FACULTY OF SCIENCES

Jackiw-Teitelboim (super)gravity, topological gauge theory and EOW branes

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A dissertation submitted to Ghent University in partial fulfilment of the requirements for the degree of Master of Science in de Fysica en de Sterrenkunde

Academic year: 2022 - 2023

GHENT UNIVERSITY

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mémé Clara

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Words of gratitude

After some intense months of writing this thesis, I am finally in a place to reflect on the work that has been done so far. It is safe to say that the result of this thesis would not have been possible without the help and encouragement of those around me, all of whom I would like to share my gratitude.

On a professional level, I would like to start credit where credit is due and thank my promoter *Thomas Mertens*. His input and encouragement cannot be understated, and I look up to him in many ways. He has been my professor for the course on Quantum Field Theory in the first semester of my first Master's year. In a semester full of numerical modelling and solid state physics, his course was a breath of fresh air. When I showed interest in his subject as a prospective Master's thesis, his enthusiasm about his field during our private conversation was inspiring.
A broad part of this thesis involved reading through the (often vast and daunting) literature and making it my own. Almost all of the literature involved in this thesis is contributed directly by him. It really amazes me how much he has already contributed to his field over a relatively short period of time. I would like to thank him for giving me the opportunity to pursue my passions in the years to come and for putting his trust in me from early on.
The original contributions of this thesis are the fruit of his encouragement throughout the year. In this journey, it has been a pleasure to collaborate with *Francesca Mariani*. We could work together on a shared topic and our shared inputs and perspectives have greatly influenced the result of this work in many ways. It really demonstrates that science is at its core a social activity, where scientific results are often the result of the synergy between different perspectives and opinions.
I would also like to thank a handful of professors that are most dear to me. Starting with *Didar Dobur*; I would like to thank her for her enthusiasm and encouragement throughout the years. She has been a true source of inspiration and a joy to talk to. Her persona did not facilitate the choice between theory and experiment. Although I have chosen the path towards theoretical physics, my passions for particle physics are still as strong as ever and I will always look back to her courses with a certain nostalgia and joy.
I would also like to thank *Michal Heller*. He has been my professor for specialized elective courses in the fields of holography and quantum gravity. His passion for the field inspired me to continue in this direction. The result is this thesis. Most importantly, his visions and perspectives on science have been really refreshing. In particular, he has encouraged his students from the start to dare to think for themselves, which is perhaps aside from the content-related material of his course, one of the most important lessons.

Op persoonlijk niveau zou ik graag mijn naasten willen bedankten. In het bijzonder mijn ouders die mij door dik en dun zullen blijven steunen. De impact van die steun op deze thesis en andere uitdagingen kan niet genoeg benadrukt worden. Zo ook is de steun van mijn zus onmiskenbaar.
Langs de andere kant heb ik gedurende mijn opleiding ook vele mensen moeten afgeven. De opstapeling van verliezen op korte tijd heeft emotioneel een sterke tol geëist. De aanwezigheid van mijn ouders heeft ook in deze verwerking een onmiskenbare rol gespeeld.
In het bijzonder zou ik mijn mémé Clara willen herinneren in dit werk. Van jongs af heeft haar aanmoediging mij telkens weer een boost gegeven om steeds het beste van mezelf te geven. Haar onconditionele liefde en vertrouwen heeft geleid tot wie ik ben en waar ik nu sta. Het valt me dan ook zwaar dat ik dit laatste

moment van afstuderen niet met haar kan delen. Hoe dan ook heb ik mij voor deze thesis keihard ingezet om haar nog een laatste keer trots te maken.
Ik zou voorts ook mijn vrienden willen bedanken, waarvan ook hun aanwezigheid soms te vaak vanzelfsprekend lijkt. Hen op geregelde basis ontmoeten zorgde voor de nodige afwisseling en maakte het schrijven van deze thesis des te aangenamer. In het bijzonder wil ik Eben en Helena vernoemen, waarmee ik vele van mijn fijnste herinneringen koester.
Ten slotte wil ik ook Jeroen bedanken, die mij door vele ups en downs heeft geholpen dit jaar. De wetenschap dat ik telkens op hem kon terug vallen heeft de kwaliteit van deze thesis in aanzienlijke mate beïnvloed.

I would also like to thank Emmy, who has invested her time in reviewing a first draft of this work and assisted me with linguistic advice. To be clear, due to a shortage of time near the end of this semester, this work is certainly not free of grammatical errors or unnecessary typos in the equations, for which I apologize beforehand.

Finally, I would also like thank you, the reader, for taking the time to go through (some parts) of this extensive work. I hope you enjoy it at least one bit as much as I have enjoyed investigating and writing it.

June 2023,
Andreas.

Plato: *The Allegory of the Cave*, from The Republic[1]

Socrates And now, I said, let me show in a figure how far our nature is enlightened or unenlightened: Behold! human beings living in a underground cave, which has a mouth open towards the light and reaching all along the cave; here they have been from their childhood, and have their legs and necks chained so that they cannot move, and can only see before them, being prevented by the chains from turning round their heads. Above and behind them a fire is blazing at a distance, and between the fire and the prisoners there is a raised way; and you will see, if you look, a low wall built along the way, like the screen which marionette players have in front of them, over which they show the puppets.

Glaucon I see.

Socrates And do you see, I said, men passing along the wall carrying all sorts of vessels, and statues and figures of animals made of wood and stone and various materials, which appear over the wall? Some of them are talking, others silent.

Glaucon You have shown me a strange image, and they are strange prisoners.

Socrates Like ourselves, I replied; and they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave?

Glaucon True, he said; how could they see anything but the shadows if they were never allowed to move their heads?

Socrates And of the objects which are being carried in like manner they would only see the shadows?

Glaucon Yes, he said.

Socrates And if they were able to converse with one another, would they not suppose that they were naming what was actually before them?

Glaucon Very true.

Socrates And suppose further that the prison had an echo which came from the other side, would they not be sure to fancy when one of the passers-by spoke that the voice which they heard came from the passing shadow?

Glaucon No question, he replied.

Socrates To them, I said, the truth would be literally nothing but the shadows of the images.

Glaucon That is certain.

. . .

[1] This fragment is taken from the Benjamin Jowett translation (Vintage, 1991)

List of Abbreviations and Acronyms

JT	Jackiw-Teitelboim gravity
SYK	Sachdev-Ye-Kitaev
EOW	End-of-the-world brane
QFT	Quantum field theory
CFT	Conformal field theory
(A)dS	(Anti-)de Sitter
GHY	Gibbons-Hawking-York boundary term
SUGRA	Supergravity
SUSY	Supersymmetry
EOM	Equation of motion
RN	Reissner-Nordström solution
BH entropy	Bekenstein-Hawking entropy
EH action	Einstein-Hilbert action
EM action	Einstein-Maxwell action
GR	General Relativity
TFD	Thermofield double state
OTOC	Out-of-time-order-commutator
TOC	Time-ordered-commutator
YM2	two-dimensional Yang-Mills theory
BF	Background Field theory
WZW	Wess-Zumino-Witten model

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Abstract

The issues regarding the non-renormalizability of the Einstein-Hilbert action can be averted in theories of two-dimensional gravity, where the Newton's constant is dimensionless and does not set the scale for new physics. By formulating the theory in the context of holography, we can study quantum solutions of gravity in this setting. A particularly attractive model is 1+1d Jackiw-Teitelboim gravity (JT), which captures the near-horizon region of a large class of higher-dimensional nearly extremal black holes.
The amount of exact solvability in this model is unprecedented. In particular, its quantum gravitational solutions are exact to all orders in perturbation theory. This allows to make real quantitative predictions to long-lasting problems. Most notably, coupling the theory to matter, it is able to shed a new light on the Hawking information paradox. By including non-perturbative corrections to the gravitational path integral, the result is a completely unitary Page curve of the entropy of the Hawking radiation.

Important objects to model the black hole microstates in the evaporation process are the End-of-the-World branes (EOW). These act as probe particles that follow geodesic trajectories between two points on the boundary of a hyperbolic universe. Other topological solutions of EOW branes have also been obtained at the neck of a Euclidean wormhole. The explicit quantum amplitudes of these objects have been obtained in the boundary-particle formalism of JT gravity so far.
In this thesis, we complement these computations by providing an alternative perspective on these quantum amplitudes in the framework of a $\mathfrak{sl}(2,\mathbb{R})$ topological BF gauge theory. We follow the perspective of the quantum description of JT gravity in terms of the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ of positive group elements. By identifying the free particle action with a Wilson operator insertion, we obtain precisely the same quantum amplitudes of the boundary-particle formalism in the geodesic approximation. This perspective neatly unites the two seemingly different answers for different topologies.

