapter on programming in Python was included after the Chapter on logic.

- "Nando" is another series by publisher Die Keure. Similarly to VBTL, they publish separate books per subject. There are two new workbooks for logic for grade 9, one for pupils with 4 hours of mathematics per week, and one for pupils with 5 hours. They can both be found here (Carreyn et al., 2021a,b). The books are identical. In the books it is hinted at logic being part of Nando for grade 10.

  The book starts with analyzing several statements in natural language in a few exercises, after which the symbols $\Rightarrow$, $\Leftrightarrow$ and $\forall$ are introduced. Truth tables are not handled in a systematic way yet; they explicitly say they only treat cases of $p \Rightarrow q$ where p is true. The implication gets a separate Section to show the difference between implication and equivalence with examples. After that, the other operators are introduced by considering a virtual game and pseudo code. So logic gates are used as a stepping stone for the operators, and there truth tables are given. In the end, there's a Section providing a clear link with set theory. Finally, exercises are included that reflect the character of the different Sections. In conclusion this book has a weak link with natural language, rather strong with set theory and a strong link with technology and computational thinking.

- "Level" is a workbook series by publisher Plantyn. This is the only mathematics book series of this publisher which will include logic for grade 9 and has new books online. An other big mathematics series of Plantyn called "Delta" will put logic in grade 10. There is one book for students with 4 hours of mathematics per week, and one book for students with 5 hours per week, which can both be found online here (Lev, 2021a,b). Those books contain all the mathematical content for that year, not just logic. They seem to contain the same parts for logic. It is harder to tell, because logic does not only get a separate Section in the beginning, but is also treated later.

  The book starts with a Chapter on the Pythagorean theorem which includes a Section on logic. This Section introduces the concept of logical statements and truth tables with limited attention to $\neg$. After this there is a lot of attention towards implication and contraposition. Truth tables are analyzed to make the upcoming definition of $\Rightarrow$ with a truth table more accessible and to construct the law of contraposition. However, cases of $p \Rightarrow q$ were p is false are discarded as unimportant for mathematical reasoning, while also using this case later for constructing the law of contraposition. The concept of implication and contraposition is then applied to several mathematical exercises related to the Pythagorean theorem. Much later, in the Section on properties of real numbers a, quantifiers are introduced with an interactive example based on properties of pupils in the classroom. In this introduction $\forall x : P(x)$ and $\exists x : P(x)$ are wrongfully assumed to be exclusive. The introduction is followed up with a proof that $\sqrt{2}$ is irrational, and with many exercises on quantifier notation, truth value of $p \Rightarrow q$ and simple proving exercises using counterexamples. It is fair to assume the other operators and logic gates will be treated in grade 10, since they are not included yet in Level for grade 9 and students have to see these topics in the second stage. There's some connection to set theory, when quantifiers are introduced. No link with history of logic is present. Natural language is used in the introduction, but is not very present. The link with mathematics is very strong, because of logic reappearing several times with several mathematical topics. There is no link to technology and computational thinking.

2.2.2 Thematic analysis

There are a few remarks I would want to make about the available teaching material listed in Subsection 2.2.1. Firstly, it varies quite strongly what introductions for logic and logic concepts are used. In the literature of Section 1.2 often riddles and puzzles, formulated in natural language, are used. Also in the materials in the first list of Subsection 2.2.1 (apart from (Polé, 2016)) mostly riddles or puzzles are used as an introduction. In the books by educational publishers, there's in general two options: several simple, artificial statements/arguments in natural language are presented to the student, where they have to decide on the truth value/validity of the statements/arguments. For example VBTL starts their book with a long list of syllogisms that need to be judged on validity. This is in line with the phrasing of the learning objective for logic in the second stage: "6.20: The students determine the truth value of logical statements.". The second option is introductions that are heavily based on technology and computational thinking. The best example is provided by Nando, where the logic operators are introduced from logic gates and pseudo code. This is the only book from the last list that uses logic gates as an introduction, and not as an application, which is quite remarkable since using it as an introduction was suggested by the principles of the logic curriculum in the second stage. In conclusion, the introductions used by the books of Flemish educational publishers are quite different from what can be found in the other available teaching material.

Secondly, a shortcoming of the curriculum comes up again in at least one of the workbooks studied. In the advanced book of Matrix, there is an extra Section about mathematical reasoning. In this Section the difference between individual statements and arguments is dealt with, by explaining intuitively and with examples what the difference is. For example it gives an example of Modus Tollens by saying: "In other words, $(\neg q \wedge (p \Rightarrow q)) \Rightarrow \neg p$ is a tautology. Written as as reasoning: $\neg q, p \Rightarrow q \rightarrow \neg p$." Separate symbols are introduced for arguments, for example $\rightarrow$ to draw a conclusion.

I claim the introduction of these separate symbols is not warranted in this context. The logic curriculum most likely has a role to play in the choice of the authors to introduce separate symbols, as the curriculum hints at a difference between statement and argument. The use of these new symbols is not a good idea, because in propositional logic it is not needed. As you can already see from the quote: any reasoning can be formulated as an individual statement, and vice versa. Patton Burgess (2009) also explains this.

Apart from that, the curriculum and many of the books in the last list in Subsection 2.2.1 do not make a strict difference between syntactically proving in argument form and semantically proving with individual statements. VBTL treats both of these kinds of proof, and uses $\Rightarrow$ for both.

Another reason I present for not introducing these specific symbols, is because then $\rightarrow$ is used in different ways in the textbook. In Matrix, $\rightarrow$ is both a symbol between statements (for example: $\neg q, p \Rightarrow q \rightarrow \neg p$) and a symbol between mathematical objects (for example: $f : \mathbb{R} \rightarrow \mathbb{R}$). This can create confusion for students. The use of these separate symbols is a specific example of teaching material where an aspect of the curriculum that is not perfected, has an influence.

Lastly, another not so good aspect of the curriculum shows in the books of the educational publishers. Recall that in Subsection 2.1.3 we saw that the logic curriculum suggests (mathematical) contexts throughout the second and third stage, but does not demand them. It leaves the option open to treat logic in a very abstract way, without context, of which we know it is a bad idea from Subsection 1.2.3.[19] For example, VBTL defines almost all of the logic content of second and third stage in the first half of their Chapter on logic with almost no context. After that, some Sections with context follow where logical riddles, logic gates and mathematical reasoning are discussed. The structure of this book does not align with the intention of the curriculum, but technically meets the requirements set by the learning objectives. It is another example of how the not so good part of the logic curriculum can have a very real impact on teaching.

[19] But be reminded that a (text)book does not completely determine the way logic is taught. Textbooks with less context can also simply be used as a summary of the lesson, or a tool to study for students.

Education, for most people, means
trying to lead the child to resemble the typical adult of his society.
But for me, education means making creators.
You have to make inventors, innovators, not conformists.

— Piaget, Jean

3
Designed Teaching material

After the literature review in Chapter 1 and the assessment of the curriculum and available teaching material in Chapter 2, we can discuss the designed teaching material that was generated in this thesis. Rather than designing, yet another, mathematics book for teaching logic,[1] I chose to design separate Sections for teaching logic. To choose what to design, I kept the question "Given the state of the curriculum and the most commonly used mathematics books, what can I design to improve logic teaching in this context?" in mind. The general idea of the teaching material is to **support** teaching logic in the second stage, since it is urgently needed for this stage as their curriculum starts taking effect in September 2021. However, there are also some Sections that focus on **innovation** for teaching logic.

In the end of the material, there are solutions for the exercises. An accompanying explanatory text for teachers that includes extra advice suggestions for using the teaching material, is also part of the material. The complete material (in Dutch) can be found in Appendix B; here I will describe thematically what the material contains while following its structure per Section.

In the first Section there is a list of differences (between predicate logic) and natural language. In the curriculum only two differences between logic and natural language are listed. Also the books from Flemish educational publishers only mention those differences. This contrasts strongly with the fact that the differences between natural language and logic was listed as one of the main reasons for difficulties in mathematical reasoning in Subsection 1.2.1. So in this Section, I made an extensive list of all the differences that I came across in all the sources I studied for this thesis. Teachers can use this list to discuss more differences explicitly in class. It is also useful as a tool to more quickly understand why a student is struggling with logic, since the problem is likely to originate from one of the differences.

[1] that almost surely would be outcompeted by the books of Flemish educational publishers

In the second Section I focus on the implication $\Rightarrow$ in propositional logic. According to Subsection 1.2.4 this is the hardest operator in propositional logic. It also plays a central role in many mathematical theorems and mathematical reasoning in general. The curriculum also introduces $\Rightarrow$ and $\Leftrightarrow$ already in the first stage to really train the use of and differences between these symbols. However, some books in Section 2.2 treat $\Rightarrow$ with the same level of attention as the other operators like $\wedge$ and $\vee$. This can definitely cause problems. There already plenty of exercises available that focus on calculating the truth value of statements involving an implication, so that's not what I focused on. Instead I compiled a list of arguments that can be used to make the truth table of $\Rightarrow$ more acceptable and less problematic. The idea of this Subsection is decidedly not that pupils are not intelligent or mathematically proficient enough to understand the truth table. From a philosophical and psychological perspective, it is not surprising that pupils do not want to accept the truth table of $\Rightarrow$, since this truth table has been criticized to not align well with conditional reasoning in everyday language. For example Stenning and van Lambalgen (2004) criticized the use of the $\Rightarrow$ to judge performance in psychological experiments on deductive reasoning. In Chapter 0 I presented some extensions of propositional logic, which can be used to better model the 'intuitive' meaning of the implication in mathematical reasoning. I briefly discussed this in Subsection 1.1.2. So rather than talking from up to down in the list of arguments for the truth table of the implication, the idea is to assume that the students **are** intelligent enough to understand the truth table, but that they, rightfully so, find it weird and unacceptable. These arguments for example apply to the need for synchronization between the truth table of $\Rightarrow$ and the other operators, so that handy logical laws in propositional logic hold (Heylen, 2019). They also explain that propositional logic is a **simplified model** of reasoning, not a full description.

In the second Subsection of the Section on $\Rightarrow$, the problem is flipped upside down.[2] Rather than convincing students to accept the truth table of $\Rightarrow$, this Subsection is intended for students who have already accepted this truth table, but are curious about the weird properties of $\Rightarrow$ in propositional logic. This Subsection is not written for teachers, but contains an exercise written for students with more explanation for teachers in the accompanying text in the end. Students examine for example the truth value and meaning of $(p \Rightarrow q) \vee (q \Rightarrow p)$.

The third Section contains two suggestions for teachers to use visualizations when teaching logic. In Subsection 1.2.3 we already saw there are many different options for visualizing logical structures. Also the books in Section 2.2 use some visualizations: electrical circuits, occasionally Venn diagrams, ... However, during my research I came across one visualization method I had never seen before in a didactical context, though it could be very valuable. I also came up with on visualization method that I have never seen anywhere, but that could also be useful for teachers. So that is why I included them in the Section, to possibly provide something innovative to teaching logic in mathematics education.

The first one is about constructing 'truth matrices' rather than truth tables. Basically, for a statement involving two propositional letters p and q, instead of organizing all the combinations of the truth values of p and q row after row, something else is possible. By

[2] The argumentation for the Section with advanced exercises later in this Chapter also applies here.

putting the truth values of p in the vertical dimension, and the truth values of q in the horizontal dimension, we get a 2D matrix that also displays all the combinations of the truth values of p and q. The benefit of this method, is that properties of matrices (for example: invariant under transposition) are linked with logical properties of the statement considered (for example: commutativity).

The second method is a visual tool for students to remember how to construct the contraposed statement of $p \Rightarrow q$. The statements $p, q, \neg p$ and $\neg q$ are organized in a square[3] and contraposition is interpreted as the action of 'mirroring'. This visual tool can come in handy, because there are already many actions you can do with $\Rightarrow$ like reversing the implication, negating the implication and taking the contraposition. The full explanation of both these visualization methods is in the teaching material.

Similarly to the third Section, the fourth Section contains innovative material for teaching logic. As can be expected, this fourth Section demanded much more effort than the other Sections. While analyzing the available teaching material in both lists of Subsection 2.2.1, I noticed a pattern that I also explained in Subsection 2.2.2. There were in general three categories for introductions to logic (concepts): riddles/puzzles, simple/artificial statements/reasonings and technology/computational thinking. The differences between predicate logic and natural language were in general treated as an anecdote. They were in general not used as a starting point. In other words, reflection on everyday reasoning was done after logic was introduced from a different angle. Personally, when reading these introductions, the question "But **why** do we **need** to introduce formal logic?" was not answered in my perspective. Riddles and puzzles are indeed easier to solve with truth tables (examples can be found in VBTL), but you could still solve them without truth tables. Also in the technological introductions, it is similar. Using logic to understand logic gates is an interesting connection, and use of logic, but a separate system for electrical gates could have been designed without the connection to reasoning. These two types show the use(s) of logic, but not its need. Something similar can be said about using simple/artificial statements/arguments as an introduction. Let it be clear I do not want to devalue these introductions, as they can very accessible, show the usefulness of logic, ...

I have tried to design an introduction for all the operators (quantifiers included) that more clearly lets students experience the **need** for formal logic. I started by making a list per operator of possible differences between logic and natural language that could cause a realistic communication/reasoning problem. For this I looked at the first Section of the teaching material. I intended to make a few dialogues/statements per operator that would cause discussion/confusion in the class of pupils (based on one of these differences). This would then be discussed in a classroom conversation, led by the teacher, who will push them towards finding out the reason of confusion. Experiencing there's a, so far inexplicable, problem with reasoning is a much better motivation for symbolic logic, than being told there are possible problems with reasoning in natural language versus logic. That was the idea. However, this list only contained many conceptual ideas on what the studied statements/arguments need to have as logical structure. It was no easy process to come up with appropriate, realistic, convincing statements/arguments in natural language that are comparable to a mathematical similar statements/argument and follow the

[3]This is inspired by the logical square of Aristotle for syllogisms (de Pater and Vergauwen, 1993).

needed logical structure. There are certainly improvements possible, and suggestions are also included in the accompanying text for teachers who want to further develop this Section. Be aware that none of the teaching material, including this Section, has undergone experimental testing because of the timing of my master thesis. After coming up with the statements/arguments I put everything together in a small text intended for pupils. The text also contains guiding question for the classroom discussion.

Lastly, there is a significant shortage in interesting, advanced logic exercises for pupils in secondary education. In most of the books the 'advanced' exercises simply study more complex logical statements, which is mostly 'advanced' in terms of calculations needed to solve the exercise. As we saw in Subsection 2.1.2, the logic curriculum also has a mechanical view on logic sometimes. That is why, in the fifth Section of the teaching material, I included three interesting, advanced exercises about logic. Each of them explores a theme that is not part of the logic content in the core curriculum. The first one introduces students to a logical system with three truth values, instead of just true and false. Students explore the appropriate new definitions for $\wedge, \vee$ and $\Rightarrow$ in this new system, using the truth matrices of the third Section. This exercise was also published in Uitwiskeling (a magazine for mathematics teachers in Flanders) (Eggermont et al., 2021). The second exercise investigates how you can change a logical statement, while keeping its truth table the same in propositional logic. The idea for this exercise originated from noting that the concept of tautologies and contradictions is in the logic curriculum, but two logical statements being equivalent is not. Two logical statements A and B are equivalent iff their truth tables are the same, or in other words, iff $A \Leftrightarrow B$ is a tautology. There is clearly a connection, and this exercise tries to answer that. The third exercise is based on online (Hofstede, 2021). It investigates how you can define one logical operator in term of other operators. The final conclusion of this exercise is that all the operators in propositional logic can actually be defined by one single operator!

"I refuse to prove that I exist," says God,
"for proof denies faith, and without faith I am nothing."
"But," says Man, "The Babel fish is a dead giveaway, isn't it?
It could not have evolved by chance. It proves you exist,
and so therefore, by your own arguments, you don't. QED."
"Oh dear," says God, "I hadn't thought of that,"
and promptly vanishes in a puff of logic.

— Adams, Douglas

Conclusion and Discussion

Let's conclude this thesis by summarizing what we have achieved with our research, and what could have been improved.

We started of with an extensive literature review in Chapter 1. There was both a focus on theoretical perspectives from mathematics, philosophy, psychology and cultural psychology on logic, and on applied pedagogical research. In Section 1.2 I listed several factors that have been shown to influence the logic performance, and also summarized the body of research into points of advice for teaching logic. This Chapter was useful within this thesis as a general framework. However, it is also a concise overview for the readers interested in research about the teaching of logic.

Chapter 2 is second in this thesis. This Chapter lays out the analysis/assessment of the curriculum and available teaching material based on Chapter 1. It goes into much detail for analyzing the curriculum, and is more concise when it comes to the teaching material. This analysis was useful for deciding what teaching material to design. However, this Chapter can also be considered part of the creative component of this thesis, not just exposition of literature, as such an extensive analysis of specifically new logic curriculum for Flemish secondary mathematics education had never been done before. The points of improvement of the curriculum can be used in the future[4] to potentially modify the curriculum or can be used by teachers as an interesting background for their teaching. Also the analysis of available teaching material, specifically of books by Flemish educational publishers, has an external use: mathematics teachers can use the concise description to make an informed decision on what book to use in their teaching.

Taking the results of Chapter 1 and Chapter 2 I developed extra teaching material for logic in the given context which is described in Chapter 3. Some parts were written with the intention of supporting mathematics teachers next year. With the explanation in the

[4]The current curriculum cannot be changed anymore before the part for the second stage starts taking effect on September first. Private Catholic education has started a court case to change the current curriculum, but the constitutional court decided to not abolish the current curriculum for now. So the current curriculum will definitely take effect in September 2021 (AHOVOKS, 2021).

accompanying text in the teaching material, teachers can use these parts without extra hassle in their lessons. There are also two parts that were written with the intention of innovation. They could also be used by teachers, but need to be treated with caution as they have not been experimentally tested. They can definitely provide inspiration for other teaching material and for future research.

Apart from discussing what was achieved and how it is useful, I would also like to provide some finishing remarks on how this thesis could have been improved. Firstly, more attention towards a computer science perspective on logic would have been useful. When analyzing the curriculum and the available teaching material, it became clear the view on logic was more inspired by technology and computer science, rather than natural versus mathematical language. An extra Subsection in Section 1.1 with a theoretical perspective on logic would have come in handy. Also in Section 1.2 more pedagogical sources that use logic within a computer science setting, would have been helpful to better understand this view.

However, the current analysis of the curriculum was already detailed enough to provide inspiration for what teaching material to design. Also, it was not really possible to know about this presence of computer science in the logic curriculum, before actually performing the analysis. In the start of this thesis a choice had to be made in what theoretical perspectives to discuss, as there are many possibilities. The current perspectives were chosen based on how different they were from the common mathematical view on logic, to get a wider picture as I am a mathematics student. A perspective from computer science was expected to more similar to a mathematical view than the currently chosen perspectives, which is why I did not include it.

A second point that is open for improvement is the difference between the analysis of the available teaching material in Section 2.2 and the analysis of the curriculum in Section 2.1. The analysis of the curriculum is more detailed and in depth than the analysis of the available teaching material. Providing a more elaborated analysis and assessment of the teaching material would be even more useful to mathematics teachers.

However, the current intensity of the analysis and assessment in Chapter 2 already provided plenty of ideas on what teaching material to develop, and is thus from this perspective, detailed enough for internal use in the thesis.

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A
Full translation of logic curriculum in mathematics

In this appendix I provide the full translation of all learning objectives and principles related to the building block "Reasoning and abstracting taking into account the coherence and structure of mathematics". This includes in total 10 learning objectives, 2 for the first stage, 2 for the second stage, 2 for the third stage and 4 in the specific learning objectives for the third stage. These last 4 objectives are only relevant for some pupils, depending on their study program. Each stage here also starts with a translation of the stage-specific principles of the learning objectives of this building block. To start, I provide a translation of the general principles of the whole mathematics curriculum, followed by the general principles of this building block. The original Dutch version of all the learning objectives can be found online.

A.1 General principles

A.1.1 General principles in mathematics curriculum

The idea that mathematics is abstract and formal and that it is actually separate from reality, is correct to some extent. Mathematics education revolves around the meaningful development and construction of mathematical knowledge and reasoning. This requires a pedagogical approach that pays sufficient attention to finding meaning in the more abstract mathematical concepts. This is why, when interpreting the learning objectives, one needs to not only take mathematics as a subject into account but also the pupil and the society in which that pupil will function. The mathematical contents should therefore preferably be applied in various situations. A provision relating to the context was included in almost every learning objective. In learning objectives where the (use of) context is not specified, one can choose to realize them with or without context.

A number of subjects in the mathematics curriculum are introduced that will not be completed immediately in their entirety. Some of these components will be treated again at a higher level in later years. Knowledge acquired in previous years is also further developed.

When reading the building blocks, one should keep in mind that communication in the mathematics classroom needs use mathematical language as much as possible, which consists of a basic set of instruments of everyday language, subject-specific terminology, definitions, symbols and representations. The pupil is at least expected to be able to actively use the concepts, symbols, standard forms and formulas listed under factual knowledge. The letters used to indicate coefficients, variables, unknowns or parameters can be chosen freely, only the form is fixed. For example, the standard form for a first-degree function can be denoted as $f(x) = ax + b$ and just as well as $f(x) = mx + q$.

The use of technical aids offers added value and support in researching and/or solving problems. "Technical aids" include a set square, compass, ruler but also IT. "IT" is used as an all-encompassing name for digital tools such as a (graphing) calculator, a computer with a software package, a tablet, a smartphone,. . . It must always be decided in a thoughtful way when and which aid to use. The idea should be to use the aid to illustrate mathematical concepts and simplify calculations or other operations.

In each element of procedural knowledge in the learning objectives, it is indicated whether or not it should be realized with IT. The following system is used for this:

- No mention of IT use: this element must be realized without IT. However, the fact still holds that illustration with IT can provide added value in the pedagogical process.

- Mentioning of "with and without IT": this element must be realized both with and without the use of IT.

- Mentioning of "with functional use of IT": certain aspects of this element are easy to realize without IT, while other aspects cannot be efficiently calculated manually. The complexity of a specific test question may also play a role. IT should therefore consciously and effectively be used for every aspect of this element where IT is necessary.

For calculations that are not explicitly mentioned in a procedural knowledge element, one is referred to learning objective 6.1 for IT use: feasible calculations are done without IT, complex calculations with IT.

A.1.2 Principles in 'Reasoning and abstracting taking into account the coherence and structure of mathematics'

Mathematical insight is developed and problem solving skills are promoted by mathematical reasoning through formulation of a conjecture, argumentation, explanation, proof, generalization, structuring, ordering, working analogously and/or synthesizing. It is about more than just writing down a proof from rote memory. In this way the coherence can be overseen, apart from the fragmentary.

A number of mathematical thinking methods must be acquired. It is important that students are obliged to reflect on the thinking process. If necessary, adjustments can be made and if progress remains absent, the causes have to be sought after.

A.2 First stage

A.2.1 Principles for the first stage

In the first stage of secondary education we start reasoning and abstracting taking into account the coherence and structure of mathematics with learning objective 6.17. In this learning objective we make a small start in logic: students must know the distinction between an 'if-then' relation (implication) and an 'if-and-only-if' relation (equivalence). Using the correct arrows with these relations is important. It is more important here that the students can use the relations than that they can explain in words what they mean. In addition, students see for the first time how mathematical knowledge can be acquired by constructing simple proofs. The proofs that will be covered, relate to the other mathematics learning objectives in the first stage. In the building block 'Developing insight into and managing space and shape: geometry', these proofs relate to congruence (proving that two triangles are congruent), properties about transformations, triangles and quadrilaterals (for example proving that the diagonals in a square have the same length) and in the building block 'Developing insight into and managing relation and change: such as algebra, calculus and discrete structures' deals with the special products (from product to polynomial) and some simple number theory proofs (for example the square of an even number is even). It is of course useful to set up a learning trajectory in this regard and to address this learning objective at different times if the proofs considered correspond to other learning objectives in the curriculum.

There is also a small portion of elementary set theory in the curriculum (learning objective 6.18). Students determine whether an element belongs to a particular set or not, they can consider subsets of sets, and determine the intersection, union and difference of two sets. Using the correct symbols associated to these operations is also important. This learning objective is more about performing these operations on (merely two) simple sets (for example the even numbers less than 10), than about performing the operations on the number sets known up to that point ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}$). Giving meaning to $\mathbb{Z} \cup \mathbb{N}$ for example is not a requirement.

A.2.2 Learning Objectives

6.17: The students give a mathematical reasoning or an argumentation for mathematical properties.
Including knowledge:

- Factual knowledge:
  - Symbols: $\Rightarrow, \Leftrightarrow$
- Conceptual knowledge:
  - 'if-then' relation (implication), equivalence
  - Mathematical properties mentioned in the learning objectives of stage 1 A-stream such as congruence properties for triangles, properties of transformations, properties of triangles and quadrilaterals, special products $(a+b)^2$ en $(a+b)(a-b)$, simple and concrete number theory proofs

Including context:

- The learning objective is realized both with and without context.

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "evaluate"

6.18: The students perform operations on two sets.
Including knowledge:

- Factual knowledge:
  - Symbols: $\cap, \cup, \setminus, \in, \notin, \subset, \not\subset$
- Conceptual knowledge:
  - Element, subset, intersection, union, difference
- Procedural knowledge:
  - Element, subset, intersection, union, difference

Including context:

- The learning objective is realized both with and without context.

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "apply"

A.3 Second stage

A.3.1 Principles for the second stage

The learning trajectory for logic, which was started in a limited way in the first stage, is further expanded with an introduction to propositional logic (learning objective 6.20). Using truth tables, students learn how to determine whether compound logical statements are always true or not. The students are introduced to the logical connectives and their meaning. It's important to emphasize the difference with the corresponding expressions in everyday language. The logic gates, which were already covered in the technology learning objectives in the first stage, are a nice stepping stone for this subject.

In addition, the students' reasoning skills are further developed by paying attention to providing arguments for mathematical reasonings and statements (learning objective 6.21). As the description of this building block and the procedural knowledge for this learning objective clearly indicate, it is not just about simply writing down a proof. By covering the various elements of procedural knowledge to a greater or lesser extent, it is possible to differentiate in teaching based on the students' profile. The mathematical statements, reasonings and proofs that are the subject of this learning objective relate to the other mathematics learning objectives of the second stage. A number of suggestions were therefore included in the learning objective. It is important to not cover this learning objective in isolation, but to address it in an integrated way with the other learning objectives of this stage, whenever those other learning objectives provide interesting properties, theorems, statements, ...

A.3.2 Learning Objectives

6.20: The students determine the truth value of logical statements.
Including knowledge:

- Factual knowledge:
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$
- Conceptual knowledge:
  - Logical statement
  - Truth value
  - Meaning of negation, conjunction, disjunction, implication and equivalence in logic, including differences with the meaning of 'or' and 'if ... then ... ' in everyday language
- Procedural knowledge:
  - Constructing a truth table
  - Translating a statement in words to a statement in symbols

Including context:

- The learning objective is realized both with and without context.
- At least the following context is covered: logic gates.

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "apply"

6.21: The students provide arguments for mathematical reasonings and statements
Including knowledge:

- Factual knowledge:
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$
- Conceptual knowledge:
  - Implication, equivalence
  - Necessary and sufficient condition
  - Logic from the learning objectives in the second stage
  - Mathematical properties from the mathematics learning objectives in the second stage such as similarity properties of triangles, the theorem of Pythagoras, the theorem of Thales, relative orientation of lines and planes in space, irrationality of $\sqrt{2}$, properties of graphs (from graph theory)
- Procedural knowledge:
  - Exemplifying a statement

  - Verifying the correctness of a mathematical statement
    * Constructing a simple mathematical reasoning
    * Disproving a statement using a counterexample
  - Providing arguments for reasoning steps in a given mathematical reasoning
  - Reconstructing covered proofs in an altered situation with for example different symbols, in a specific case

Including dimensions of the final learning objective:

- Cognitive dimension: competence level “evaluate”

A.4 Third stage

A.4.1 Principles for the third stage

Propositional logic from the second stage is further expanded. Students in that stage determined the truth value of a logical statement; now in the third stage they analyze a logical derivation or reasoning, in other words they investigate a sequence of logical statements (learning objective 6.14). Students explain the steps in the reasoning and determine whether the reasoning or derivation is valid. In order to do that, they are introduced to various logical laws and logical inference rules. Attention to frequently occurring fallacies is inextricably linked to this. For certain study programmes, the link with argumentation theory is immediately clear.

In addition, the students’ reasoning skills are further developed by paying attention to providing arguments for mathematical reasonings and statements (learning objective 6.15). As the description of this building block and the procedural knowledge for this learning objective clearly indicate, it is not only about simply writing down a proof. By covering the various elements of procedural knowledge to a greater or lesser extent, it is possible to differentiate in teaching based on the students’ profile. The mathematical statements, reasonings and proofs that are the subject of this learning objective relate to the other mathematics learning objectives of the third stage. A number of suggestions were therefore included in the learning objective. It is important to not cover this learning objective in isolation, but to address it in an integrated way with the other learning objectives of this stage, whenever those other learning objectives provide interesting properties, theorems, statements, . . .

A.4.2 Learning Objectives

6.14: The students analyze logical derivations and reasonings.
Including knowledge:

- Factual knowledge:
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$
- Conceptual knowledge:
  - Logical statement

- Truth value
- Tautology, contradiction
- Meaning of negation, conjunction, disjunction, implication and equivalence in logic
- Necessary and sufficient condition
- Logical laws, inference rules, valid argumentation forms such as
  - The law of double negation
  - Contraposition of an implication, proof by contradiction
  - De Morgan's laws
  - The law of excluded middle
  - Equivalence as conjunction or disjunction
  - Distributivity, commutativity, associativity
  - Modus ponens
  - Modus tollens
- Fallacies such as wrongly inverting an implication, confusing an implication with an equivalence

• Procedural knowledge:
  - Determining the validity of a logical reasoning
  - Explaining a step in a logical reasoning

Including context:

• The learning objective is realized both with and without context.