We further exploit this perspective to formulate, for the first time, EOW branes in the context of $\mathcal{N}=1$ JT supergravity (SUGRA). In particular, we work out the relevant geodesic differential geometry in superspace and obtain an explicit expression for the extrinsic supercurvature in terms of a properly defined covariant derivative. Using these definitions, we write down the action of EOW branes in superspace. Using reasoning similar to the bosonic case, we solve the explicit quantum amplitudes in the group theoretical $\mathfrak{osp}(1|2,\mathbb{R})$ BF formulation of $\mathcal{N}=1$ JT SUGRA. The result again critically depends on the topology, where now also the periodicity of the fermionic sector has to be taken into account. We find that for periodic fermions, the spurious UV divergence of the bosonic case seizes in applications of supergravity. Thus, the incorporation of supersymmetry cures the diverges of the bosonic case.
We end with a look towards the future for $\mathcal{N}=2$ JT supergravity.

Abstract

De problemen met betrekking tot de renormalisatie van de Einstein-Hilbert actie kunnen afgewend worden in modellen van tweedimensionale zwaartekracht. In de context van holografie kunnen we kwantum oplossingen van zwaartekracht bestuderen in deze opstelling. Een bijzonder vruchtbaar model is 1+1d Jackiw-Teitelboim zwaartekracht (JT), dat de dynamica nabij de horizon beschrijft van een brede waaier aan bijna extremale zwarte gaten in ons universum.
De exacte oplosbaarheid van dit model is ongezien. In het bijzonder kan men de kwantummechanische amplitudes exact formuleren tot op elke orde in perturbatietheorie. Dit laat toe om onopgeloste problemen van kwantumgravitatie te herinterpreteren. Door het model te koppelen aan materie lost men de informatieparadox van Hawking op door niet-perturbatieve Euclidische wormgaten in de gravitationele padintegraal te betrekken. Het resultaat is een volkomen unitaire Page curve die de entropie van de Hawking straling beschrijft.

Belangrijke objecten om deze evaporatie mee te modelleren zijn EOW branen die de microtoestanden van het zwart gat beschrijven. Deze dienen als testdeeltjes die geodetische paden beschrijven tussen twee ankerpunten aan de rand van een hyperbolisch universum. Andere topologieën van deze EOW branen zijn ook reeds bestudeerd, zoals een EOW braan dat eindigt op de nek van een Euclidisch wormgat. Deze kwantumamplitudes zijn origineel beschreven in het vrije-deeltjes model van JT zwaartekracht.
In deze thesis werpen we een nieuwe blik op de kwantisatie van EOW branen in de context van een topologische $\mathfrak{sl}(2,\mathbb{R})$ BF ijktheorie. We volgen de kwantumbeschrijving in termen van de halfgroep $\mathrm{SL}^+(2,\mathbb{R})$. Door de vrije-deeltjes actie te identificeren met een Wilson operator insertie, kunnen we de bekomen kwantumamplitudes identificeren met deze uit het vrije-deeltjesmodel in de geodetische approximatie. Dit perspectief verbindt de verschillende topologische oplossingen in één gezamenlijke beschrijving.

We extrapoleren dit perspectief om voor het eerst EOW branen in de context van $\mathcal{N}=1$ JT superzwaartekracht te definiëren en te beschrijven. In het bijzonder werken we de relevante geodetische supermeetkunde uit om een definitie van de superextrinsieke kromming te bekomen in termen van een gepaste covariante afgeleide. Met deze definities kunnen we uiteindelijk de actie van een EOW braan in de supermeetkunde neerschrijven. Gebruik makende van de groepentheoretische beschrijving van $\mathcal{N}=1$ JT superzwaartekracht in termen van een $\mathfrak{osp}(1|2,\mathbb{R})$ topologische ijktheorie, lossen we de resulterende kwantumamplitudes op. Het resultaat hangt opnieuw af van de topologie, maar eveneens van de periodiciteit van de fermionische sector rond de cirkel. We concluderen dat de periodieke sector zich regulier gedraagt in het UV, in tegenstelling tot het bosonische geval.
We sluiten af met een blik naar de toekomst in de context van $\mathcal{N}=2$ supergravitatie.

Chapter 0

Content and Overview

"On the walls of the cave, only the shadows are the truth."

Plato, The Allegory of the Cave

0.1 Orientation and motivation

Since the development of the Standard Model [1][2] as a spontaneously broken $SU(2)_L \otimes U(1)_Y \otimes SU(3)_c$ gauge theory, physicists have been able to explain and predict almost all observable processes in experimental high-energy experiments with unprecedented precision. The theory unifies the electromagnetic, weak, and strong interactions in the framework of a local, Lorentz-invariant Quantum Field Theory (QFT).
It is rather disturbing that there exists a consistent description of these three fundamental forces at the level of the path integral, while we still lack a basic framework to describe the most familiar gravitational interaction Quantum Mechanically.

The main difficulties in defining a UV-complete quantum theory of gravity can be categorized in technical and conceptual problems. The former includes the non-renormalizabilty of the Einstein-Hilbert action in $D > 2$. The latter are conceptual problems about how to even think about a quantized theory of gravity. The Weinberg-Witten theorem [3] rules out the possibility that the graviton emerges as a low-energy degree of freedom in a local, Lorentz-invariant Quantum Field Theory, analogously to how pions are emergent degrees of freedom in QCD. More specifically, the theorem states that massless spin-2 particles, such as gravitons, cannot be produced as asymptotic states in an underlying QFT theory with interacting particles [3]. Inspired by the fact that quantum gravity has no local observables that do not reach the boundary, the majority of modern approaches to Quantum Gravity are being developed in the framework of the Anti-de Sitter/Conformal Field Theory duality, often called AdS/CFT [4] for short. In this framework, spacetime itself is emergent and is an approximate, collective description of some underlying degrees of freedom in a lower-dimensional theory without gravity.

This is a concrete realization of a holographic universe envisioned by 't Hooft and Susskind in the early '90s [5][6].

The starting point of this duality was hinted in the search for a consistent microscopic quantum description of black holes. After the discovery of black hole evaporation by S.W. Hawking [7], it was quickly realized that black holes are thermal systems, whose temperature leads to a thermal entropy that scales linearly with the area of the Horizon in Planck units [8]. Quantum Mechanically, this entropy should, of course, be related to the number of degrees of freedom of the black hole system. However, since the black hole entropy scales with the boundary area rather than the volume, this suggests a holographic description of black holes [5][6] where a theory of quantum gravity must secretly live in lower dimensions than our observed spacetime.
The AdS/CFT correspondence was the first concrete realization of such a duality by J. M. Maldacena in the context of D-brane string theory [4][9]. It motivates that there exists an exact relationship between any theory of quantum gravity in asymptotically AdS spacetimes and ordinary CFTs without gravity on a lower dimensional hypersurface at the asymptotic boundary. E. Witten [10] proposed the concept of Witten diagrams, and introduced an operational dictionary for this duality, which was independently formulated by Gubser, Klebanov and Polyakov [11]. It states that any physical (gauge-invariant) quantity that can be computed in one theory can also be computed in its dual description.

A strategy to attack the first class of technical problems concerning renormalizability, would be to work in lower dimensions. In particular, when $D = 2$, the Newtons constant is dimensionless and does not set the scale of new physics. By formulating the theory in the framework of holography, we furthermore have a preferred anchoring point to deal with the second class of conceptual problems.
1+1d gravity is special however, since the Einsteins equations are trivially satisfied, and the Einstein-Hilbert action is topological. Additionally coupling the theory to matter, the equations of motion can only describe the ground state configurations. This is related to the backreaction problem raised in [12], which makes holography in 1+1d in a sense more challenging than its higher dimensional cousins. To go beyond and investigate generic matter excitations, Jackiw-Teitelboim gravity (JT) has proven to be a particularly fruitful gravitational model. It belongs to the class of 2d dilaton gravity models, whose dynamical degrees of freedom are the 2d metric field $g_{\mu\nu}$ and a scalar dilaton field Φ. The JT model in particular is characterized by a linear dilaton potential $U(\Phi) = -2\Phi$, whose corresponding Euclidean action is

$$I_{JT}[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\,\sqrt{g}\Phi(R+2) \;-\; \frac{1}{8\pi G_N}\oint\limits_{\partial\mathcal{M}} d\tau\,\sqrt{h}\Phi_{bdry}(K-1). \tag{1}$$

The last term is the familiar Gibbons-Hawking-York (GHY) boundary term, introduced in this context to yield a proper variational problem for an action with second-order derivatives of the metric. This action was first proposed independently by R. Jackiw [13] and C. Teitelboim [14] in the 1980s, but was rediscovered in a holographic context in [15], where the classical equations of motion of the dilaton and the metric were derived. Variation with respect to the dilaton field fixes all metric solutions to different coordinate frames of an AdS_2-manifold, while variation with respect to the metric yields equations of motion of the dilaton in terms of (non-zero) matter sources. This procedure fixes the backreaction of the metric in terms of the variation of the

dilaton.
Next to being a valuable model to study lower-dimensional quantum gravity, the model describes the spherical symmetric s-wave sector in the near-horizon region of nearly extremal higher-dimensional black holes. In particular, it captures the near-horizon throat of asymptotically flat nearly extremal higher-dimensional black holes relevant in our universe.