Including dimensions of the final learning objective:

• Cognitive dimension: competence level "analyze"

6.15: The students provide arguments for mathematical reasonings and statements.
Including knowledge:

• Factual knowledge:
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$

• Conceptual knowledge:
  - Implication, equivalence
  - Necessary and sufficient condition
  - Logic from the learning objectives in the third stage
  - Mathematical properties in the other mathematics learning objectives in the third stage such as derivative functions of $f(x) = x$ and $f(x) = x^2$, properties of sequences

- Procedural knowledge:
  - Exemplifying a statement
  - Verifying the correctness of a mathematical statement
    * Constructing a simple mathematical reasoning
    * Disproving a statement using a counterexample
  - Providing arguments for reasoning steps in a given mathematical reasoning
  - Reconstructing covered proofs in an altered situation with for example different symbols, in a specific case

Including dimensions of the final learning objective:

- Cognitive dimension: competence level “evaluate”

A.5 Third stage: specific learning objectives

There are also principles online for these specific learning objectives, but they were not included in this translation since they provided little extra information compared to the actual learning objectives.

A.5.1 Specific learning objectives: extended math in service of science

6.2.12: The students provide arguments for mathematical reasonings and statements.
Including knowledge:

- Factual knowledge:
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$
- Conceptual knowledge:
  - Implication, equivalence
  - Necessary and sufficient condition
  - Concepts from logic
  - Mathematical properties, calculation rules and formulas in the specific learning objectives 6.2.1 up to and including 6.2.11, such as properties of matrix operations, derivatives of elementary functions, properties in calculus, de Moivre’s formula, trigonometric identities
- Procedural knowledge:
  - Exemplifying a statement
  - Verifying the correctness of a mathematical statement
    * Constructing a simple mathematical reasoning

        - Disproving a statement using a counterexample
    - Providing arguments for reasoning steps in a given mathematical reasoning
    - Reconstructing covered proofs in an altered situation with for example different symbols, in a specific case

Including context:

- The specific learning objective is realized using knowledge elements related to logic from the learning objectives in the second and third stage.

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "evaluate"

A.5.2 Specific learning objectives: extended mathematics in service of economics

6.3.10: The students provide arguments for mathematical reasonings and statements.

Including knowledge:

- Factual knowledge:
    - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$
- Conceptual knowledge:
    - Implication, equivalence
    - Necessary and sufficient condition
    - Concepts from logic
    - Mathematical properties, calculation rules and formulas in the specific learning objectives 6.3.1 up to and including 6.3.9, such as properties of matrix operations, derivatives of elementary functions, properties in calculus
- Procedural knowledge:
    - Exemplifying a statement
    - Verifying the correctness of a mathematical statement
        - Constructing a simple mathematical reasoning
        - Disproving a statement using a counterexample
    - Providing arguments for reasoning steps in a given mathematical reasoning

Including context:

- The specific learning objective is realized using knowledge elements related to logic from the learning objectives in the second and third stage.

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "evaluate"

A.5.3 Specific learning objectives: advanced mathematics

6.4.15: The students examine mathematical statements using predicate logic.
Including knowledge:

- Factual knowledge:
  - Terminology intrinsic to the delineation to this specific learning objective
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$
- Conceptual knowledge:
  - Argument, predicate, quantifier
  - Quantified logical statement
  - Negation of a quantified statement
    - Meaning of the negation
    - Negation theorems
  - Reversing the order of quantifiers in a given statement
    - Difference in meaning
    - Theorems about reversing the order of quantifiers
- Procedural knowledge:
  - Translating mathematical statements in words to quantified logical statements and vice versa
  - Determining the truth value of a given statement involving quantifiers
  - Determining if the order of quantifiers is allowed to be reversed in a given statement
  - Constructing and simplifying the negation of a given statement

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "analyze"

6.4.16: The students prove mathematical statements.
Including knowledge:

- Factual knowledge:
  - Terminology intrinsic to the delineation to this specific learning objective
  - Symbols: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$
- Conceptual knowledge:
  - Implication, equivalence
  - Necessary and sufficient condition
  - Concepts from logic

  - Axiom
  - Proof techniques: direct proof, proof by induction, proof by cases, proof by contradiction, proof by counterexample
  - Mathematical properties, calculation rules and formulas in the specific learning objectives 6.4.1 up to and including 6.4.15, such as derivatives of elementary functions, connection between continuity and differentiability, fundamental theorem of calculus for continuous functions, de Moivre's formula, properties of relative orientation of lines and planes in space, properties of matrix operations, binomium of Newton, properties of summations of sequences, trigonometric formulas and identities, convergence of a sequence to a certain limit using the $\epsilon - N$ definition, uniqueness of the neutral element in a group, uniqueness of the inverse of an element in a group

- Procedural knowledge:
  - Reconstructing covered proofs
    - In the covered situation, combined with providing arguments for the reasoning steps
    - In an altered situation with for example different symbols or in a specific case
  - Constructing proofs that weren't covered

Including context:

- The specific learning objective is realized using knowledge elements related to logic from the learning objectives in the second and third stage and the specific learning objective 6.4.15.

Including dimensions of the final learning objective:

- Cognitive dimension: competence level "create"

B
Designed Teaching Material

Extra Lesmateriaal Logica

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Inleiding

Dit document bevat ondersteunend lesmateriaal voor logica in de wiskundeles. Het doelpubliek en de moeilijkheidsgraad variëren en worden aangegeven in de begeleidende tekst in Sectie 6. Dit materiaal werd ontwikkeld in het kader van mijn Masterthesis. Deze pdf kan u terugvinden op de persoonlijke website van mijn promotor. Originele LaTeX-bestanden kunnen verkregen worden door te mailen naar alexanderholvoet23@gmail.com.

Inhoudsopgave

1 Lijst van verschillen met omgangstaal

In deze Sectie is een lijst van verschillen tussen de alledaagse omgangstaal en de formele logicataal opgenomen. We ordenen de verschillen volgens de connectieven in de predicatenlogica.

- $\neg$ De logische negatie lijkt op het eerste zicht redelijk gelijkaardig aan de negatie in de omgangstaal. Er zijn veel problemen met hoe de negatie zich verhoudt tot andere connectieven (negatie van een implicatie, negatie van kwantoren, ... zijn moeilijk), maar deze lijken niet onmiddellijk te liggen aan een verschil in gebruik qua taal. Wat wel een verschil is, is dat in de omgangstaal veel zaken niet zwart-wit zijn, terwijl dat in propositielogica wel zo is. In propositielogica is $\neg\neg p$ equivalent met p, maar “Ik ben niet kwaad.” betekent niet hetzelfde als “Ik ben blij.”

- $\vee$ Het belangrijkste verschil met de omgangstaal is het exclusief, dan wel inclusief zijn. In propositielogica is $p \vee q$ waar wanneer minstens één van de componenten waar is, dus ook als p en q beiden waar zijn. Dit noemen we de inclusieve of (ook wel *disjunctie* genaamd). Daartegenover staat de exclusieve “of” die waar is als exact één van de componenten waar is. In de omgangstaal wordt vaak “of” voor de exclusieve of gebruikt, maar soms ook voor de inclusieve. Bijvoorbeeld in “Ze heeft goed gestudeerd; ze heeft goed gestudeerd of gemakkelijke vragen gekregen.” wordt de inclusieve of gebruikt. In “Wil je koffie of thee?” wordt een exclusieve of gebruikt.

- $\wedge$ Het meestgenoemde verschil met de omgangstaal voor dit connectief is de symmetrie. De logische en (ook wel *conjunctie* genaamd) is symmetrisch, dus $p \wedge q$ is equivalent met $q \wedge p$. In de omgangstaal kan er echter een tijdsverschil zitten tussen de twee componenten. Denk aan bijvoorbeeld “Ik zal studeren en een aflevering van mijn favoriete serie bekijken.”, wat volgens de ouders van deze leerling niet overeen komt met “Ik zal een aflevering van mijn favoriete serie kijken en studeren.”. Naast een verschil in tijdsvolgorde kan er ook een verschil in belangrijkheid zijn; het belangrijkste wordt eerst vermeld.

  Een minder voorkomend maar wel interessant verschil is dat “en” soms voor inclusieve “of” gebruikt wordt. Bijvoorbeeld met “Ik bracht de groene en de rode ballen mee.” wordt “Ik bracht de ballen mee die groen of rood zijn.” bedoeld.

Een laatste verschil dat vrij subtiel is, is dat in het Nederlands “en” ook gebruikt wordt tussen objecten, en niet enkel tussen uitspraken. Zowel “ Alice en ik zijn slim.” als “Alice is slim en ik ben slim.” zijn aanvaardbare uitspraken in het Nederlands met de zelfde betekenis. In predicatenlogica kan een $\wedge$ tussen twee objecten echter niet, en is dus eigenlijk enkel de tweede zin uit te drukken. In het voorbeeld geeft dat niet echt een probleem, aangezien “slim zijn” een eigenschap is die doelt op zowel “Alice” als “ik” afzonderlijk. Bij de volgende zin is het echter moeilijker: “Alice en ik hebben veel knikkers.”. Die zin is niet hetzelfde als “Alice heeft veel knikkers en ik heb veel knikkers.”.

$\Rightarrow$ Dit connectief is vaak het moeilijkst voor de leerlingen, en daarmee gepaard ook het meest frappant verschillend met de omgangstaal. In de omgangstaal wordt met de “als …, dan … ” soms een equivalentie, soms enkel een implicatie uitgedrukt. Dit is duidelijk door de context. De zin “Als je goed studeert, dan krijg je een cadeau.” bevat in principe enkel een implicatie, maar iedereen zal in die situatie de zin begrijpen als equivalentie: “Als je goed studeert, en enkel dan, krijg je een cadeau.”. In de wiskunde en logica zijn deze twee echter strikt gescheiden, en wordt met $\Rightarrow$ altijd enkel een implicatie bedoeld.

Naast het wisselen tussen implicatie en equivalentie is er nog een ander belangrijk verschil tussen de omgangstaal en de implicatie in propositielogica. Propositielogica kijkt enkel naar de waarheid van de deeluitspraken, en niet naar de inhoudelijke verbanden. Een zin als “Als de maan gemaakt is van kaas, dan ben ik de beste wiskundige ter wereld.” is waar volgens de propositielogica. In de omgangstaal verwachten/eisen we een inhoudelijk verband (en dan nog liefst een oorzakelijk verband) tussen de aanname en de conclusie van een implicatie. In Sectie 2 gaan we verder in op deze verschillen.

$\forall, \exists$ De betekenis van kwantoren is niet zo strikt in de omgangstaal als in de wiskunde/logica. Als we in logica $\forall$ schrijven, dan bedoelen we effectief **alle** gevallen. Gelijkaardig bedoelen we met $\exists$ minstens **één** geval (wat heel weinig kan zijn in vergelijking met de volledige verzameling). In de omgangstaal is deze betekenis losser ingevuld. Bij gebruik van “alle” of “iedere” kan men “de meeste” of “elk geval dat ik individueel controleerde” bedoelen. Gelijkaardig betekent “sommige” of “er zijn … ” vaak dat er niet slechts één, maar wel een redelijk deel van de volledige verzameling voldoet. Als je naar het strand gaat, en zegt dat “alle” plaatsen bezet zijn, dan bedoel je hoogstwaarschijnlijk dat de meeste plaatsen bezet zijn en het moeilijk is om zelf een plaatsje te vinden. Je bedoelt niet dat **elke** mogelijke plaats bezet is. Gelijkaardig betekent de uitspraak “Sommige mensen waren kleurrijk gekleed.”, afhankelijk van de context, waarschijnlijk dat er meer dan één persoon kleurrijk gekleed was.

Het lidwoord “een” kan zowel gebruikt worden voor $\exists$ als voor $\forall$. Vergelijk hiervoor “Een mens heeft wel eens ontspanning nodig.” met “Een leerling van mij is op reis gegaan naar Ierland.”.

Naast dat verschil in hoe strikt de betekenis is, wordt er ook losser omgesprongen met het verschil in betekenis bij een andere volgorde van kwantoren, en is het opnieuw vaak duidelijk uit de context. In wiskunde betekent $\forall x \in \mathbb{N} : \exists y \in \mathbb{N} : x < y$ iets helemaal anders dan $\exists y \in \mathbb{N} : \forall x \in \mathbb{N} : x < y$. In de omgangstaal zegt het spreekwoord “Op ieder potje past een dekseltje.” dat iedereen iemand kan vinden die bij hem/haar past. Mocht je echter “Er is een passend dekseltje voor ieder potje.” zeggen, dan zou nog steeds iedereen hetzelfde verstaan. Logisch gezien beweert deze laatste uitspraak echter dat er één magisch dekseltje is dat op iedere pot past, aangezien de volgorde van kwantoren wordt omgedraaid.

Tot slot is er iets dat zowel in de wiskunde als in de omgangstaal iets anders aangepakt wordt dan in de logica. De logische uitspraak $(P(x) \Rightarrow Q(x))$ heeft technisch gezien geen waarheidswaarde (waarbij P en Q twee predicaten zijn), aangezien x nog niet ingevuld is of gebonden is aan een kwantor. In de omgangstaal en in de wiskunde wordt met dergelijke uitspraken vaak $\forall x : P(x) \Rightarrow Q(x)$ bedoeld, wat wel een waarheidswaarde heeft, aangezien x gebonden is aan de kwantor. Het verschil/probleem komt boven, wanneer $(P(x) \Rightarrow Q(x))$ waar is voor sommige x, maar niet voor allemaal. Neem bijvoorbeeld $P(x) =$“x is deelbaar door 2” en $Q(x) =$“x is deelbaar door 6”. Dan wordt $(P(x) \Rightarrow Q(x))$ gelijk aan “Als x deelbaar is door 2, dan is x deelbaar door 6.”. Logici zouden zeggen dat die uitspraak geen waarheidswaarde heeft. Wiskundigen zouden beweren dat die uitspraak vals is, aangezien er gevallen zijn voor x waarbij $(P(x) \Rightarrow Q(x))$ vals is. Ze lezen het dus als $\forall x : P(x) \Rightarrow Q(x)$. Leerlingen zouden zeggen dat de waarheidswaarde afhangt van x, wat eigenlijk dichter aansluit bij de logici.

2 Over de implicatie in propositielogica

We bespraken in Sectie 1 reeds enkele verschillen tussen logica en omgangstaal. Dit geeft aanleiding tot problemen met de implicatie, en onder andere ook bij de introductie van zijn waarheidstabel. We starten dus met een lijst van argumenten die de waarheidstabel van $\Rightarrow$ aannemelijker maken. Vervolgens is er een oefening ter verdieping die enkele vreemde eigenschappen van $\Rightarrow$ onderzoekt.

2.1 Motivatie

We vertrekken in deze lijst van de uitspraak $p \Rightarrow q$, indien niet gespecifieerd. Afhankelijk van het gebruikte argument hieronder kun je gepaste zinnen voor p en q kiezen. Deze argumenten zijn vooral gericht op de waarheidstabel. Meer algemene problemen, zoals het omdraaien van implicaties, behandel ik hier niet.

- **Denk aan hoe je een implicatie ontkracht** Stel dat iemand een 'als ..., dan ...' uitspraak doet en jij gaat er niet mee akkoord. Hoe kan je dan aantonen aan je gesprekspartner dat het niet klopt wat hij zegt? De enige mogelijkheid die je hierbij hebt, is om te tonen dat p waar is, en q niet waar. 'als p, dan q' zegt niets over het geval dat p niet waar is, en daarmee kan je dus ook deze uitspraak niet ontkrachten. Je gesprekspartner zou direct reageren met: "Ja, maar ik zei: 'ALS p'." Om die reden staat er dus enkel een 0 in de waarheidstabel van $p \Rightarrow q$ bij p waar en q niet waar.

- **Gedachtenexperimenten** Het is geen goed idee om $p \Rightarrow q$, bij p niet waar, als onwaar te definiëren. Hiermee zouden gedachtenexperimenten namelijk niet meer mogelijk zijn. Uitspraken als 'als de aarde plat is, dan zou ik nog steeds naar school gaan.' en 'als de aarde plat is, dan zou de aarde een rand hebben waar je af kan vallen.' zouden onwaar zijn. Gedachtenexperimenten worden echter vaak gebruikt. Denk bijvoorbeeld aan als je meer zakgeld wil krijgen van je ouders en je zegt: 'als ik 50 euro zakgeld per maand zou krijgen, dan zou ik genoeg geld hebben om meer uit eigen zak te betalen.' De aanname van deze uitspraak klopt eigenlijk (nog) niet, maar toch gaat iedere leerling hiermee akkoord.

- **Synchronisatie met andere connectieven in propositielogica** Het is hoofdzakelijk de waarheidstabel van de implicatie die problemen oplevert, in tegenstelling tot die van de andere connectieven. Dit kan handig gebruikt worden om de waarheidstabel van de implicatie op te bouwen aan de hand van de andere connectieven. Ik geef enkele voorbeelden. U kan dit klassikaal bespreken of de leerlingen zelf de waarheidstabellen laten vergelijken. Zo lijkt het bijvoorbeeld wenselijk dat als $p \vee q$ waar is, dat dan ook $\neg p \Rightarrow q$ waar is. U kan hierbij zelf een voorbeeld verzinnen. Dit geeft onderstaande waarheidstabel waarbij de gekleurde 1'tjes voortkomen uit onze eis.

p	q	p	$\vee$	q	$\neg$	p	$\Rightarrow$	q
0	0	0	0	0	1	0		0
0	1	0	1	1	1	0	1	1
1	0	1	1	0	0	1	1	0
1	1	1	1	1	0	1	1	1

Ook is het wenselijk dat *modus ponens* een geldige redeneervorm is. Modus ponens betekent dat je uit de aannames $p \Rightarrow q$ en p de uitspraak q mag afleiden. Als we willen dat modus ponens een geldige redeneervorm is, dan moet het zo zijn dat als de aannames $p \Rightarrow q$ en p waar zijn, dan ook q waar is. We kunnen deze redenering ook omdraaien via contrapositie: als q niet waar is, dan moet $p \Rightarrow q$ of p niet waar zijn. Hiermee ligt dus de onderstaande gekleurde 0 vast.

p	q	p	$\Rightarrow$	q	p	q
0	0	0		0	0	0
0	1	0		1	0	1
1	0	1	0	0	1	0
1	1	1		1	1	1

Bovenstaande waarheidstabellen geven dan uiteindelijk de waarheidstabel van $\Rightarrow$. Er zijn ook nog andere opties van waarheidstabellen die u kan vergelijken om tot de waarheidstabel van $\Rightarrow$ te komen. Zo kan u bijvoorbeeld eisen dat $p \Leftrightarrow q$ altijd dezelfde waarheidswaarde heeft als $(p \Rightarrow q) \wedge (p \Leftarrow q)$ (als de waarheidstabel van $\Leftrightarrow$ aanvaard is.) Ook is de wet van contrapositie interessant om op te leggen aan de implicatie; dus dat $p \Rightarrow q$ altijd dezelfde waarheidswaarde moet hebben als $\neg q \Rightarrow \neg q$.

Deze interne verbanden kunnen dus gebruikt worden om de waarheidstabel aannemelijk te maken. Ik voeg hierbij toe: het wordt in de vakdidactische literatuur afgeraden om implicaties met een onware aanname te negeren. Men bespreekt best de volledige waarheidstabel, zonder het moeilijke punt van de onware aanname onder de mat te vegen. Er zijn verschillende problemen die zich kunnen voordoen bij het onvolledig bespreken. Ten eerste, als bij $p \Rightarrow q$ het geval met p onwaar genegeerd wordt, dan is $p \Rightarrow q$ niet te onderscheiden van $p \wedge q$ of van q. Wanneer leerlingen een zin in omgangstaal moeten omzetten naar symbolen, komt het dus voor dat ze $p \wedge q$ schrijven in plaats van $p \Rightarrow q$. Ten tweede heb je de volledige waarheidstabel nodig om contrapositie te staven aan de hand van de waarheidstabel. Contrapositie wordt veelvuldig gebruikt in de meetkunde.

- **Maar dat is onzin?**

  Een meer algemeen probleem met propositielogica is dat het enkel de waarheid van uitspraken bekijkt. In communicatie speelt er echter meer dan enkel de waarheid een rol. En dat probleem komt vooral naar boven bij $p \Rightarrow q$, omdat we intuïtief een inhoudelijk/oorzakelijk verband willen tussen p en q. Een uitspraak als 'Het regent of mijn haar is paars.' is vreemd, maar aanvaardbaar voor de leerlingen, terwijl 'Als het niet regent, dan is mijn haar paars.' meer reactie uitlokt. Er zijn enkele manieren om te reageren op de verwachting van een inhoudelijk/oorzakelijk verband bij $\Rightarrow$.

  Ten eerste is het een probleem van heel het systeem van de propositielogica. Bij alle andere connectieven kijken we ook enkel naar de waarheidswaarde van een uitspraak. Meer zelfs, 'logische uitspraak' wordt vaak gedefinieerd als een uitspraak waarvan je kan zeggen dat hij waar of vals is. Het systeem van de propositielogica is in zekere zin de gemakkelijkste manier om het redeneren te onderzoeken. Je zou het kunnen verbeteren, en beter laten aansluiten op de alledaagse manier van redeneren, maar verderop in het derde punt zien we dat de eenvoudigheid ook voordelen heeft.

  Ten tweede is een inhoudelijk/oorzakelijk verband tussen p en q eigenlijk een moeilijk ding. Om te oordelen of er een verband is of niet, moet je gebruik maken van je achtergrondkennis. Bijvoorbeeld bij 'als het regent, dan valt er water op de straat.' is gemakkelijk aanvaardbaar, omdat we weten wat het betekent om te regenen en omdat we weten dat de regen overal valt, zowel op als naast de straat. Er zijn echter uitspraken met een 'als ..., dan ...' en een inhoudelijk verband, die toch heel vreemd kunnen klinken. Ik geef enkele wiskundige voorbeelden: 'als n een Fermat priemgetal is, dan is de regelmatige n-hoek construeerbaar met een passer en liniaal (zonder markeringen op).', 'als de som van de kwadraten van twee zijden in een driehoek gelijk is aan het kwadraat van de derde zijde, dan is die driehoek rechthoekig.' en 'als je twee even getallen optelt, dan krijg je opnieuw een even getal.'. Deze voorbeelden lijken, afhankelijk van je voorkennis, geen inhoudelijk verband te bevatten. Zo zie je dus dat het niet zo eenvoudig is om het anders aan te pakken.

Ten derde is het soms net voordelig om enkel naar de waarheid van de uitspraken te kijken bij een implicatie. Als een specialist in het nieuws een moeilijke uitspraak doet in de vorm van $p \Rightarrow q$, dan begrijp je er misschien niet alles van, maar je weet wel dat als p waar is, dat q dan waar zal zijn. Zonder iets te weten over het vakgebied kan je zelfs redeneren dat als q niet waar is, dan p ook niet waar kan zijn. Ook in wiskunde is dit toepasbaar. Je kan redeneren over de voorbeelden die ik daarnet gaf, ook als je niet over de wiskundige voorkennis beschikt.

2.2 Pijnpunten

Zoals je waarschijnlijk zelf al merkte, gedraagt het connectief $\Rightarrow$ zich soms nogal vreemd. Je hebt de waarheidstabel wel gezien, en daarmee is de implicatie dus gedefinieerd, maar soms levert het vreemde situaties op. Zo is "Als de maan van kaas gemaakt is, dan is er geen paus." een ware uitspraak. Controleer zelf dat dit klopt.

In deze oefening ga je zelf op verkenning welke vreemde eigenschappen het connectief $\Rightarrow$ binnen de propositielogica heeft. Noteer je antwoorden op een apart blad.

1. De eerste eigenschap die we bekijken is de uitspraak $(p \Rightarrow q) \vee (q \Rightarrow p)$.
   - (a) Controleer met een waarheidstabel dat deze uitspraak altijd waar is.
   - (b) Neem p ="Ik eet graag brownies." en q ="Ik ben goed in wiskunde.". Wat betekent $(p \Rightarrow q) \vee (q \Rightarrow p)$ dan?
   - (c) Neem p ="Er zijn veel natuurlijke getallen." en q ="Er is wereldvrede.". Wat betekent $(p \Rightarrow q) \vee (q \Rightarrow p)$ dan?
   - (d) Kan je formuleren wat er vreemd is aan $(p \Rightarrow q) \vee (q \Rightarrow p)$? *Tip:* Je uitleg kan starten met 'Als je twee willekeurige uitspraken p en q hebt, die mogelijk niets met elkaar te maken hebben, dan ...'.
2. De tweede eigenschap die we bekijken is de uitspraak $p \Rightarrow (q \Rightarrow p)$.
   - (a) Controleer dat deze uitspraak altijd waar is.
   - (b) Kies voor p een uitspraak waarvan je weet dat hij waar is, zoals "De aarde is rond." Kies q ="Het regent.". Wat betekent $p \Rightarrow (q \Rightarrow p)$ dan?
   - (c) Houd dezelfde uitspraak voor p. Kies voor q nog een andere uitspraak. Wat betekent $p \Rightarrow (q \Rightarrow p)$ dan?
   - (d) Kan je formuleren wat er vreemd is aan $p \Rightarrow (q \Rightarrow p)$? *Tip:* Je uitleg kan starten met 'Als je een ware uitspraak p hebt, dan wordt die geïmpliceerd door ...'.
3. De laatste eigenschap die we bekijken is de uitspraak $(q \wedge \neg q) \Rightarrow p$.
   - (a) Controleer dat deze uitspraak altijd waar is.

(b) We zullen opnieuw enkele uitspraken invullen om de gegeven uitspraak beter te verstaan. Neem voor p een uitspraak waarvan je heel graag zou willen dat die waar is. Bijvoorbeeld p =“Ik ben rijk.”. Kies vervolgens voor q enkele willekeurige uitspraken. Schrijf je keuze voor p en q telkens op en formuleer wat $(q \wedge \neg q) \Rightarrow p$ dan betekent.

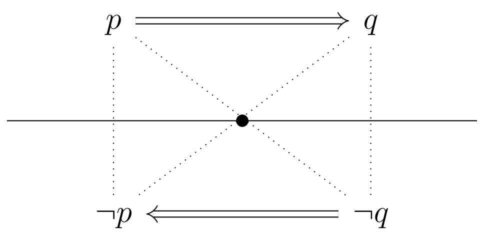
(c) Kan je formuleren wat er vreemd is aan $(q \wedge \neg q) \Rightarrow p$? Het kan handig zijn om te weten dat $(q \wedge \neg q)$ ook wel een *contradictie* genoemd wordt. De uitspraak $(q \wedge \neg q)$ is een contradictie omdat twee uitspraken elkaar tegenspreken (in dit geval q en $\neg q$).
Tip: Wat volgt er volgens de eigenschap uit een contradictie?

3 Visualisatie

In deze Sectie stel ik kort enkele mogelijkheden voor om visualisatie te gebruiken in propositielogica.

3.1 Vierkant voor Contrapositie

In vele handboeken worden de logische symbolen ingevoerd rond de stelling van Pythagoras. Daar, en in meetkunde in het algemeen, worden deze symbolen vaak gebruikt bij contrapositie. Contrapositie is een manipulatie van een implicatie, waarbij $p \Rightarrow q$ wordt omgevormd naar $\neg q \Rightarrow \neg p$. Deze twee uitspraken zijn equivalent. Dit "omkeren" van de implicatie kan voor verwarring zorgen, aangezien er ook veel aandacht gaat naar het niet zo maar omdraaien van de implicatie. Hiervoor stel ik onderstaande kleine schema voor. Dit kan natuurlijk nog visueel prettiger uitgewerkt worden, met eventueel aparte kleuren voor p, q en voor $\Rightarrow$.



Figuur 1: In bovenstaand schema staat bovenaan $p \Rightarrow q$. Daarna is er een volle lijn, die een spiegel voorstelt en in het midden een bol om rond te spiegelen. Onderaan krijgen we de implicatie na contrapositie, namelijk $\neg p \Leftarrow \neg q$. De verticale stippellijnen geven het "spiegelen" van p en q weer. De diagonale stippellijnen het "spiegelen" van $\Rightarrow$.

In bovenstaande Figuur 1 wordt contrapositie voorgesteld als spiegeling van de originele implicatie $p \Rightarrow q$. Om contrapositie correct uit te voeren, moet je alles "spiegelen". De propositieletters worden vervangen door hun negatie bij het spiegelen door de grote spiegel. Het symbool $\Rightarrow$ wordt rond het centrale punt gespiegeld en wordt zo $\Leftarrow$. Merk op dat dit, voor $\Rightarrow$, overeen komt met de realiteit: een getekende pijl zou bij de puntspiegeling effectief een $\Leftarrow$ geven. Deze Figure kan een hulpmiddel zijn om ten eerste het onderscheid te maken tussen contrapositie en simpelweg omdraaien van de implicatie. Ten tweede kan het ook als schema gebruikt worden op papier om de contrapositie van een gegeven uitspraak op te stellen. Daarvoor vul je simpelweg p en q in, en je kan gewoon de structuur volgen.

3.2 Waarheidsmatrices

Deze optie voor visualisatie komt voort uit mijn uitwerking bij de oefening in Subsectie 5.1. Bij het opstellen van een waarheidstabel wordt meestal een verticaal gefocuste Tabel gebruikt. Daarbij worden links alle combinaties van waarheidswaarden voor de propositieletters in rijen opgeschreven. Vervolgens wordt stap voor stap de waarheidswaarde van de logische uitspraak bepaald, zoals u hieronder kan zien.

p	q	p	$\Rightarrow$	$(q$	$\wedge$	$\neg$	$q)$
0	0	0	1	0	0	1	0
0	1	0	1	1	0	0	1
1	0	1	0	0	0	1	0
1	1	1	0	1	0	0	1

Deze methode is vooral handig als er veel (≥ 3) propositievariabelen of een ingewikkelde logische uitspraak in het spel zijn. Via deze methode kan je dat namelijk puur mechanisch oplossen door zoveel kolommen en rijen toe te voegen als je wil.
Ook het connectief $\wedge$ wordt klassiek gedefinieerd met zo'n waarheidstabel.