In the limit of perfect AdS_2, the system exhibits an exact time reparameterization symmetry, which arises as the asymptotic symmetry of the AdS_2-manifold. Since the JT model keeps the leading order correction away from AdS_2 by the presence of the dilaton, this symmetry is broken down to the isometry $\mathrm{SL}(2,\mathbb{R})$ subgroup of the AdS_2-manifold, both spontaneously (by the presence of the asymptotic boundary) and explicitly (by UV effects). The corresponding (pseudo-) Goldstone bosons are the reparameterization modes of the thermal circle $f \in \mathrm{diff}(S_1)$, which can be thought of as boundary gravitons. The simplest local action that measures these boundary modes in the breaking pattern $\mathrm{diff}(S_1) \to \mathrm{SL}(2,\mathbb{R})$ and that is still invariant under the isometry subgroup $\mathrm{SL}(2,\mathbb{R})$, is given by the Schwarzian boundary action:

$$S = -C \int dt \left\{ \tan\left(\frac{\pi}{\beta} f\right), t \right\}, \qquad \{F, t\} \equiv \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 . \tag{2}$$

It was shown in [16] and [17], both in real time and at the level of the Euclidean action respectively, that due to the presence of the GHY boundary term in the JT action, one indeed recovers the Schwarzian boundary action. The backreaction of the dilaton on injected matter controls the shape of the boundary curve of the AdS_2 universe. Therefore, the breaking of pure AdS_2 by the presence of the dilaton in turn breaks the exact conformal reparameterization symmetry of the boundary curve. The quantity C quantifies the amount of conformal symmetry breaking. This resolves the tension surrounding the backreaction problem raised in [12], by deforming the AdS_2/CFT_1 correspondence into a deformed Nearly AdS_2/ Nearly CFT_1 correspondence. There is a large degree of universality in the quantum description of JT gravity, since it is ultimately specified completely in terms of the specific symmetry breaking pattern $\mathrm{diff}(S_1) \to \mathrm{SL}(2,\mathbb{R})$.

This pattern of symmetry breaking was widely appreciated among the high-energy community since it is related to a certain limit of a class of 0+1d Sachdev-Ye-Kitaev models (SYK). These are Quantum Mechanical systems of N 0+1d Majorana fermions satisfying the Clifford algebra, with random all-to-all q-fermion couplings. This was first studied by Sachdev and Ye in 1993 [18], and rediscovered in a series of seminal talks by Kitaev in 2015 [19]. Due to the random couplings, the SYK model can be formally identified with an ensemble of random Hamiltonians. In particular, one can study the correlation functions in the limit of large N using an effective field description of the Schwinger-Dyson equations. In the low-energy limit, the correlation functions gain a conformal reparameterization symmetry of the thermal circle, which is explicitly broken down to $\mathrm{SL}(2,\mathbb{R})$ by UV effects. The leading order action describing the small fluctuations is again found to be the Schwarzian action.
In this sense, the low-energy, large N limit of 0+1d SYK models can be thought of as the boundary holographic description of 1+1d JT gravity.

The amount of exact solvability of many observables in JT quantum-gravity beyond leading order in perturbation theory, is unprecedented. In particular, using localization arguments, Witten and Stanford proved that the one-loop Euclidean path integral is exact to all orders in perturbation theory in [20]. The bulk-boundary correlators of the holographic dictionary were first written in perturbation theory in [17], and later solved exactly in [21], using a suitable double-scaled limit of the Virasoro algebra. The results can be summarized elegantly in a set of pictorial rules.

Another approach has been made in [22], and independently in [23], by formulating JT gravity in its first-order BF formulation. This is a topological gauge theory involving the $\mathfrak{sl}(2,\mathbb{R})$ gauge algebra for case of JT gravity. Although the algebra is fixed on shell to $\mathfrak{sl}(2,\mathbb{R})$, the transition to the quantum description requires the precise exponentiation of the global gauge group. This involves some additional subtleties since the gravitational density of states of JT gravity does not match with the normalization of the global $\mathrm{SL}(2,\mathbb{R})$ representation matrices. In particular, this density violates the classical exponential scaling at large energies. The two approaches of [22] and [23] arrive at the correct density of states by using a different global gauge group; the former uses the BF description of the subsemigroup of positive group elements $\mathrm{SL}^+(2,\mathbb{R})$, while the latter used an $\mathbb{R}$ extension of the universal covering group $\tilde{\mathrm{SL}}(2,\mathbb{R})$. More evidence was provided in favour of the former [24] since this choice automatically excludes singular metrics in the path integral. By treating the boundary bilocal operators in the Schwarzian perspective as boundary-anchored Wilson lines, both studies were able to obtain the same diagrammatic rules using techniques of 2d Yang-Mills theory.
Since the bulk is topological, it is independent of the metric. This leads to a holographic description in terms of a particle on the $\mathrm{SL}(2,\mathbb{R})$ group manifold. Under constrained conditions does this holographic description reduce to the Schwarzian description of JT gravity. The path that we will follow in this thesis is the first order description of JT gravity in terms of a constrained $\mathrm{SL}^+(2,\mathbb{R})$ topological BF gauge theory.
Solving the gravitational path integral and holographic correlation functions exactly to all orders are tremendous achievements in a gravitational context, where appropriately defining the gravitational path integral is notoriously difficult, and where correlations functions are usually approximated by semi-classical gravity. Before the discovery of the holographic properties of the JT model, the only completely solvable models of gravity were topological in nature. The explicit dynamics on the Nearly AdS_2-boundary in the JT context allows us to phrase and calculate open problems in quantum gravity, and sometimes even solve them exactly.

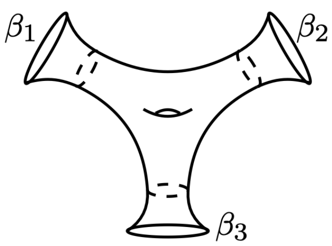


Figure 1: Partition function of a hyperbolic Euclidean AdS_2-universe of genus 1, with three boundary regions of inverse temperature β_i [25].

The amount of exact solvability of JT gravity allows to deduce that JT black holes [17] saturate the maximal bound on chaos [26], study gravitational shockwaves at the black hole horizon and virtual black hole intermediate states [27], et cetera.
One of the main unanswered questions in black holes physics is the information paradox, raised by Hawking in the 70s [28]. By doing Quantum Field Theory in a curved spacetime, black holes have been found to evaporate by emitting radiation [7]. This was quickly found to be inconsistent with quantum unitarity. In the context of holography, this should be resolved since the dual quantum mechanical description is always unitary.

The tension manifests itself roughly in two areas.

Firstly, since the density of states of pure JT gravity is found to be continuous, the entropy in the micro-canonical ensemble is infinite, and information can be lost inside a black hole. Indeed, in Quantum Mechanics, a continuous spectrum is usually associated to a non-compact target space; but in a 0+1d boundary theory, there is no spatial dimension, and the spectrum can only be continuous if it is infinite. It was readily realized that a continuous spectrum can arise from pure states if the boundary theory is a random matrix ensemble in the context of SYK [29]. In a seminal paper by Saad, Shenker and Stanford [30], it was argued that JT gravity is indeed a matrix integral, by considering a non-perturbative genus expansion of multi-boundary universes, connected by non-trivial topologies (Euclidean wormholes, see figure 0.1). This conjecture was made by realizing that Mirzakhani's topological recursion relation matches order by order with the corresponding recursion relation of random matrices for the spectral density of JT gravity.