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Wanneer er sprake is van twee propositievariabelen en slechts een simpele logische uitspraak, kan het echter handig zijn om één van de propositievariabelen op een aparte dimensie te zetten. In plaats van alles verticaal uit te werken, staan dan bijvoorbeeld de waarheidswaarden van p in de meest linkse kolom, de waarheidswaarden van q in de bovenste rij, en in de andere cellen staat de uiteindelijke waarheidswaarde van de logische uitspraak. We krijgen dan de onderstaande *waarheidsmatrix* van $p \wedge q$.

$\wedge$	0	1
0	0	0
1	0	1

Voor de duidelijkheid duidde ik in bovenstaande waarheidsmatrix de waarheidswaarden van p aan met groen en die van q met rood. Voor $\wedge$ maakt dit niet zo veel uit, maar bij $\Rightarrow$ zal dat wel heel belangrijk zijn.

Het nadeel hiervan is natuurlijk dat er dan twee soorten "waarheidstabellen" gebruikt worden. Aparte terminologie kan hierbij al helpen. Ook is het mogelijk om deze notatie met de matrices pas na een tijd te introduceren. Het voordeel is echter dat we nu een matrix hebben van waarheidswaarden. We kunnen logische eigenschappen van $\wedge$ vertalen naar meer visuele eigenschappen van matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Er zijn namelijk volgende connecties zichtbaar.

- De logische 'en' is commutatief en dus is A symmetrisch rond de diagonaal.
- De waarheidstabel van $\neg(p \wedge q)$ verkrijg je door alle elementen van A te vervangen door hun negatie.
- De negatie nemen van het linkerargument van $\wedge$ (dus in de linkse kolom) komt overeen met het wisselen van de rijen in A. De negatie van het rechterargument van $\wedge$ wisselt de kolommen.

Al deze connecties samen zorgen ervoor dat je bijvoorbeeld een wet van De Morgan visueel kan inzien. We starten met A als de waarheidstabel van $p \wedge q$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \xmapsto{p \mapsto \neg p} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xmapsto{q \mapsto \neg q} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We starten met de waarheidstabel van $\wedge$. Vervolgens wisselen we de rijen en de kolommen. Deze laatste matrix komt dus overeen met de waarheidsmatrix van $\neg p \wedge \neg q$. Die matrix herken je gemakkelijk als de negatie van de waarheidsmatrix B van $p \vee q$ waarbij dus $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Dit toont de visuele mogelijkheden van deze waarheidsmatrices aan, en we gaan hier verder op in bij Subsectie 5.1.

4 Instap voor logica vertrekkend van omgangstaal

We zullen in dit stuk iets leren over "logica". We zullen het hebben over hoe je kan "logisch redeneren", hoe je een onomstotelijk argument kan geven, hoe je de waarheid achterhaalt van een uitspraak, ... In de tweede graad zien jullie de beginselen van logica en zullen jullie analyseren of individuele uitspraken waar zijn of niet. Dit zal ook goed van pas komen wanneer we wiskundig moeten nadenken! In de derde graad zullen jullie langere argumenten of redeneringen bekijken.

Laat ons nu stap voor stap kennismaken met die "logica".

4.1 Inleiding op logica

Kan jij "logisch" redeneren?
Wel, laat ons eens kijken. In de kadertjes hieronder vind je enkele discussies in die zouden kunnen voorkomen in de klas. Jij moet bij iedere discussie bepalen of je akkoord gaat of niet met Alice en waarom. We zullen dit vervolgens bespreken in een klasgesprek.

Tom: Iedere gsm is eigenlijk een ICT hulpmiddel.

Karel: En sommige ICT hulpmiddelen zijn nuttig voor de les, zoals bijvoorbeeld een smartboard.

Alice: Dus, iedere gsm is nuttig voor de les!

Tom: Elke score van tijdens het jaar staat op je rapport.

Karel: Sommige punten op je rapport zijn slecht.

Alice: Oei, dus elke score tijdens het jaar is slecht!

Tom: Alles wat je kan dragen, staat vermeld in de dresscode van de school.

Karel: Je kan op carnaval een kostuum van The Joker dragen.

Alice: Maar dan zeg je dat het kostuum van The Joker vermeld staat in de dresscode van de school?

Tom: Ieder vierkant is een rechthoek.

Karel: En □ is een vierkant.

Alice: Dus is □ een rechthoek.

Volgende vragen kunnen aan bod komen in het klasgesprek. Jullie leerkracht heeft misschien nog andere vragen voor jullie.

1. Overloop met de klas bij elke discussie wie akkoord gaat met Alice.

2. Beargumenteer ook telkens waarom je wel of niet akkoord gaat.
   Kies per kadertje iemand uit de klas die er wel akkoord mee is, en iemand die er niet akkoord mee is, en laat ze voor de klas elkaar overtuigen van hun mening.

3. Zijn er leerlingen die bij bepaalde kadertjes niet akkoord gaan met Alice met als reden dat Tom of Karel iets fouts zei?
   Zijn er dus mensen die van mening zouden veranderen, als je moet aannemen dat alles wat Tom en Karel zeggen waar is?

4. Zijn er discussies die op elkaar lijken?
   In welke zin "lijken" ze op elkaar?

Vermoedelijk was er wel enige discussie over wanneer Alice nu correct was, en wanneer niet. Logica is de studie van het correct redeneren en dus komt dat mooi uit! We zullen de conflicten uit de weg kunnen ruimen.

4.2 Implicatie

In de eerste graad gebruikten jullie al de symbolen $\Rightarrow$ en $\Leftrightarrow$.

Vraag 1. Kunnen jullie nog uitleggen wat deze symbolen betekenen? Wat heeft het woord ‘implicatie’ daarmee te maken?

Er staan enkele uitspraken in een kadertje voor je klaar. Neem even een kijkje.

> Als het regent, dan worden de straten nat.

> Als het zomervakantie is, en enkel dan, ga ik op reis.

> Als een vlakke figuur vier gelijke zijden heeft, dan is ze een vierkant.

Bespreek even in de klas met welke uitspraken je akkoord gaat.

Bekijk daarna onderstaande uitspraken. Betekenen ze hetzelfde als de uitspraken hierboven?

> Als de straten nat worden, dan regent het.

> Als ik op reis ga, en enkel dan, dan is het zomervakantie.

> Als een vlakke figuur een vierkant is, dan heeft ze vier gelijke zijden.

Daarnet ging je waarschijnlijk niet met alle uitspraken akkoord. Dat betekende dus dat je dacht dat niet al die uitspraken met een "als ..., dan ..."-constructie waar waren. Laat ons enkele voorbeelden bekijken om te achterhalen wanneer zulke uitspraken waar zijn.

> Als een rechthoek vier gelijke zijden heeft, dan is het een vierkant.

> Als je de plantjes iedere dag water geeft, dan zullen ze blijven leven.

> Als ik miljardair was, dan zou ik een heel groot huis kopen.

> Als het regent, dan bliksemt het.

> Wanneer ik met pensioen ben, dan zal ik regelmatig sporten.

> Als de koffiezetmachine niet was uitgevonden, dan zou de webcam veel later uitgevonden zijn.

Overleg opnieuw in de klas welke uitspraken kloppen en welke niet.

Vraag 2. Laat het ons vervolgens ietsje wiskundiger aanpakken om te achterhalen wanneer bovenstaande zinnen waar zijn. We zullen daarom letters gebruiken om uitspraken voor te stellen. Zo kunnen we bijvoorbeeld $p =$"Het regent." en $q =$"Het bliksemt." gebruiken. De vierde uitspraak hierboven ziet er in symbolen dan dus uit als $p \Rightarrow q$. Zie je in waarom? Wat betekende het symbool $\Rightarrow$ weer?

Vraag 3. Zie je een gelijkenis tussen de uitspraken?
We hebben nu al één uitspraak symbolisch geschreven als $p \Rightarrow q$, maar kun je dit ook doen voor de andere uitspraken (weliswaar met andere betekenissen van p en q)?

Vraag 4. Bekijk nu even de Tabel verderop. We kijken niet meer naar de concrete betekenis van de letters p of q, maar enkel naar of ze waar zijn of niet. Er staat een uitspraak per kolom. Een 0 staat voor het vals zijn van de uitspraak, en een 1 voor het waar zijn van de uitspraak. Staan in de Tabel alle mogelijke gevallen voor de waarheid van p en q?

Vraag 5. Overleg in de klas hoe je de Tabel zou invullen. In de laatste kolom moet er dus op iedere rij een 0 of een 1 komen die weergeeft of $p \Rightarrow q$ in dat geval waar of vals is. Bekijk opnieuw de voorbeelden om een keuze te maken.

p	q	$p \Rightarrow q$
0	0	
0	1	
1	0	
1	1	

4.3 Negatie

We gaan verder op onze tocht in de logica. We zijn nu gekomen bij de 'negatie'; dat is de naam in de logica voor zinnen met een "niet" in. Bijvoorbeeld: de negatie van "Het regent." is "Het regent niet.". We noteren de "niet" met $\neg$. Dus als "Het regent." voorgesteld wordt met p, dan noteren we $\neg p$ voor "Het regent niet."

We zullen opnieuw kijken wanneer uitspraken met een "niet" waar of vals zijn. Dat lijkt op het eerste zicht eenvoudig, maar kijk maar eens naar volgende gesprekken!

Tom: Heb je zout in de soep gedaan?

Alice: Ja!

Tom: Heb je geen zout in de soep gedaan?

Alice: Ja!

Tom: Is 10 niet deelbaar door 3?

Alice: Ja!

Vraag 1. Merk je gelijkenissen op tussen de gesprekken? Welke? Denk ook aan gelijkenissen qua structuur, en niet enkel over de inhoud.

Is er iets verwarrends aan de reactie van Alice? Is het verwarrend dat ze bij de eerste twee gesprekken telkens "Ja!" antwoordt? Overleg hierover met de klas.

Vraag 2. Om verwarring bij wiskundig denken te vermijden, en om zeker te zijn dat we correct redeneren, hebben we duidelijkheid nodig over wanneer een uitspraak met "niet" nu waar is of niet. Vul daarom onderstaande Tabel in.

p	$\neg p$
0	
1	

4.4 Disjunctie

We vervolgen ons onderzoek in de logica. ‘Disjunctie’ is een moeilijk woord uit de logica voor een uitspraak met een “of” in. Je krijgt hieronder enkele gesprekken waar er telkens een “of” gebruikt wordt. Het is aan jou om te beslissen bij ieder gesprek of je akkoord gaat met de conclusie van Alice. **Neem aan dat de andere uitspraken in het gesprek zeker kloppen. Het gaat enkel over de conclusie van Alice.**

Alice: Heb je al gehoord van de punten van Karel?
Hij heeft heel goeie punten voor zijn examen.
Hij heeft dus goed gestudeerd of gemakkelijke vragen gekregen.

Tom: Ik had hetzelfde examen!
Het waren heel gemakkelijke vragen.

Alice: Ow! Karel zal dus wel niet goed gestudeerd hebben.

Alice: Ik moet een cadeau van €5 kopen voor Karel.
Ik zal snoep voor €5 of een cadeaukaart van €5 van bol.com kopen,
maar ik weet nog niet welk van beide.

Tom: Karel zou zeker blij zijn met snoep voor €5!

Alice: Okay! Dan zal ik geen cadeaukaart kopen.

Alice: Weet jij nog welke vlakke figuren vier rechte hoeken hebben?

Tom: Als een figuur vier rechte hoeken heeft, dan is ze een vierkant of rechthoek.

Alice: Okay! Dus als het een vierkant is, dan is het geen rechthoek.

Overleg in de klas over de verschillende gesprekken. Er zijn twee vragen die jullie moeten beantwoorden.

1. Er zijn gelijkenissen tussen de gesprekken. Welke? Denk aan gelijkenissen qua structuur, en let op de “of”.

2. Waarom gaan jullie soms wel en soms niet akkoord met Alice?

Je merkte tijdens het klasgesprek waarschijnlijk op dat een "of" eigenlijk niet zo eenduidig is. Als we willen communiceren in wiskunde, willen we echter geen misverstanden of onduidelijkheden hebben. We gebruiken dus het symbool $\vee$ voor de "of" in de wiskunde, en we gebruiken onderstaande waarheidstabel. Die noemen we de inclusieve "of".

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Vraag 1. Naast de inclusieve "of" heb je ook de exclusieve "of". Bij de exclusieve "of" mogen de twee delen apart waar zijn, maar niet allebei. Kan je zeggen bij welke gesprekken hierboven de inclusieve "of", dan wel de exclusieve "of" gebruikt wordt?

4.5 Conjunctie

Als we spreken over de “of”, dan moeten we het ook hebben over de “en”. Een zin met “en” noemen logici een ‘conjunctie’ en daarvoor gebruiken ze het symbool $\wedge$. Laat ons een voorbeeld bekijken. Wanneer heeft Karel in onderstaand gesprek gelijk?

Tom: Alice en ik zijn slim.

Karel: Dat is niet waar!

Meer specifiek, bij welk van de twee onderstaande uitspraken heeft Karel gelijk? Overleg hierover in de klas.

Tom is niet slim en Alice is niet slim.

Tom is niet slim of Alice is niet slim.

Vraag 1. Laat ons nu invullen in een waarheidstabel wanneer een zin met een “en” niet waar is. Op die plaatsen moet dus een 0 staan. Dan weten we ook meteen dat bij de andere gevallen de zin wel waar is, en er dus een 1 moet staan.

p	q	$p \wedge q$
0	0	
0	1	
1	0	
1	1	

4.6 Kwantoren

Het laatste logische concept dat we bekijken, zijn kwantoren. Er zijn hieronder opnieuw enkele gesprekken. In welke gesprekken vind je de reactie van Alice begrijpbaar? Waarom wel/niet?
Overleg met de klas.

Tom:	O nee! Alle tickets zijn al uitverkocht!
Alice:	Ik zal toch nog eens proberen of ik één vind.

Karel:	Ik ben net gaan kijken en alle plaatsen in de bib zijn bezet.
Alice:	Ik zal nog eens kijken voor de zekerheid.

Karel:	We hebben in wiskunde bewezen dat bij alle driehoeken de som van de hoeken gelijk is aan 180°.
Alice:	Ik zal toch nog eens op zoek gaan naar een driehoek waar dat niet zo is.

Vraag 1. In wiskunde gebruiken we het symbool $\forall$ bij een uitspraak met "(voor) alle" of "iedere". Zoals je daarnet merkte wordt dit anders gebruikt in de omgangstaal. Kunnen jullie uitleggen wat het verschil is?

Vraag 2. Stel dat we nu volgende uitspraak met "alle" bekijken.

> Alle priemgetallen zijn oneven.

Hoe zou je aantonen dat deze uitspraak **niet** klopt?

Een ander symbool dat in wiskunde gebruikt wordt is $\exists$. Dit symbool betekent "er is/zijn" of "sommige". Hou dus goed in het achterhoofd dat $\exists$ al tevreden is van zodra er **één** geval is waar de uitspraak klopt. Als ik zeg "Er zijn natuurlijke getallen kleiner dan 1.", dan is één voorbeeld (bijvoorbeeld het getal 0) voldoende om die uitspraak waar te maken.

5 Verdiepingsoefeningen

5.1 True, False of "Unknown"? Voorbij de propositielogica

De wet van de uitgesloten derde zegt dat een uitspraak p ofwel waar ofwel niet waar is ($p \vee \neg p$). Er zijn echter contexten denkbaar waar die wet niet geldt. Denk maar aan bijvoorbeeld subjectieve uitspraken als "Wiskunde is leuk!" of uitspraken waarvan het niet geweten is of ze waar zijn of niet zoals "Op 5 januari 2083 zal het regenen.". In propositielogica worden deze uitspraken niet beschouwd als 'logische uitspraak'. Stel dat we echter onze logicataal zouden uitbreiden met een derde mogelijkheid voor de waarheidswaarde zoals "nog niet geweten" of "niet te bepalen", dan zouden we meer uitspraken kunnen analyseren. Hoe zouden onze klassieke logische connectieven er dan uit zien? Dit brengt ons tot de zogenaamde driewaardige logica.

In dit gedeelte gebruiken we de notatie T van 'True' voor waar, F van 'False' voor vals en U van 'Unknown' voor onbepaald. Dit doen we om echt duidelijk te maken dat het over een verschillende logicataal gaat. Voor de definitie van de logische connectieven zullen we ervoor zorgen dat ze zich met de waarden T en F net zoals vroeger gedragen. Het is enkel als er een U in het spel is, dat we iets nieuws zullen moeten bedenken. De negatie van U moet opnieuw U zijn. Als je namelijk niet weet of een uitspraak waar of vals is, dan weet je ook niets over de negatie van die uitspraak. Samengenomen, wordt de logische negatie dus als volgt gedefinieerd:

p	$\neg p$
T	F
U	U
F	T

Vraag 1. In deze oefening zullen we vervolgens voor de andere logische connectieven ($\wedge$, $\vee$ en $\Rightarrow$) op zoek gaan naar een goede definitie. Aangezien we nu een extra waarheidswaarde U hebben, zijn er meer mogelijkheden voor de waarheidswaarden van twee uitspraken. Hoeveel mogelijkheden zijn er?

Aangezien het meer mogelijkheden zijn, zullen we ze overzichtelijker weergeven dan in de normale waarheidstabel. In plaats van alle mogelijke waarheidswaarden rij per rij op te schrijven, zullen we nu ook de kolommen gebruiken. Bij het definiëren van bijvoorbeeld de $p \wedge q$ zetten we de mogelijke waarheidswaarden van p in de linkse kolom en de mogelijke waarden van q in de bovenste rij. De andere cellen horen dan bij alle mogelijke combinaties. De waarde van $F \wedge U$ in onderstaande Tabel zal bijvoorbeeld in de vierde rij en derde kolom komen. Deze voorstelling noemen we geen waarheidstabel, maar een *waarheidsmatrix*.

$\wedge$	T	U	F
T			
U			
F			

Conjunctie

We zullen nu ook de $\wedge$ definiëren.

We willen dat de conjunctie zich gedraagt als vroeger voor uitspraken die T of F zijn. Je kunt dus al de cellen aanvullen waar er geen sprake is van U.

$\wedge$	T	U	F
T			
U			
F			

Vraag 2. Laat ons nu opnieuw intuïtief nadenken zoals daarnet bij de waarheidsmatrix van de negatie. Herinner je, de waarde U betekent “Unknown”; je weet niet of de uitspraak waar of vals is. Vul nu de rest van de waarheidsmatrix van $\wedge$ in en beargumenteer.

Disjunctie

1. Definieer de $\vee$ zoals daarnet bij de $\wedge$. Maak opnieuw gebruik van de betekenis van U als “Unknown” om de waarheidsmatrix op te stellen.

2. Verifieer dat de wet van De Morgan nog altijd geldt. Controleer dus dat $p \vee q$ altijd dezelfde waarheidswaarde heeft als $\neg(\neg p \wedge \neg q)$ bij drie mogelijke waarheidswaarden (T,U,F).

$\vee$	T	U	F
T			
U			
F			

Implicatie

Om $\Rightarrow$ te definiëren zou je opnieuw kunnen intuïtief redeneren op de betekenis van U zoals je gedaan hebt bij de vorige connectieven. Bij de $\vee$ controleerden we dan achteraf of een logische wet nog steeds klopte voor onze definitie. We draaien nu de volgorde om: in plaats van eerst intuïtief te redeneren en vervolgens een wet te verifiëren, zullen we de $\Rightarrow$ rechtstreeks definiëren door een logische wet.

Vraag 3. In propositielogica is er een logische wet die zegt dat $p \Rightarrow q$ altijd dezelfde waarheidswaarde als $\neg p \vee q$. Stel dus een waarheidsmatrix van $\neg p \vee q$ op en definieer daarmee de $\Rightarrow$ zodat aan die wet voldaan is.

$\Rightarrow$	T	U	F
T			
U			
F			

Praktische toepassing

Ik werk een voorbeeld uit waar deze logica met drie waarheidswaarden (T, U en F) belangrijk is.

Stel dat je een analyse doet over de werkzaamheid van een bepaald medicijn. Tijdens je onderzoek wil je specifiek bekijken hoe de medicatie werkt op rokers. Aangezien het een onderzoek is op lange termijn, kan het echter zijn dat sommige deelnemers overlijden tijdens het onderzoek. Je wilt dus je dataset filteren tot bijvoorbeeld alle rokers die niet gestorven zijn. De programmeertaal die je daarvoor gebruikt, zal kijken welke mensen in de steekproef de uitspraak “de persoon rookt $\wedge\neg$ de persoon is gestorven” waar maken. Daarbij kunnen we dus $p =$“de persoon rookt” en $q =$“de persoon is gestorven” nemen. Hoewel zowel p op q op het eerste zicht waar of vals lijken, is er in de praktijk nog een derde mogelijkheid. Die derde mogelijkheid is dat je het niet weet! Vele datasets zijn namelijk niet volledig, en er zal op verschillende plaatsen dan “NA” (van “Not Available”) te vinden zijn. Dit komt overeen met onze U. Dit kan bijvoorbeeld voorkomen als je een dataset hebt die opgesteld is in verschillende landen, waarbij één land wel registreerde wie er rookte (dan staat er T of F voor p), maar een ander land niet (en die mensen dus U zullen hebben voor p).

Vraag 4. Stel dat enkel de personen uit de dataset worden meegenomen die $p \wedge \neg q$ waar maken (dus waarheidswaarde T). Worden in dat geval mensen van wie niet geweten is of ze roken meegenomen in de dataset of niet? Kijk hiervoor naar de definitie van $\wedge$ van daarnet!

5.2 Identieke operaties

In deze oefening gaan we op zoek naar operaties die eigenlijk niets doen. Wat bedoelen we hiermee? Ik geef een voorbeeld. Je start met p. Vervolgens zetten we daar $\neg\neg$ voor (dit is onze operatie). We krijgen de uitspraak $\neg\neg p$. In een waarheidstabel kunnen we dan controleren dat deze nieuwe uitspraak altijd dezelfde waarde heeft als p (dus of de operatie "identiek" is).

p	$\neg$	$\neg$	p
0	0	1	0
1	1	0	1

Dit klopt dus effectief! De uitspraak $\neg\neg p$ heeft altijd dezelfde waarheidswaarde als p, en dus is $\neg\neg$ voor een uitspraak zetten een voorbeeld van een identieke operatie. In het algemeen start een identieke operatie met een uitspraak p, wordt er vervolgens iets met die p gedaan (de zogenaamde *operatie*) en krijgen we een uitspraak terug die altijd dezelfde waarheidswaarde heeft als p (hij is dus in zekere zin *identiek* met p).

1. We starten met p en de operatie die we uitvoeren is er $(q \Rightarrow q) \wedge$ voor zetten. We krijgen na deze operatie dus $(q \Rightarrow q) \wedge p$ als uitspraak. Ga na of dit een identieke operatie is.
   Tip: Kijk naar het voorbeeld hierboven.

2. We starten met p en de operatie die we uitvoeren is er $(q \vee \neg q) \wedge$ voor zetten. We krijgen dus $(q \vee \neg q) \wedge p$. Ga na of dit een identieke operatie is.

3. We voeren een nieuw symbool in: $\top$ wordt gebruikt voor een uitspraak die altijd waar is. Het doet er dus niet toe of en welke propositieletters in die uitspraak zitten; het enige belangrijke is dat $\top$ altijd waar is (in het groen). De waarheidstabel van $\top \Rightarrow (q \Rightarrow r)$ is dan bijvoorbeeld:

q	r	$\top$	$\Rightarrow$	$(q$	$\Rightarrow$	$r)$
0	0	1	1	0	1	0
0	1	1	1	0	1	1
1	0	1	0	1	0	0
1	1	1	1	1	1	1

Kijk naar de vorige twee vragen: denk je dat $\top\wedge$ voor een uitspraak zetten een identieke operatie is? Beargumenteer met een waarheidstabel.

4. Het symbool $\top$ is voor een uitspraak die altijd waar is. Het symbool $\bot$ wordt gebruikt voor een uitspraak die altijd vals is. Opnieuw: het doet er niet toe of en welke propositieletters in die uitspraak zitten: $\bot$ is altijd onwaar.
   Denk je dat $\bot\vee$ voor een uitspraak zetten een identieke operatie is? Beargumenteer!

5.3 Connectieven uitdrukken met andere connectieven

In deze oefening vragen we ons af of bepaalde connectieven kunnen uitgedrukt worden met andere connectieven. Laat ons starten met een voorbeeld.

In je cursus logica zag je misschien de exclusieve "of" en voerde je daar een apart symbool voor in. Hier gebruiken we het symbool $\otimes$ voor de exclusieve "of". Voor leerlingen die dit nog niet gezien hebben, voeg ik hieronder de waarheidstabel toe van $\otimes$.

p	q	$p \otimes q$
0	0	0
0	1	1
1	0	1
1	1	0

De exclusieve "of" is dus waar als een van de twee delen waar is, maar niet beide.
Welnu, de exclusieve of $\otimes$ kan uitgedrukt worden met andere connectieven! De uitspraak $p \otimes q$ betekent dat p of q waar is, maar niet beiden tegelijk. We kunnen dat laatste letterlijk vertalen in $(p \vee q) \wedge \neg(p \wedge q)$. In een waarheidstabel controleren we dat deze uitspraken altijd dezelfde waarheidswaarde hebben.

p	q	p	$\otimes$	q	$(p$	$\vee$	$q)$	$\wedge$	$\neg$	$(p$	$\wedge$	$q)$
0	0	0	0	0	0	0	0	0	1	0	0	0
0	1	0	1	1	0	1	1	1	1	0	0	1
1	0	1	1	0	1	1	0	1	1	1	0	0
1	1	1	0	1	1	1	1	0	0	1	1	1

Dat klopt! We kunnen dus $\otimes$ uitdrukken met de gekende symbolen $\vee, \neg$ en $\wedge$, omdat we een uitspraak vonden met enkel $\vee, \neg$ of $\wedge$ die altijd dezelfde waarheidswaarde heeft als $p \otimes q$. De volgende vraag is natuurlijk of je ook andere connectieven kan uitdrukken met $\neg, \vee$ en $\wedge$.

1. Je kan $\Rightarrow$ uitdrukken aan de hand van $\neg$ en $\vee$. De uitspraak $p \Rightarrow q$ heeft namelijk altijd dezelfde waarheidswaarde als $\neg p \vee q$. Controleer dit met een waarheidstabel.

2. Omgekeerd is er echter ook een verband. Je kan $\vee$ uitdrukken met $\Rightarrow$. De uitspraak $p \vee q$ heeft namelijk altijd dezelfde waarheidswaarde als $(p \Rightarrow q) \Rightarrow q$. Controleer dit met een waarheidstabel.

3. Kan je een manier bedenken om de $\vee$ uit te drukken met $\neg$ en $\wedge$?
   Tip: Denk aan de wetten van De Morgan.

4. We hebben nu al allerhande verbanden gevonden. De $\Rightarrow$ kunnen we uitdrukken met de $\neg$ en de $\vee$, en de $\vee$ kunnen we uitdrukken met de $\neg$ en de $\wedge$. We kunnen dus eigenlijk elk connectief dat je kent uitdrukken met $\neg$ en $\wedge$! Dat is al indrukwekkend, aangezien we dat kunnen doen met slechts twee connectieven. Zou het ook lukken met één connectief?

   Het antwoord is ja! De zogenaamde 'Sheffer Stroke' met als symbool $|$ kan ieder ander connectief uitdrukken. Dit is de waarheidstabel van $|$:

p	q	$p \mid q$
0	0	1
0	1	1
1	0	1
1	1	0

   Je kan zien dat $p \mid q$ eigenlijk de negatie is van $p \wedge q$.
   Kan je een manier bedenken om $\neg$ uit te drukken met $|$?

5. Kan je een manier bedenken om $p \wedge q$ uit te drukken met $|$?

6 Begeleidende Tekst voor Leerkrachten

In deze Sectie geef ik meer uitleg bij de vorige delen. Het gaat hierbij voornamelijk om concreet didactisch advies om het lesmateriaal te gebruiken, en niet zo zeer over de theoretische achtergrond (waarvoor u mijn thesis kan lezen). Ik gebruik de zelfde structuur als hierboven. Doorheen het lesmateriaal merkte je waarschijnlijk dat er nogal veel witruimte is. Zo kunnen de delen apart gebruikt worden.

Sectie 1

Deze lijst met verschillen spreekt redelijk voor zich. Ze is opgesteld voor leerkrachten. Uit de vakdidactische literatuur blijkt dat veel moeilijkheden met de formele logica voortkomen uit verschillen met gewone omgangstaal. Ook in de eindtermen voor de tweede graad is het verschil met de omgangstaal (in beperkte mate) opgenomen. Onze volledigere lijst kan u gebruiken als persoonlijk hulpmiddel: u kan dus simpelweg de lijst lezen en u zo bewust zijn tijdens de lessen over logica wat mogelijke problemen kunnen zijn. Daarnaast kan deze lijst ook inspireren voor het maken van eigen lesmateriaal (zoals ik zelf heb geprobeerd in Sectie 4). Er zijn nog andere problemen met logica die niet onmiddellijk voortkomen uit verschillen met de omgangstaal. Deze zijn hier niet opgenomen.

Sectie 2

Deze Sectie bestaat uit twee delen. Het eerste deel is geschreven voor leerkrachten en bevat een lijst van argumentaties voor de waarheidstabel van de implicatie, zonder twijfel het moeilijkste connectief. Dit toont zich bijvoorbeeld in problemen met het verschil tussen equivalentie en implicatie, of problemen met omkeren van de implicatie. Daarenboven kunnen er ook problemen optreden bij het aanleren van de waarheidstabel van $\Rightarrow$. Die problemen kunnen verstaan worden als het niet begrijpen van de waarheidstabel, maar ook als het niet aanvaarden van die waarheidstabel. De lijst argumenten gaat eerder uit van die tweede optie, en probeert de gegeven Tabel aannemelijker te maken. Het kan dus nuttig zijn om deze opgelijste argumenten in het achterhoofd te houden, wanneer u hierover les geeft.