Secondly, coupling the theory to matter, black holes have been found to evaporate in the presence of absorbing boundary conditions at the boundary. If the evaporation is unitary and the initial state is pure, the black hole microstates can only purify as much as their own entropy. The von Neumann entropy of the radiation grows steadily during evaporation, due to the entanglement with the interior black hole partners. On the other hand, the entropy of the black hole itself decreases while evaporating, following roughly the quasi-static decreasing Bekenstein-Hawking entropy. The Page curve is the minimum of these two entropies, and describes a unitary evaporation process. At the Page time, the entropy description changes from the fine-grained von Neumann entropy to the coarse-grained Bekenstein-Hawking entropy contribution. In a series of papers [31][32], a new rule was formulated to find a unitary Page curve for the entropy of the radiation by adding island regions behind the black hole horizon in the standard holographic prescription [33]. Later in 2019, the famous west [34], and east coast [35] papers, named after their respective university in Stanford and Princeton, provided a gravitational proof of this new rule by incorporating higher topologies in the gravitational path integral calculation. One considers n replicas of the original black hole and calculates the Rényi entropy $S_R = \text{Tr}(\rho^n)$ of the density matrix ρ. By putting $n = 1$, one recovers the original fine-grained von Neumann entropy. Before the analog of the Page time, the dominant topology in the Rényi entropy consists of n disconnected copies. After the Page time, the dominant topology is connected, with an n-boundary Euclidean replica wormhole, connecting all the different replicas.

0.2 Goals of this thesis

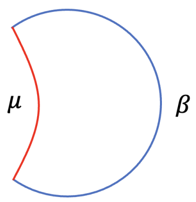
In the last paragraph, it is mentioned how the inclusion of Euclidean wormholes in the semiclassical entropy calculation accounts for a decreasing slope in the Page curve. In [34], one models the total system by a maximally entangled state between the radiation and the black hole microstates. The latter are described by the internal states of an end-of-the-world (EOW) brane. These are fictitious geodesic boundaries of the Euclidean hyperbolic universe that are anchored on both sides to the boundary.
It is well known that eternal black holes in AdS are dual to a maximally entangled thermofield double (TFD) state between two copies of a CFT [9]. On the gravity side, this eternal black hole couples two parallel universes

by the presence of the black hole interior itself (ER=EPR conjecture). Half of the hyperbolic disk in Euclidean signature prepares this dual state in Lorentzian signature.
By performing a $\mathbb{Z}_2$-quotient along the geodesic fixed points of the EOW brane, one mods out one entangled pair of the solution, effectively purifying the system. In the gravity setting, the total action including these EOW branes is formulated in [36], and extends the JT action Eq 1 with an additional free particle contribution:

$$I_{JT}[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\,\sqrt{g}\Phi(R+2) - \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\tau\,\sqrt{h}\Phi(K-1) - \int_{EOW} dv\,\sqrt{-g_{vv}}(\Phi K-\mu). \quad (3)$$

Here, v is an affine parameter and K is the trace of the extrinsic curvature along the EOW brane. μ is interpreted as the mass tension of the brane. Physically, we can interpret an EOW brane as the dynamics of a probe particle of mass μ moving along the trajectory set by the EOW brane. The disk amplitudes with an EOW brane attached to the asymptotic boundary were calculated in the boundary-particle formalism in [34]. These are consequently used in the computation of the Page curve. The result is pictorially denoted by:

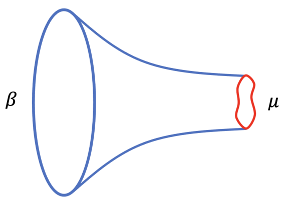
$$Z_{EOW}(\beta) =$$



$$= \int_0^\infty dk\; k\sinh 2\pi k\; e^{-\beta k^2} 2^{2\mu-2}|\Gamma(\mu-\frac{1}{2}+ik)|^2. \quad (4)$$

Here, the blue boundary describes the asymptotic UV boundary of the hyperbolic disk with length β, while the EOW particle moves between two points on the boundary along the red-shaded curve. It is only because this amplitude can be calculated exactly in the context of JT gravity that one can obtain a quantitative solution of the Page curve that obeys quantum unitarity.
The analysis in [36] extends this discussion to non-trivial topologies. By deforming the natural boundary conditions of the time reparameterization mode to a non-trivial monodromy, one can include operational defects in the disk partition function [37]. In particular, the class of hyperbolic defects deforms the disk partition function to a wormhole topology, where the asymptotic boundary is connected to a geodesic boundary at the neck of a single trumpet. In the discussion of [36], one considers EOW branes attached at the neck of this single trumpet amplitude. Again using the boundary-particle formalism, they have obtained the following exact quantum amplitude:

$$Z_{EOW}(\beta) =$$



$$= \int dk\; e^{-\beta k^2}\int_0^\infty \frac{db}{\sinh(b/2)}\cos(kb)e^{-\mu b}. \quad (5)$$

Computations in the boundary-particle formalism obscure much of the immediate geometrical interpretations of these expressions. In particular, it is not at all obvious what is the origin of the correction to the classical geodesic saddle $e^{-\mu b}$ in the last answer. A first hint was noted in [38], with the observation that the combination

$$\frac{e^{-\mu b}}{\sinh(b/2)} \tag{6}$$

looks like a lowest-weight character of a discrete series module of $\mathrm{SL}(2,\mathbb{R})$. We aim to make this argument more precise. In particular, we follow the bulk quantization perspective of [22] [24] in terms of a topological $\mathfrak{sl}(2,\mathbb{R})$ BF gauge theory. Using the arguments of [23], we note that the EOW brane action Eq 3 in the metric formulation of JT gravity acts as the insertion of a Wilson operator in the BF path integral. This result critically depends on the topology of the Wilson operator. For open paths, the free particle path integral evaluates to a Wilson *line* between two lowest weight states. This is precisely a discrete series matrix element in the representation theory of $\mathrm{SL}^{(+)}(2,\mathbb{R})$. A path integral over closed loops on the other hand instructs to take a trace over all vectors in the lowest weight module, and the Wilson operator evaluates to a Wilson *loop* instead. This precisely identifies with a character of $\mathrm{SL}^{(+)}(2,\mathbb{R})$. Going beyond the analysis of [38], we note that the extrinsic trace in the action Eq 3 vanishes due to the dilaton equations of motion. This effectively localizes the free particle path integral to describe geodesic trajectories. The identification between a discrete series character of $\mathrm{SL}^{(+)}(2,\mathbb{R})$ and Eq 6 becomes exact in this limit.
In addition, we show that the matrix element of the Wilson line operator is identified exactly with the on-shell value of the free particle path integral in the geodesic limit. Using this perspective, it is tempting to interpret the denominator in the answer Eq 6 for closed paths as a one-loop correction to this saddle.

This perspective on the calculation of EOW brane amplitudes neatly unites the two seemingly different answers for the disk and trumpet amplitude Eqs 4, 5 into a common framework in group theory. Furthermore, we can extend the notion of EOW branes to theories of $\mathcal{N}=1$ JT supergravity (SUGRA). To our knowledge, these objects have not yet been considered before in this context.

The $\mathcal{N}=1$ JT SUGRA action in superspace has been formulated in [39]:

$$I_{JT}^{\mathcal{N}=1} = \frac{1}{4}\left[\int_{\Sigma} d^2z d^2\theta \; E\Phi(R_{+-}+2) + 2\int_{\partial\Sigma} d\tau d\vartheta \; \Phi K\right]. \tag{7}$$

Besides the bosonic coordinates $z,\ \overline{z}$, the bulk superspace is equipped with an additional fermionic (Grassmann) pair $\theta,\ \overline{\theta}$. These superspace coordinates are bundled in terms of coordinates Z^M on a $2|2$-dimensional manifold. Φ, R_{+-}, E in the bulk action serve as the superdilaton field, the supercurvature and determinant superframe field respectively. These are expanded in a Grassmann expansion of the fermionic coordinates $\theta, \overline{\theta}$. Using the superdilaton equations of motion, the bulk term is again seen to vanish, and describes patches of the Poincaré super upper-half plane (SUHP). The remaining dynamics is captured by the super-conformal symmetry breaking pattern along the boundary. Essentially, the boundary curve is $1|1$-dimensional and is described in terms of a bosonic τ and fermionic ϑ affine parameter. We can image these curves as infinitesimally thickened sheets along the fermionic ϑ-direction. The definition of the super extrinsic supercurvature K in this boundary term is fine-tuned to yield precisely the super-Schwarzian derivative [39].