In het tweede deel is er een oefening die geschreven is voor leerlingen. Het heeft ook een oplossingsleutel in Subsectie 7. Dit is een moeilijkere oefening voor leerlingen die verdieping willen. Het is geen verplichte leerstof. In deze oefening gaan leerlingen zelf op onderzoek (met begeleiding) welke vreemde eigenschappen de implicatie in propositielogica heeft. Er zijn drie eigenschappen, die quasi geordend zijn volgens moeilijkheidsgraad/ondersteuning. U kan deze oefening zelfstandig laten oplossen, maar ik vermoed dat er wel een beetje ondersteuning van de leerkracht nodig zal zijn. Wat de leerkracht ook in de gaten zal moeten houden bij deze oefening, is de attitudinale reactie op de materie. Het is vanuit wiskundig perspectief heel interessant, maar kan ook een reactie te weeg brengen als “Logica klopt niet.”. In dat geval kan u bijvoorbeeld deze kadering geven: “Deze oefening betekent niet dat alles van wiskunde en logica om zeep is. Het betekent enkel dat propositielogica een vrij simpel **model** is van hoe wij wiskundig denken. Er zijn ook meer uitgebreide systemen (zoals predicatenlogica) die dichter proberen aan te leunen bij onze gewone manier van redeneren.”

Sectie 3

Deze volledige Sectie is geschreven voor leerkrachten. In mijn thesis bespreek ik enkele mogelijkheden om visualisatie te gebruiken bij het aanleren van logica. Daaruit bleek dat visualisatie, althans van de logische structuur en niet van de inhoud, effectief kan helpen. In Sectie 3 geef ik daarom twee suggesties voor het gebruik van visualisatie. Ten eerste is er een visueel schema voor contrapositie gebaseerd op spiegelen. Het “invullen” van dit schema (met specifieke p en q) kan leerlingen een houvast bieden om geen fouten te maken bij de contrapositie nemen van een uitspraak. Daarnaast kan het ook nuttig zijn om conceptueel het verschil tussen contrapositie en omkeren van de implicatie te onthouden. Dit schema lijkt me geschikt voor alle leerlingengroepen.

Ten tweede introduceer ik een andere soort waarheidstabellen, die ik ‘waarheidsmatrices’ noem. Dit lijkt me eerder iets voor de wiskundig sterkere groepen (aangezien ze waarheidstabellen en waarheidsmatrices goed uit elkaar zullen moeten houden). Het linkt matrixmanipulaties aan eigenschappen van logische connectieven, wat een verrassend en interessant verband is.

Sectie 4

Deze Sectie verdient zonder twijfel de langste begeleidende tekst. In deze Sectie maakte ik een (beperkte) leerlingentekst om logica te introduceren. Hij is niet bedoeld als volledige cursus voor logica. Zo zal er bijvoorbeeld meer aandacht moeten zijn voor het invoeren van propositieletters en het idee van een waarheidstabel dan wat in dit lesmateriaal neergeschreven staat. Wel bevat het per logisch connectief (en één keer in het algemeen voor “logica”) aantrekkelijke instappen. Deze zijn telkens gebaseerd op realistische redeneringen/uitspraken in de omgangstaal. De gedachte is dat in de plaats van logica te introduceren met toepassingen uit technologie of met artificiële, simpele uitspraken, het ook interessant is om te starten vanaf realistische redeneringen. De verschillen met de omgangstaal worden dus gebruikt vanaf de start, om het nut en de noodzaak van logica interactief te tonen, in plaats van een deductief systeem voor te stellen en achteraf op te merken dat het verschilt van de gewone omgangstaal. Dit is interessant voor iedere leerlingengroep, maar vooral voor richtingen die niet techologisch noch wiskundig georiënteerd zijn. Omdat er niet gestart wordt van simpele uitspraken, kan het ook wel iets moeilijker zijn voor de leerlingen, maar zal de start des te aangrijpender zijn. U kan als leerkracht zelf kiezen hoe u na deze start de overgang naar formalisatie maakt. Zo kan u na de realistische redeneringen eerst enkele simpelere uitspraken geven om de waarheidstabel op te stellen.

Per connectief zijn er enkele redeneringen die in een klasgesprek vergeleken moeten worden. De redeneringen zijn zo gekozen dat ze discussie te weeg zullen brengen, en die reactie kan dan gebruikt worden om de waarheidstabel van het connectief te introduceren. De redeneringen kunnen natuurlijk misschien nog beter gekozen worden. Ik nodig u uit om eventuele alternatieven te bedenken! Er is geen oplossingsleutel voorzien, aangezien het bedoeld is voor een klasgesprek dat u als leerkracht zal moeten begeleiden. U zal sowieso een beetje moeten sturen of gepaste vragen stellen. Daarover later meer in onderstaande Subsecties. Na de instap per connectief maken de leerlingen best enkele oefeningen met eenvoudigere uitspraken om het meer in de vingers te krijgen. Ik raad de voorgestelde volgorde van instappen/connectieven aan. Zo hangt de uitleg van conjunctie in Subsectie 4.5 sterk af van de voorgaande Subsectie 4.4 over disjunctie. Het is zeker mogelijk om slechts enkele van de voorgestelde instappen te gebruiken, of bijvoorbeeld instappen te combineren.

Subsectie 4.1

Deze Subsectie draait om vier redeneringen. Impliciet wordt het verschil tussen geldige en ongeldige redeneringen geïntroduceerd. Geldigheid als zelfstandig concept is geen deel van de eindtermen, en zeker niet voor de tweede graad, maar is wel zo fundamenteel voor logica dat het een ideaal inleidingsonderwerp is.

De twee eerste redeneringen zijn ongeldig, aangezien ze de structuur volgen van “alle A zijn B, sommige B zijn C, dus alle A zijn C.”. De eerste redenering zal echter verleidelijk zijn om mee akkoord te gaan voor de leerlingen, en de tweede zeker niet. U kan de leerlingen dus tegen elkaar uitspelen door in vraag te stellen waarom ze wel met de ene, maar niet met de andere van deze twee redeneringen akkoord gaan.

De laatste twee redeneringen zijn geldig, en hebben een structuur als “alle A zijn B, deze specifieke C is A, dus deze specifieke C is B.”. Daarvan is de eerste verleidelijk om niet akkoord mee te gaan, en de tweede wel.[1] Hier kan u leerlingen dus ook confronteren, als ze niet de zelfde keuze maken bij deze twee redeneringen. De vraag is of je akkoord gaat met Alice. Er zullen waarschijnlijk leerlingen zijn die niet akkoord gaan, door iets wat Tom of Karel zei. Dit kan echter aan bod komen tijdens het klasgesprek.

Naar het einde van het klasgesprek gaat het over “gelijkenissen” tussen de redeneringen. Een optie hiervoor is dat sommige redeneringen dezelfde logische structuur hebben. Daarnaast kan je het ook algemener hebben over geldigheid van redeneringen. U kan als leerkracht zelf kiezen hoeveel u hierbij zelf opbrengt. Een mogelijke uitleg is: “De twee eerste redeneringen lijken op elkaar, omdat ze *ongeldig* zijn. Zelfs al ga je akkoord met alle aannames van Tom en Karel, dan kan het nog steeds zijn dat wat Alice zegt, niet klopt. Bij de twee laatste is het anders. Als je akkoord gaat met de aannames, dan **moet** je wel akkoord gaan met de conclusie van Alice. Deze twee redeneringen noemen we dus *geldig*. Je kan dus geldig redeneren, zonder te weten wat waar is en wat niet. Dit is de kracht van de logica: je kan van een redenering zeggen dat ze geldig is of niet, zelfs al weet je niet wat er waar is of niet. In wiskunde willen we enkel geldig redeneren. Daarom gaan we nu verder met onze studie van de logica.”

Subsectie 4.2

De Subsectie over de implicatie start met een herhaling van de eerste graad. Er worden enkele voorbeelden opnieuw in een klasgesprek besproken, waarna gecontroleerd wordt wanneer je de uitspraken mag ‘omdraaien’. Vervolgens zijn er nog enkele voorbeelden die gericht zijn op het opstellen van de waarheidstabel. Er zijn doelbewust ook enkele moeilijkere uitspraken. Deze dienen als voorbeeld bij de eigenschap dat de implicatie waar is, als de aanname vals is. Er is ook één voorbeeld (over een koffiezetmachine) dat de leerlingen

[1] Toegegeven, bij die eerste redenering zit er nog een extra addertje in het gras: “kan” wordt eigenlijk in twee betekenissen gebruikt zodat de leerlingen de aannames vlotter aanvaarden. Technisch gezien is die eerste redenering dus een drogredenering met equivocatie.

doelbewust op het verkeerde spoor brengt, en gelinkt is aan mijn uitleg over 'verband' in Subsectie 2.1. Er hoeft geen **inhoudelijk** verband te zijn tussen de twee uitspraken die verbonden worden door $\Rightarrow$. Bij de uitspraak over de koffiezetmachine lijkt er totaal geen inhoudelijk verband te zijn, maar online op deze site kan u nalezen dat er eigenlijk wel een verband is. Dit maakt duidelijk dat 'verband' of 'relevantie' moeilijk is om op voorhand te zeggen, en we dus enkel naar de waarheid kijken van de deeluitspraken.

In dit stuk worden propositieletters en waarheidstabellen vrij snel ingevoerd in vergelijking met de snelheid van de rest van de instap. Het is echter een belangrijk stuk om verder te kunnen. Hier zal u dus misschien extra ondersteuning/materiaal moeten voor voorzien. Op het einde benadrukt u ook best nog eens de waarheidstabel van $\Rightarrow$, aangezien die soms moeilijk aanvaard wordt door de leerlingen. Denk bijvoorbeeld aan het feit dat $p \Rightarrow q$ altijd waar is, als p onwaar is. Argumenten om die waarheidstabel aannemelijker te maken kan u vinden in Subsectie 2.1.

Subsectie 4.3

In deze Subsectie gebruiken we negatieve vragen als startpunt. Deze kunnen namelijk, zelfs in de omgangstaal, voor onduidelijkheid zorgen en zijn dus een goeie aanleiding voor formalisering. Het voorbeeld van "Is 10 niet deelbaar door 3" lijkt misschien uit de lucht gegrepen, maar denk aan (wat de leerlingen nog zullen zien) "Is $\sqrt{2}$ irrationaal?" en "Is $\sqrt{2}$ niet rationaal?". Deze twee laatste vragen zien er logisch equivalent uit, en je zou dus ook hetzelfde antwoord erop verwachten.

Er waren ook andere ideeën om de nood van formalisatie van de negatie te introduceren. Ik lijst ze hier ter inspiratie.

- Een Vals dilemma is een drogredenering waarbij twee alternatieven verkeerdelijk als exhaustief worden voorgesteld. Denk bijvoorbeeld aan "Je bent ofwel mijn vriend ofwel mijn vijand.". Dit staat in contrast met de logische wet $p \vee \neg p$ in propositielogica. Een tautologie zou dus kunnen zijn "Je bent mijn vriend of niet mijn vriend.". Het contrast tussen deze twee, afhankelijk van de situatie, kan aanleiding geven tot reflectie en formalisatie van de negatie.

- Naast negatieve vragen zijn ook uitspraken met dubbele negaties soms onduidelijk. Er zijn soms zelfs uitspraken met drie of vier negaties! Door enkele dergelijke uitspraken te vergelijken kan je ook de negatie invoeren.

- Het Engels heeft zogenaamde 'tag questions', waarbij op het einde van een uitspraak bijvoorbeeld nog een ",isn't it?" komt. Daarnaast heeft het Engels geen "jawel", wat in het Nederlands gebruikt wordt om eenduidig op negatieve vragen te beantwoorden. Het vergelijken van een Engels en Nederlands gesprek met negatieve vragen kan dus ook als startpunt ontwikkeld worden. In mijn opzoekingswerk zijn 'tag questions' in het Engels echter nog niet gekend door leerlingen uit de tweede graad, dus wees u bewust van uw doelpubliek bij deze optie.

Subsectie 4.4

De drie redeneringen zijn qua logische structuur alledrie equivalent. Ze gebruiken alledrie de redeneervorm $(p \vee q) \wedge p \Rightarrow \neg q$. Die is ongeldig voor de inclusieve "of", maar geldig voor de exclusieve "of"! Bij de eerste en laatste redenering is er een inclusieve "of", bij de tweede een exclusieve. Na een klasgesprek over waarom ze soms wel en soms niet akkoord gaan met de redenering, draaien de leerlingen het blad om en vinden ze de waarheidstabel van de inclusieve "of". Ze moeten vervolgens nog eens nagaan bij welke redenering nu de inclusieve of exclusieve "of" gebruikt wordt.

Subsectie 4.5

Een moeilijkheid met de conjunctie, die ook kan voorkomen in de omgangstaal, is de negatie ervan nemen. Dit hebben we gekozen als basis van het bestudeerde gesprek. Merk op, dit stuk moet zeker na het behandelen van de "of" komen, aangezien hier de inclusieve "of" echt wel nodig is.
Denk bij dit deel voor de $\wedge$ ook aan het verschil met de omgangstaal over "en" tussen uitspraken en "en" tussen objecten. Dit staat uitgelegd in Sectie 1. Ik heb geprobeerd dit duidelijk te houden door zo snel mogelijk over te stappen op enkel een "en" tussen uitspraken.

Subsectie 4.6

We hebben meerdere opties overwogen om kwantoren in te leiden. Ik geef hier opnieuw de lijst om u misschien te inspireren.

- Je kan samen met de leerlingen 'proberen' om syllogismen te formaliseren met propositielogica. Dit zijn redeneervormen die gecategoriseerd zijn in een systeem van Aristoteles. Bij Subsectie 4.1 zijn bijvoorbeeld alle redeneringen syllogismen. Meer info kan u online over dit soort redeneringen vinden.

Het zal echter niet lukken om syllogismen te formaliseren, aangezien je een “voor alle” moet kunnen uitdrukken, wat niet lukt in propositielogica. Het is een activerende inleiding, maar vraagt wel extra ondersteuning aangezien het redelijk moeilijk is.

- In de omgangstaal wordt er losser omgesprongen met de volgorde van kwantoren. Dit komt voor in bijvoorbeeld spreekwoorden zoals “Op ieder potje past een dekseltje.”. Door spreekwoorden met omgedraaide versies te vergelijken zou je kwantoren kunnen introduceren. Het is opnieuw een mogelijkheid om te vergelijken met spreekwoorden uit een andere taal, zoals Frans of Engels.
- Voor de geavanceerde studenten kan u de kwantoren invoeren als oneindige variant van de conjunctie en disjunctie. Een uitgewerkt voorbeeld vindt u hier.