We argue that this boundary behaviour is not readily appropriate to describe EOW branes in superspace. Instead, we should consider genuine $1|0$-dimensional curves that are described solely in terms of a single bosonic affine parameter s. These curves should again localize onto geodesics. Somewhat surprisingly, the mathematical and physical literature on differential geometry in superspace is relatively scarce. In particular, an appropriate definition of the extrinsic curvature along $1|0$-dimensional curves is not readily available. We set out to construct it from first principles, following the textbook development of bosonic general relativity. We deduce an exact expression for the geodesic equations in superspace and automatically obtain the definition of the extrinsic supercurvature in terms of the variation of the normal vector field n_M along the curve:

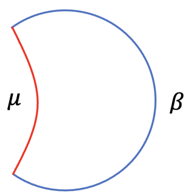
$$K = \dot{Z}^M \dot{Z}^N \nabla_N n_M. \tag{8}$$

We also define an appropriate definition of the covariant superderivative in this context. Significant care has to be taken in the ordening of the vectors and covectors in superspace since they in general do not commute. We find a consistent set of conventions by considering coordinate invariant contractions in the north-west - south-east direction (NW-SE): $\dot{Z}^M g_{MN} \dot{Z}^N = \dot{Z}^N \dot{Z}_N$. The anticommutative nature of these vectors leads to the emergence of additional sign factors compared to the bosonic answer.
We propose the following action for EOW branes in superspace, as the natural generalization of the bosonic answer Eq 3:

$$\boxed{I_{EOW} = \int ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N} (\mu - \phi K)}. \tag{9}$$

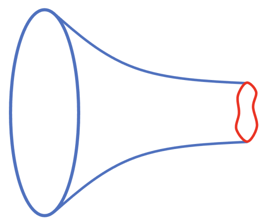
This is one of the key results of this thesis.

To obtain explicit quantum amplitudes of the EOW branes in superspace, we rely on the group theoretical description of $\mathcal{N} = 1$ JT SUGRA in terms of a topological $\mathfrak{osp}(1|2,\mathbb{R})$ BF gauge theory. In particular, we extend the discussion of [23] and identify the path integral of a free particle in terms of a super-Wilson operator insertion in the BF path integral.
The result again depends on the topology, and we obtain the answer for a Wilson line/loop in terms of the matrix elements/characters of $\text{OSp}^+(1|2,\mathbb{R})$, determined earlier in [40]. The result for the disk-shaped amplitude reads:



$$= \int_0^\infty dk \ \cosh(2\pi k) e^{-\beta k^2} \int_{-\infty}^{\infty} d\phi \left(K_{1/2+2ik}(2e^\phi) + \epsilon_- \epsilon_+ K_{1/2-2ik}(2e^\phi) \right) e^{2\ell\phi}.$$

, in terms of the modified Bessel functions of the second kind. As an additional subtlety, the fermionic sector of the group $\text{OSp}(1|2,\mathbb{R})$ decomposes into a periodic Ramond (**R**) sector and an anti-periodic Neveu-Schwarz (**NS**) sector for rotations around the circle. Thus, we obtain two separate answers depending on the sector for closed EOW brane loops:



$$Z^{\mathbf{NS}}_{EOW}(\beta) = \int_0^\infty dk\, e^{-\beta k^2} \int_0^\infty db \cos(bk) \frac{e^{-\mu b}}{2\sinh(b/4)},$$

$$Z^{\mathbf{R}}_{EOW}(\beta) = \int_0^\infty dk\, e^{-\beta k^2} \int_0^\infty db \sin(bk) \frac{e^{-\mu b}}{2\cosh(b/4)}.$$

Notably, the result for the **R** sector does not yield the spurious UV divergence for small geodesic lengths $b \to 0$ that was present in the bosonic solution Eq 5 obtained in [36].

We conclude that we have not only obtained an alternative approach on the computation of EOW brane amplitudes in group theory, we have also been able to generalize the concept of EOW branes to $\mathcal{N} = 1$ JT SUGRA and obtain explicit amplitudes using this perspective.

0.3 Contents of this thesis

The contents of this thesis cover a wide range of the literature involving the classical and quantum description of JT gravity. However, the main goal of this thesis is a thorough investigation in the quantum description of EOW branes, for which we resort to the first-order gauge theoretical formulation of JT gravity.
Since this thesis should be accessible for wide variety of readers not directly involved in the field of quantum gravity, a large amount of this thesis is devoted to elaborate on the existing literature involving JT gravity. Instead of summarizing the results of different papers, I have opted to work out as much of the obscured non-trivial derivations explicitly, which are often left open in the existing publications. I believe that this not only increases honesty towards the reader, it also makes it more interesting for both an expert and non-expert audience.
On the other hand, this makes for a rather long thesis. To make the text more digestible, please take into account the following notes.

We start with a broad summary of the classical solutions of JT gravity in chapter 1. Although I felt it was necessary to include it for completeness and consistency, readers who are already familiar with the basics of JT would be forgiven if they were to skip this chapter. The chapter serves as an introduction and motivation to study this particular gravity model.
The content covers the headaches of the backreaction problem of AdS_2/CFT_1 holography, and how to go beyond by introducing a non-trivial dilaton profile. This leads automatically to a definition of the JT gravity model. The first part also aims to explain the physical relevance of this model in the context of a dimensional reduction of near-extremal black holes in our $3+1$-dimensional universe. This discussion is largely based on

[12] [41] [42], where I again work out the derivations more in detail.
The next sections involve the classical solutions of this model, after [15] and [25]. This leads to the holographic Schwarzian description of [17] and [16]. The chapter ends with the setup of the quantum gravitational description of JT gravity. In particular, we derive explicitly the one-loop exact partition function in the perspective of coadjoints orbits of the Virasoro algebra [20]. This involves a technical discussion on symplectic manifolds and the Virasoro algebra which is averted to appendix C.
We finally study the quantum incorporation of matter via the standard holographic dictionary. For readers who are unfamiliar with the holographic dictionary and the laws of black hole thermodynamics, I have additionally included an appendix D which serves as a crash course with the necessary prerequisites to follow this thesis. The content broadly covers the related course by M. P. Heller on quantum black holes [43].
As elaborated before, readers who are solely interested in the content directly related to the original research may choose to skip this chapter and its associated appendices altogether. On the other hand, due to its very explicit derivations, this chapter serves nicely as a convenient introduction to JT.

Chapter 2 is rather long, and covers in detail the exact quantization of JT gravity in its first order gauge theoretical perspective after [22], [24], [23], [44], [45] and [37]. We start by deriving the on-shell equivalence between the action of JT gravity in its metric formulation Eq 1 and a topological $\mathfrak{sl}(2,\mathbb{R})$ BF gauge theory. We do this at the level of the action and at the level of the equations of motion. In particular, I demonstrate explicitly that the variational solutions of the BF theory exactly coincide with those obtained in JT gravity.
We further investigate the quantum solutions of this model with multiple Wilson line insertions in the case of a general compact gauge group, using an open channel slicing developed in [22]. In particular, taking inspiration from 2d Yang-Mills [45], the topological nature of this theory allows to cover the disk partition functions directly by a Hamiltonian evolution. This leads to a set of diagrammatic rules analogous to [21].
We finally arrive at the quantum disk amplitudes of JT gravity in terms of a constrained non-compact $\mathrm{SL}^+(2,\mathbb{R})$ gauge theory. The chapter ends with an extension to non-trivial topologies after [37], by inserting suitable operational defects in the bulk. In the gauge theory, these are equivalent to insertions of suitably normalized characters of $\mathrm{SL}(2,\mathbb{R})$.
Some technical details about the representation theory of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}^+(2,\mathbb{R})$ are averted to appendix A, by combining the analysis of [24] [22] and [40]. The main text should be followable without it. Whenever necessary, I make reference to the appropriate equations.

Chapter 3 comes to the core essence of this thesis, where we define EOW branes and obtain quantum amplitudes for them directly in the gauge theoretical description of the previous chapter. For the most part, this chapter tights together loose ends in the existing literature into a coherent story, which is in this light part of the original work.
The chapter ends with a section 3.5 on the physical application of EOW branes in the Hawking evaporation process. This part is a review of [34] and can be skipped according to the reader's wishes. The rest of the thesis does not depend on this discussion.

The first parts of chapter 4 extend the discussion of bosonic JT gravity to $\mathcal{N}=1$ JT SUGRA using the per-

spectives of [39] and [40]. The former starts from the superspace action Eq 7 and pins down the holographic description in terms of the super-conformal symmetry breaking pattern described by the super-Schwarzian action at the holographic $1|1$-dimensional boundary.
A recent paper by Fan et al. [40] extends the group theoretical quantization of bosonic JT gravity to the gauge formulation of $\mathcal{N} = 1$ JT SUGRA in terms of an $\mathfrak{osp}(1|2, \mathbb{R})$ gauge algebra. The precise exponentiation is again found to be the subsemisupergroup $\mathrm{OSp}^+(1|2, \mathbb{R})$. The representation theory on both this superalgebra and supergroup is averted to the appendix B.

Near section 4.6 of this chapter, we start to investigate EOW branes in superspace. We proceed by generalizing the geodesic equations to superspace and come to the same description of [46]. We also define an appropriate covariant derivative acting on both vectors and covectors in superspace. From this follows the definition of the extrinsic curvature in superspace Eq 8, which allows us to write down the action of EOW branes in JT SUGRA Eq 9.
Using an extension of the quantization perspective developed in chapter 3, we deduce the quantum amplitudes of these objects involving various topologies.

The thesis ends with a summary and conclusion in chapter 5. In addition, we make some preliminary steps to write down an action for EOW branes in $\mathcal{N} = 2$ JT SUGRA. These results are however still preliminary and involve recent research in the group.

To settle conventions, throughout in this thesis, I will use the east-coast signature of the metric tensor $(-, +, \ldots, +)$, appropriate in applications of quantum gravity. The transition to Euclidean signature proceeds by Wick rotating $t \to -i\tau$. Euclidean signature actions are usually denoted with I, while Lorentzian signature actions are denoted S. They are connected under $S = iI$ and $\mathscr{L}_E = -\mathscr{L}$. For more on these conventions, see D.4.

Chapter 1

Introduction to JT gravity

"Simplicity is the ultimate sophistication."

Da Vinci, Leonardo

The majority of modern approaches to Quantum Gravity are being developed in the framework of the Anti-de Sitter/Conformal Field duality, often called AdS/CFT [4] for short, which is a concrete realization of a holographic universe envisioned by 't Hooft and Susskind [47][6]. In the example of Jackiw-Teitelboim (JT) gravity especially, the generic holographic dictionary is used to introduce operators in the putative nearly CFT, dual to a massive scalar fields propagating in the bulk of a nearly AdS_2 spacetime. The generic holographic dictionary is used to introduce operators in the putative nearly CFT, dual to a massive scalar field propagating in the bulk of a nearly AdS_2 spacetime. One of the main achievement of JT gravity is that the correlators of these dual operators, including quantum gravity effects, do not only make sense on a perturbative level, but are exactly solvable at all orders [21]. This is a tremendous achievement in a gravitational context, where an appropriate definition of a gravitational path integral, and subsequent UV divergences are notoriously difficult [48]. JT gravity furthermore allows to find clues to the doubly non-perturbative gravitational origin of the underlying discreteness in the dual quantum mechanical theory [30] [29].

Most approaches to holography are in a sense still semi-classical in nature. In ordinary QFT, one fixes the spacetime manifold beforehand, and path-integrates over the fields defined on it. This manifold is usually the flat Minkowksi spacetime. In the discussion of Hawking radiation, the manifold is taken to be a black hole instead. This ultimately leads to an observer dependent definition of the vacuum state. Here too, one fixes the manifold and performs regular QFT calculations on this background geometry.
In quantum gravity however, we must integrate over the geometry itself. This poses some immediate difficulties since the Einstein-Hilbert action is not renormalizable and the Euclidean path integral is unbound from below. More often than not, one cannot actually perform the path integral, and one must restrict to the different saddles of the classical solutions instead.
To investigate the renormalizability of the action, one can perform a dimensional analysis. The metric tensor

itself has no proper dimensions in the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Since the Ricci scalar R consists of second order derivatives of the metric, the (Euclidean) Einstein-Hilbert action

$$I_{EH} = -\frac{1}{16\pi G_N}\int d^D x\sqrt{g}R \tag{1.1}$$

has a coupling constant G_N which scales as $[G_N] = \text{Length}^{D-2}$. This is consistent to yield a dimensionless number in the exponents of the path integral. One can only hope for this action to be renormalizable[1] if the coupling constant is dimensionless. In particular for $[G_N] > 0$, the theory is non-renormalizable. This imposes the proper spacetime dimension of $D = 2$. This promises a fruitful avenue to work in a lower-dimensional setting.

1.1 Pure gravity in $D = 2$

Pure gravity is described by the Einstein-Hilbert action without matter sources. In $D = 2$, this model exhibits no explicit dynamics. Additionally coupling this model to matter imposes the matter stress tensor to vanish $T^m_{\mu\nu} = 0$, and no energy flows can exist.

More precisely, in D-dimensional general relativity, the Einstein equations are built from the Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. Since the geometry is ultimately determined by the full Riemann tensor, the Einstein equations can still yield interesting dynamical solutions for the metric. In D dimensions, the number of independent components of the Riemann tensor $R_{\mu\nu\rho\sigma}$ is [50]

$$C_D = \frac{1}{12}D^2(D^2-1).$$

For $D = 2$, the full Riemann tensor is completely specified by one number, which can be taken to be the value of the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. Thus, in this case, there are no additional degrees of freedom and the metric solutions are entirely fixed by the value of the Ricci tensor[2].

By contracting with the metric tensor, the correct parametrization of the Riemann tensor in terms of the Ricci scalar is

$$R_{\mu\nu\rho\sigma} = \frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{1.2}$$

This is indeed the only possible parametrization of $R_{\mu\nu\rho\sigma}$ in $D = 2$ with the correct symmetry properties that is built entirely from the Ricci scalar, and contracts to the latter after consecutive application of $g^{\mu\rho}g^{\nu\sigma}$. In particular, every two dimensional metric is a maximally symmetric solution. Contracting with $g^{\mu\rho}$ consequently yields the Ricci tensor:

$$R_{\mu\nu} = \frac{R}{2}g_{\mu\nu}. \tag{1.3}$$

[1] The fact that the coupling is in the denominator is subtle. Defining $2\kappa^2 \equiv 16\pi G_N$, the Einstein-Hilbert term looks similar to a gauge action $\frac{1}{4g^2}\int F^2$. Expanding around flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ leads to $S \sim \frac{1}{2\kappa^2}\int d^D x((\partial h)^2 + h(\partial h)^2 + \dots)$. Rescaling $h \equiv \kappa\tilde{h}$ leads to $S \sim \frac{1}{2}\int d^D x((\partial\tilde{h})^2 + \kappa\tilde{h}(\partial\tilde{h})^2 + \dots)$. The first term is a kinetic term, while all higher order interaction terms have coupling proportional to κ. These are indeed non-renormalizable for $D > 2$ [49].

[2] Compare this with e.g. the vacuum solutions in $D = 4$, where for a vanishing Ricci tensor a whole range of solutions is possible. This includes e.g. flat Minkowski space or the Schwarzschild black hole solution.

The Einstein equations in terms of the Einstein tensor $G_{\mu\nu}$ are therefore trivially satisfied for any choice of the metric

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv 0. \tag{1.4}$$

The latter equivalence sign "$\equiv$" emphasizes that this equation has no dynamical solutions, and any choice of metric automatically satisfies it. When coupling the theory to matter, the Einstein tensor imposes $T_{\mu\nu} \equiv 0$, and the metric tensor plays the role of a Lagrange multiplier enforcing this constraint [51].
The Euclidean Einstein-Hilbert action in $D = 2$, extended with a Gibbons-Hawking-York (GHY) boundary term

$$I_{EH} = -\frac{1}{16\pi G_N} \int d^2x \sqrt{g} R - \frac{1}{8\pi G_N} \oint ds \sqrt{h} K \tag{1.5}$$

is therefore invariant under continuous variations of the metric $\delta g_{\mu\nu}$, which leads to the trivial Einstein equations. The GHY boundary term is needed a posteriori to make sense of the model variationally on non-compact manifolds $\partial M \neq 0$, and consists of the induced metric h_{ab} and the extrinsic curvature K_{ab} (more precisely its trace $K = h^{ab} K_{ab}$) on the boundary surface. Invariance under variations of the metric implies that the action itself should be a topological invariant. This is captured by the Gauss-Bonnet theorem in differential geometry, which implies that the Euclidean Einstein-Hilbert action extended with the GHY boundary term, reduces to the Euler characteristic χ on a Riemann surface:

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R + \frac{1}{2\pi} \oint ds \sqrt{h} K. \tag{1.6}$$

This is a topological invariant $\chi = 2 - 2g - n$, with g the genus of the manifold and n the number of boundaries. This action still has important applications in string theory as it weights the different topologies of the 2d string worldsheet.
Including a cosmological constant, and gauge fixing the metric under diffeomorphisms to the conformal gauge, leads to an interesting Liouville gravity model. This model has an emergent Weyl-symmetry, and can be described in terms of a single Liouville field ϕ. The dynamics of the latter are described in terms of the Liouville action. The central charge of this effective field cancels the central charges of the possible matter systems and the bc-ghost terms. Since the central charge describes the conformal anomaly, the vanishing of the total central charge leads to a one-loop exact quantum Liouville theory of gravity coupled to matter. For more information on this model, I highly recommend the review note [50] and references therein.