Uiteindelijk viel de keuze op het verschil tussen $\forall$ wat slaat op **echt alle**, en “alle” wat in de omgangstaal vaak slaat op “de meeste”. De reden is dat dit een erg belangrijk verschil is voor wiskundige bewijzen. Vervolgens wordt $\exists$ geïntroduceerd door het idee van een tegenvoorbeeld. Het was ook mogelijk geweest om die kwantor te introduceren met redeneringen waar “er zijn” soms “er is een redelijk deel” en soms “er is letterlijk minstens één” betekent. Dit zou dan gelijkaardig zijn aan de introductie van $\forall$, maar het geven van een tegenvoorbeeld is te belangrijk om niet aan bod te laten komen. Bij echt vergevorderde klassen zou u bij $\exists$ kunnen spreken over het verschil tussen constructieve en niet-constructieve existentiebewijzen. Bij constructieve bewijzen wordt een concreet geval geconstrueerd die de uitspraak met een $\exists$ waar maakt. Bij niet-constructieve wordt enkel aangetoond dat er een geval bestaat dat voldoet, maar wordt niet achterhaald wat het geval dan concreet is. Dit verschil kan een goeie startdiscussie opleveren, aangezien niet iedereen de tweede soort onmiddellijk zal aanvaarden.

Merk op: in deze Subsectie wordt er niet gesproken over een zogenaamd ‘domain of discourse’ of over het binden van variabelen aan de kwantor. Het is dus simpelweg de symbolen $\forall$ en $\exists$ die ingevoerd worden, en (nog) niet $\forall x \in \mathbb{R}$ of $\forall x$. Als u dit aan bod wil laten komen, zal u hiervoor zelf iets moeten maken.

Sectie 5

Deze Sectie bevat drie verdiepingsoefeningen. De eerste oefening kwam ook voor in Uitwiskeling van zomer 2021. Deze oefeningen zijn zeker geen verplichte leerstof. Voor het geven van de oefening in Subsectie 5.1 bekijkt u best eens de Subsectie 3.2 over visualisatie met waarheidsmatrices. Oefening 5.3 is gebaseerd op deze website.

7 Oplossingsleutel

Oefening 2.2

“Als de maan van kaas gemaakt is, dan is er geen paus.” is een ware uitspraak. Dit komt omdat “De maan is gemaakt van kaas.” vals is. Als p vals is in $p \Rightarrow q$, dan is $p \Rightarrow q$ sowieso waar.

1. De eerste eigenschap die we bekijken is de uitspraak $(p \Rightarrow q) \vee (q \Rightarrow p)$.

   (a) De waarheidstabel is:

p	q	$(p$	$\Rightarrow$	$q)$	$\vee$	$(q$	$\Rightarrow$	$p)$
0	0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	0	0
1	0	1	0	0	1	0	1	1
1	1	1	1	1	1	1	1	1

   (b) Die uitspraak lees je dan als “Als ik graag brownies eet, dan ben ik goed in wiskunde; of, als ik goed ben in wiskunde, dan eet ik graag brownies.”.

   (c) Die uitspraak lees je dan als “Als er veel natuurlijke getallen zijn, dan is er wereldvrede; of, als er wereldvrede is, dan zijn er veel natuurlijke getallen.”.

   (d) Als je twee willekeurige uitspraken p en q hebt, die misschien niets met elkaar te maken hebben, dan is het toch zo dat er een implicatie is tussen de twee. Zonder de waarheidswaarden van p en q weet je niet in welke richting, maar er zal altijd één van de twee uitspraken de andere impliceren.

2. De tweede eigenschap die we bekijken is de uitspraak $p \Rightarrow (q \Rightarrow p)$.

   (a) De waarheidstabel is:

p	q	p	$\Rightarrow$	$(q$	$\Rightarrow$	$p)$
0	0	0	1	0	1	0
0	1	0	1	1	0	0
1	0	1	1	0	1	1
1	1	1	1	1	1	1

   (b) We hebben p = “De aarde is rond.” en q =“Het regent.”. Wat betekent $p \Rightarrow (q \Rightarrow p)$ dan? Die uitspraak lees je dan als “Als de aarde rond is, dan is het zo dat als het regent, de wereld aarde is.”.

(c) We nemen bijvoorbeeld $q =$"Ik eet graag kaas." Die uitspraak lees je dan als "Als de aarde rond is, dan is het zo dat als ik graag kaas eet, de aarde rond is.".

(d) Als je een ware uitspraak p hebt, dan wordt die geïmpliceerd door iedere uitspraak.

3. De laatste eigenschap die we bekijken is de uitspraak $(q \wedge \neg q) \Rightarrow p$.

(a) De waarheidstabel is:

p	q	$(q$	$\wedge$	$\neg$	$q)$	$\Rightarrow$	p
0	0	0	0	1	0	1	0
0	1	1	0	0	1	1	0
1	0	0	0	1	0	1	1
1	1	1	0	0	1	1	1

(b) We nemen $p =$"Ik ben rijk." en voor $q =$"Het regent.". Die uitspraak lees je dan als "Als het regent en niet regent, dan ben ik rijk.". Als we nu $q =$"$1 + 1 = 2$" nemen, dan wordt het "Als $1 + 1$ wel en niet gelijk is aan 2, dan ben ik rijk.". Met andere woorden, we kunnen de uitspraak telkens lezen als "Als er een contradictie is, dan ben ik rijk.".

(c) Uit een contradictie volgt alles. Je kan voor p gelijk welke uitspraak kiezen; als er ergens een contradictie gevonden wordt, dan is p waar. Dit wordt ook wel *the principle of explosion* genoemd, aangezien een contradictie explodeert en alles waar maakt.

Oefening 5.1

Per uitspraak zijn er drie mogelijkheden: T, U of F. Als we twee uitspraken beschouwen, zijn er dus drie mogelijkheden voor de eerste uitspraak **en** drie voor de tweede. In het totaal zijn er dus $3 \cdot 3$ mogelijkheden.

Conjunctie

De uiteindelijke waarheidsmatrix is

$\wedge$	T	U	F
T	T	U	F
U	U	U	F
F	F	F	F

Deze bekom je door de volgende redeneringen.

- Als er één van de twee uitspraken vals is in $p \wedge q$, dan weten we al zeker dat $p \wedge q$ vals moet zijn. Het maakt dan niet meer uit welke waarde de andere uitspraak heeft. Daarom zijn de onderste rij en de rechtse kolom overal F.
- Voor de waarden van $T \wedge U$ en $U \wedge T$ hangt de waarde af van die U. Moest de 'Unknown' uitspraak waar zijn, dan zou de conjunctie waar zijn; moest hij vals zijn, dan zou de conjunctie vals zijn. Dit geeft ons dus U als resultaat. Voor $U \wedge U$ kan je op een gelijkaardige manier beredeneren dat de waarheidswaarde opnieuw U moet zijn.

Disjunctie

De uiteindelijke waarheidsmatrix is

$\vee$	T	U	F
T	T	T	T
U	T	U	U
F	T	U	F

Deze bekom je door de volgende redeneringen.

- Als er één van de twee uitspraken waar is, dan weten we al zeker dat hun disjunctie waar moet zijn. Het maakt dan niet meer uit wat de andere uitspraak van waarde heeft. Daarom zijn de bovenste rij en de linkse kolom overal T.

- Voor de waarden van $F \vee U$ en $U \vee F$ hangt de waarde af van die U. Moest de 'Unknown' uitspraak waar zijn, dan zou de disunctie waar zijn; moest hij vals zijn, dan zou de disjunctie vals zijn. Dit geeft ons dus U als resultaat. Voor $U \vee U$ kan je ook het resultaat niet weten door die 'Unknown', dus weer is het resultaat U.

Vervolgens controleren we of onze bovenstaande definitie zorgt dat die logische wet gevolgd wordt. Hiervoor maak je dus onderstaande waarheidstabel. Het is hier handiger om met een waarheidstabel in plaats van een waarheidsmatrix te werken. Aangezien de groene kolommen overeen komen, klopt het dus.

p	q	p	$\vee$	q	$\neg$	$(\neg$	p	$\wedge$	$\neg$	$q)$
T	T	T	T	T	T	F	T	F	F	T
T	U	T	T	U	T	F	T	F	U	U
T	F	T	T	F	T	F	T	F	T	F
U	T	U	T	T	T	U	U	F	F	T
U	U	U	U	U	U	U	U	U	U	U
U	F	U	U	F	U	U	U	U	T	F
F	T	F	T	T	T	T	F	F	F	T
F	U	F	U	U	U	T	F	U	U	U
F	F	F	F	F	F	T	F	T	T	F

Implicatie

De uiteindelijke waarheidstabel is

$\Rightarrow$	T	U	F
T	T	U	F
U	T	U	U
F	T	T	T

Voor het oplossen van deze vraag zijn er verschillende opties. Je kan intuïtief redeneren op de betekenis van U, maar dat zal hier iets moeilijker zijn. Je kan ook de waarheidstabel van $\neg p \vee q$ opstellen en rechtstreeks de waarheidsmatrix van $\Rightarrow$ aflezen. Ik toon ook een andere optie (die ik uitleg in Sectie 3.2):

Hieronder staat de waarheidsmatrix van $p \vee q$, en die formule lijkt sterk op $\neg p \vee q$.

$\vee$	T	U	F
T	T	T	T
U	T	U	U
F	T	U	F

Om de matrix van $\neg p \vee q$ te verkrijgen, moeten we nog de negatie van de eerste uitspraak erin verwerken. Momenteel staan in de meest linkse kolom de waarheidswaarden van p. Als we eerst nog de negatie nemen van p, vooraleer we kijken naar de $\vee$, dan moeten we de rijen manipuleren. In de rij waar p de waarde U heeft, maakt de negatie niet uit. Deze rij blijft dus gewoon hetzelfde. In de eerste rij heeft p waarde T, en heeft $\neg p$ waarde F. Om de eerste rij van de waarheidsmatrix van $\neg p \vee q$ in te vullen (waar p de waarde T heeft), moet je dus de rij overnemen in bovenstaande matrix van $p \vee q$ waar p de waarde F heeft. Op die manier wisselen de bovenste en onderste rij dus effectief om, als we de negatie van de eerste uitspraak nemen. Zo krijgen we de matrix van $\Rightarrow$.

Praktische toepassing

We hebben $p =$"de persoon rookt" en $q =$"de persoon is gestorven". Als niet geweten is of de persoon rookt, dan heeft p als waarheidswaarde U. Kijk dus in de waarheidsmatrix van $\wedge$ waar p waarde U heeft.

$\wedge$	T	U	F
T	T	U	F
U	U	U	F
F	F	F	F

Je ziet dat daar voor geen enkele waarde van q de waarde T voorkomt. De mensen waarvan je niet weet of ze roken, zullen dus niet meegenomen worden in de dataset.

Oefening 5.2

1. Hiervoor moeten we een waarheidstabel maken. De Tabel vind je hieronder.

p	q	$(q$	$\Rightarrow$	$q)$	$\wedge$	p
0	0	0	1	0	0	0
0	1	1	1	1	0	0
1	0	0	1	0	1	1
1	1	1	1	1	1	1

   Aangezien de groene kolommen gelijk zijn, is het een identieke operatie.

2. Je zou hier ook een waarheidstabel kunnen maken. Het is echter zo dat deze operatie dezelfde is als de vorige. Immers, $q \Rightarrow q$ heeft altijd dezelfde waarde als $\neg q \vee q$. Dus $(q \Rightarrow q) \wedge$ voor p zetten zal altijd dezelfde waarde hebben als $(q \vee \neg q) \wedge$ voor p zetten. Aangezien de vorige operatie identiek is, is deze operatie ook identiek.

3. $\top \wedge$ voor een uitspraak zetten is inderdaad een identieke operatie. Dit kon je daarnet al zien, aangezien $q \Rightarrow q$, en ook $\neg q \vee q$, altijd waar is. De waarheidstabel staat hieronder.

p	$\top$	$\wedge$	p
0	1	0	0
1	1	1	1

4. $\bot \vee$ voor een uitspraak zetten is ook een identieke operatie. Je kan dit snel intuïtief inzien: de $\bot$ zal de $\vee$ nooit waar maken, het is enkel de p die daarop volgt die het resultaat bepaalt. Je ziet het ook in de waarheidstabel hieronder.

p	$\bot$	$\vee$	p
0	0	0	0
1	0	1	1

Oefening 5.3

1. De Tabel vind je hieronder.

p	q	p	$\Rightarrow$	q	$\neg$	p	$\vee$	q
0	0	0	1	0	1	0	1	0
0	1	0	1	1	1	0	1	1
1	0	1	0	0	0	1	0	0
1	1	1	1	1	0	1	1	1

2. De Tabel vind je hieronder.

p	q	p	$\vee$	q	$(p$	$\Rightarrow$	$q)$	$\Rightarrow$	q
0	0	0	0	0	0	1	0	0	0
0	1	0	1	1	0	1	1	1	1
1	0	1	1	0	1	0	0	1	0
1	1	1	1	1	1	1	1	1	1

3. De wetten van De Morgan zeggen dat $p \vee q$ altijd dezelfde waarde heeft als $\neg(\neg p \wedge \neg q)$. Dit betekent dat $\vee$ uitgedrukt kan worden met $\neg$ en $\wedge$.

4. Merk op in de waarheidstabel van $|$ dat wanneer zowel p als q vals zijn, dan is $p \mid q$ waar. En als zowel p als q waar zijn, dan is $p \mid q$ vals.

p	q	$p \mid q$
0	0	1
0	1	1
1	0	1
1	1	0

   Dat lijkt dus al een beetje op de negatie, maar we zitten nog met die q die niets te maken heeft met p. We willen onze observatie gebruiken om $\neg p$ uit te drukken, maar daarvoor moeten we nu een q vinden die altijd dezelfde waarde heeft als p. Neem dus voor q simpelweg de uitspraak p en dan krijg je de volgende Tabel.

p	$\neg$	p	p	$\mid$	p
0	1	0	0	1	0
1	0	1	1	0	1

   We kunnen dus inderdaad $\neg p$ uitdrukken als $p \mid p$.

5. In de uitleg van $|$ stond er dat $p \mid q$ eigenlijk de negatie is van $p \wedge q$. Dit betekent dat $p \mid q$ altijd dezelfde waarheidswaarde heeft als $\neg(p \wedge q)$. Dat lijkt al redelijk goed op $p \wedge q$; we moeten alleen nog de $\neg$ wegwerken. En de $\neg$ heb je daarnet uitgedrukt met $|$. Hieronder staat een overzicht van de overgangen.

$$\begin{aligned}
p \mid q &\text{ heeft altijd dezelfde waarde als } && \neg(p \wedge q)\\
\neg(p \mid q) &\text{ heeft altijd dezelfde waarde als } && \neg\neg(p \wedge q)\\
\neg(p \mid q) &\text{ heeft altijd dezelfde waarde als } && (p \wedge q)\\
(p \mid q) \mid (p \mid q) &\text{ heeft altijd dezelfde waarde als } && (p \wedge q)
\end{aligned}$$

Eerst schrijven we $p \mid q$ als $\neg(p \wedge q)$. Vervolgens zetten we voor die beide uitspraken een $\neg$. Daarna valt de dubbele negatie in de rechteruitspraak weg. Tot slot vervangen we in de linkeruitspraak de $\neg$ door de uitspraak met $|$ van de vorige vraag. Zo hebben we dus een uitspraak gevonden met $|$ die altijd dezelfde waarde heeft als $p \wedge q$, namelijk de uitspraak $(p \mid q) \mid (p \mid q)$.

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