The first part of this chapter will be devoted to introducing the alternative model of 2d Jackiw-Teitelboim (JT) gravity, which is seen to describe the near-horizon region of near-extremal black holes. I will also discuss the classical solutions derived from the action. This will lead to a natural holographic description of this gravity model in terms of an effective one-dimensional theory whose only degrees of freedom are located at the asymptotic boundary. In a later stage, we will look at how to incorporate quantum effects onto the classical energy spectrum in perturbation theory.
The physical implications of the JT model reviewed hereunder serve both as a review and motivation for the rest of this thesis. On the other hand, the bulk text on the quantization of JT gravity in its first order formalism, and the subsequent group theoretic description of EOW branes, starting from chapter 2 onward, can be read independently. Readers who are already familiar with the JT gravity model may choose to skip this chapter

and directly avert to the subsequent chapters.

1.2 The backreaction problem in AdS_2/CFT_1

When embedding a model of 1+1d gravity in the framework of holography, we are guided by the large amount of experience developed since the original proposal by Maldacena in [4]. While most discussions on quantum gravity are limited to first order perturbation theory due to the limited knowledge on how to compute correlators in strongly coupled CFTs and path integrals in quantum gravity, 2D/1D holography provides a promising framework to go beyond. This is largely because gravitons or gauge bosons in two dimensions have no dynamical bulk degrees of freedom [23].
The AdS_2/CFT_1 correspondence however is still an enigmatic case to study finite energy excitations on both sides of the conjecture. In general CFT, the energy-momentum tensor of any conformal field theory has a vanishing trace $T_\mu^{\ \mu} = 0$ [52]. The conformal anomaly that appears when quantizing the theory only turns up in even dimensions [25]. Since we consider 0+1 dimensions, the following conclusions are universal. For a tensor with only one index, the vanishing of the trace of the stress tensor immediately implies that the Hamiltonian itself also vanishes, and we are only able to describe the ground states or extremal states. This can also be argued by dimensional analysis in the CFT [53][25]. Since we consider 0+1d, there is no volume to scale with, and the density of states in the CFT can only take the form

$$\rho(E) = A\delta(E) + \frac{B}{E}$$

to yield the correct dimensionality of 1/[Energy]. A and B are dimensionless constants. The latter needs to vanish in a well-defined theory in the IR $B \equiv 0$. We again come to the conclusion that the vacuum is the only allowed energy eigenstate in the 0+1d CFT.
On the 1+1d AdS_2-side, Maldacena argued in [12] that the backreaction from any excitation is so strong that it destroys the asymptotic AdS_2-geometry, leaving only the zero energy or extremal states.
This should be resolved since AdS_2 is perhaps the most interesting from the point of view of black hole physics. The subsequent subsections will review the original argumentation by Maldacena, stating that it is not possible to take the near-horizon limit of an extremal black hole, while simultaneously keeping the charge, energy and temperature fixed.

1.2.1 Near-horizon region of near-extremal black holes

This subsection first motivates how AdS_2 turns up in the near-horizon limit of near-extremal black holes in $D = 4$. I will show this in the case of the electrically charged Reissner-Nördstrom (RN) black hole.
The RN black hole solution in 3+1d describes a black hole of mass M and charge Q, whose geometry is characterized by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{1.7}$$

This solution has two horizons $r_{\pm}$, located at the zeros of $f(r)$; $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$. These zeros identify the null surfaces at constant radius. The Hawking temperature was derived explicitly in section D.3, see Eq D.17. When $Q = M$, the two horizons coincide $r_+ = r_- = M$ and the Hawking temperature vanishes.
I will work out more explicitly the derivation given in [25] to identify the near-horizon near-extremal geometry. For $E \equiv M - Q \neq 0$, the near-extremal near-horizon limit is found by introducing a small excess ρ, in the order of magnitude $\rho \sim \mathcal{O}(E^{1/2})$, such that $r = Q + Q^2\rho$. The positions of the horizon in the near-extremal regime $E \approx 0 \leftrightarrow Q \approx M$ are:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} = M \pm \sqrt{(M-Q)(M+Q)} \approx M \pm \sqrt{2ME}.$$

Reparameterizing to small ρ yields:

$$\begin{aligned} f(r) &= \frac{(r-r_+)(r-r_-)}{r^2} \sim \frac{1}{Q^2}(Q + Q^2\rho - M - \sqrt{2ME})(Q + Q^2\rho - M + \sqrt{2ME}) \\ &= \frac{1}{Q^2}(-E + Q^2\rho - \sqrt{2ME})(-E + Q^2\rho + \sqrt{2ME}) \\ &= \frac{1}{Q^2}(E^2 + Q^4\rho^2 - 2EQ^2\rho - 2ME). \end{aligned}$$

Since $\rho \sim \mathcal{O}(E^{1/2})$, the second term in the last line scales as $Q^4\rho^2 \sim \mathcal{O}(E)$, while the third term is already $2EQ^2\rho \sim \mathcal{O}(E^{3/2})$. Neglecting terms higher order than $\mathcal{O}(E)$, the first and third term are seen to vanish, and we can approximate the metric by:

$$ds^2 \approx -Q^2\left(\rho^2 - \frac{2E}{Q^3}\right)dt^2 + \frac{Q^2}{\rho^2 - \frac{2E}{Q^3}}d\rho^2 + Q^2 d\Omega_2^2. \tag{1.8}$$

We can identify this as a black hole patch in the pure $AdS_2 \otimes S_2$ geometry. Rescaling distances with Q^{-1}, the canonical form can be written as

$$ds^2 = -(\rho^2 - \rho_h^2)dt^2 + \frac{d\rho^2}{\rho^2 - \rho_h^2} + d\Omega_2^2 \tag{1.9}$$

, with the horizon $\rho_h = \sqrt{\frac{2E}{Q^3}}$ at finite proper distance $\rho = \rho_h$. For extremal black holes $E \to 0$, the black hole's horizon is shifted to $\rho_h \to 0$, and the near-horizon geometry is the Poincaré patch of pure $AdS_2 \otimes S_2$ with $z = 1/\rho$;

$$ds^2 = -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} + d\Omega_2^2 \quad = \quad \frac{-dt^2 + dz^2}{z^2} + d\Omega_2^2. \tag{1.10}$$

The main difference between the extremal and near-extremal near-horizon limits is that the near-horizon region of an extremal black hole is infinitely long. This is seen explicitly for the distance towards the horizon at $\rho = 0$ for pure AdS_2 coordinates, describing the near-horizon region the extremal black hole geometry with $\rho_h = 0$:

$$ds = \int^{\rho_h} \frac{d\rho'}{\rho'} = \ln \rho_h \to -\infty. \tag{1.11}$$

For nearly extremal black holes with non-zero ρ_h, the near-horizon region $\rho \approx \rho_h$ can be identified with a Rindler patch instead. In this case, we have $\rho^2 - \rho_h^2 = (\rho - \rho_h)(\rho + \rho_h) = 2\tilde{r}\rho_h$, with $\tilde{r} \equiv \rho - \rho_h$

$$ds^2 = -2\tilde{r}\rho_h dt^2 + \frac{dr^2}{2\tilde{r}\rho_h}. \tag{1.12}$$

Defining $dx = \frac{dr}{\sqrt{2\rho_h}\sqrt{\tilde{r}}}$, we can integrate $x = \sqrt{\frac{2\tilde{r}}{\rho_h}}$, and write the metric in Rindler coordinates in terms of an angular variable $\theta \equiv \rho_h t$:

$$ds^2 = -x^2(\rho_h dt)^2 + dx^2 \equiv -x^2 d\theta^2 + dx^2. \tag{1.13}$$

Upon Wick rotating $\theta \to -i\theta_E$ to Euclidean signature, we arrive at the polar coordinates of the flat Euclidean plane. The Hawking temperature can be deduced by demanding that the Euclidean manifold is regular near the horizon. To avoid a conical singularity, a rotation in θ of 2π should correspond to a translation in it by $\beta = 1/T$. By consistency, the temperature is therefore $T = \frac{\rho_h}{2\pi}$. For more information on this calculational trick, I refer to section D.3 in the appendix.
Plugging in the explicit value $\rho_h = \sqrt{\frac{2E}{Q^3}}$, and $E \equiv M - Q$, this is indeed equivalent to the near-horizon approximation of the exact value of Hawking temperature of a RN black hole Eq D.17:

$$T = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{2\sqrt{M^2 - Q^2}}{4\pi Q^2} = \frac{1}{2\pi}\sqrt{\frac{2E}{Q^3}} \equiv \frac{\rho_h}{2\pi}. \tag{1.14}$$

1.2.2 AdS_2 as the low-energy limit of magnetically charged RN black holes

An insightful low-energy limit in the discussion of AdS_2 backreaction is the near-horizon limit of the four-dimensional magnetically charged RN black hole solution. I will use the conventions due to Maldacena [12], and introduce two dimensionful parameters of energy, being the mass M of the black hole, and the Planck mass M_p. The latter is related to the Planck length $L_p = 1/M_p$, which sets the scale of the Newton's constant in $D = 4$: $G_N = L_p^2$. These are the only dimensionful parameters in the theory.
One also introduces a dimensionless number Q, which is related to the magnetic charge of the black hole $\sim Q/L_p$, where Q is a dimensionless constant. The energy above extremality is given by

$$E = M - \frac{Q}{L_p}. \tag{1.15}$$

The full magnetically-charged 3+1d black hole solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{1.16}$$

, with

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2}.$$

$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the line-element of the two-sphere. The electromagnetic two form of a magnetically charged black hole is $F = Q\sin\theta\, d\phi \wedge d\theta$, where $\sin\theta\, d\phi \wedge d\theta$ is the volume form of the two-sphere. The positions of the horizons $r_\pm$ are identical to Eq D.5, and can be expressed in terms of the energy above extremality:

$$r_\pm = QL_p + EL_p^2 \pm \sqrt{2QEL_p^3 + E^2L_p^4}. \tag{1.17}$$

Therefore, the main difference between the electrically charged solution is the value of the electromagnetic two form (which in the case of the electrically charged solution is described by $F_{rt} = \frac{Q}{r^2}$).
The BH entropy of this solution is related to the total surface area of the horizon $A = 4\pi r_+^2$. Together with the general relation with the Hawking temperature Eq D.17, we have:

$$S_{BH} = \frac{A}{4G_N} = \frac{\pi r_+^2}{L_p^2}, \qquad T_H = \frac{r_+ - r_-}{4\pi r_+^2}. \tag{1.18}$$

In the extremal case $E \equiv 0$, $r_+ = r_- = QL_p$, the Hawking temperature vanishes:

$$T_H = 0. \tag{1.19}$$

One considers the near-horizon region by introducing a length parameter z as:

$$z = \frac{Q^2L_p^2}{r - r_+} \tag{1.20}$$

, and zooming in towards r_+ by taking $L_P \to 0$, while keeping both z and Q fixed. In the extremal case, $f(r)$ can be rewritten in terms of z:

$$f(r) = \frac{(r-r_+)(r-r_-)}{r^2} \sim \frac{(r-r_+)^2}{r_+^2} = \frac{1}{Q^2L_p^2}\frac{Q^4L_p^4}{z^2} = \frac{Q^2L_p^2}{z^2}.$$

The limit of $L_p \to 0$ (or equivalently $M_p \to \infty$) is natural from the perspective of the uncertainty principle, where large energies imply small distance scales. The near-horizon region of the extremal solution is ($dr = -\frac{Q^2L_p^2}{z^2}dz$):

$$ds^2 = L_{AdS}^2\left(\frac{-dt + dz^2}{z^2} + d\Omega_2^2\right) \tag{1.21}$$

, where we have defined a characteristic scale $L_{AdS} \equiv QL_p$ of the AdS-spacetime. This is again a product space of $AdS_2 \otimes S_2$ (see figure 1.1). I will further neglect the two-sphere contribution. We should however remember that every point on the Penrose diagram in fact represents a two-sphere. The coordinates Eq 1.21 represent the Poincaré patch of the near-horizon AdS_2 region. This geometry can be geodesically continued to global coordinates, which cover the entire near-horizon region along the causal diamonds in the RN solution. Note that one of the timelike boundaries is just outside of the black hole horizon, while the other is just inside. The global metric of AdS_2 is given by [41]:

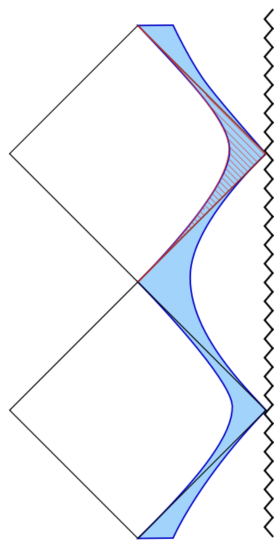


Figure 1.1: Maximally extended RN black hole Penrose diagram. The black zigzag line represents the black hole singularity. The blue shaded region represents the geodesically continued near-horizon AdS_2-patch. The dashed region is the patch covered by Poincaré coordinates Eq 1.21. Figure taken from [15].

$$ds^2 = 4\frac{L_{AdS}^2}{\sin^2(2z)}(-dt^2 + dz^2) = -4\frac{L_{AdS}^2}{\sin^2(u-v)}dudv \tag{1.22}$$

, where the range of the timelike coordinate is decompactified, while the spacelike coordinate covers a range of $0 < z < \pi/2$. In the last equation, I introduced the ingoing and outgoing lightcone coordinates $u = t + z$ and $v = t - z$ respectively. This is conformally identical to a region of 1+1d Minkowksi space, confined within two infinite timelike strips. Unlike higher dimensional AdS_D $(D > 2)$ geometries, AdS_2 has two boundaries.

1.2.3 The backreaction problem

Maldacena noted in [12] that the backreaction from any excitation in AdS_2 is so strong that it destroys the asymptotic AdS_2-geometry. Close to extremality $E \sim 0$, the energy-temperature relation becomes (combining Eqs 1.17, 1.18):

$$T_H = \frac{1}{2\pi}\left(\frac{2}{L_p Q^3}E\right)^{1/2} + \mathcal{O}(E^{3/2}).$$

Therefore, near-extremality, the energy-temperature relation is:

$$E \propto 2\pi^2 Q^3 T_H^2 L_p. \tag{1.23}$$

This energy-temperature relation necessarily pushes the energy of the excitations down to zero $E \to 0$ in the near-horizon limit $L_P \to 0$, while keeping Q fixed.

These formulas were obtained in a semi-classical analysis of black hole thermodynamics. This description must necessarily break down when the typical energy scale above extremality E is of the order of the energy of the radiation quanta. This is of the order of the Hawking temperature T_H itself. Using the previous formula, this breakdown happens at an energy of:

$$E_{gap} \simeq \frac{1}{Q^3 L_p}. \tag{1.24}$$

One expects this to be the rough magnitude of the energy gap above the ground states in the microscopic spectrum of the black hole [15]. Although this argument is rather heuristic, string theory calculations [54] show that this is indeed of the order of the lowest-lying excitations. The near-horizon limit pushes the gap to infinity $E_{gap} \to \infty$, and the only accessible states are the ground states $E = 0$. We can try to consider more general limits that are not constrained to zero temperature and zero excitation energy.

$L_p \to 0$ with E, Q fixed

From the near-extremal energy-temperature relation Eq 1.23, this limit is particularly distributing, since the Hawking temperature would have to diverge $T_H \to \infty$ in order for the energy E and Q to remain constant in near-horizon limit $L_p \to 0$. From the general temperature expression Eq D.17, an infinite temperature implies a microscopic dimensional black hole, with the horizon getting within Planckian distance towards the singularity. In this regime, the general relativistic description of these relations breaks down. This is intrinsic to the discussion of 1+1d AdS_2 geometries, where the only natural length scale of the theory comes from the Planck scale itself. For higher $(p > 0)$ dimensional AdS p-branes, the energy-temperature relations extracted from the near-extremal solutions can be scaled with the volume of the regulated transverse space V_p instead. The energy-temperature relation is then of the form $E \sim V_p T_H^{p+1}$, which has the correct units of energy. This does not involve the Planckian distance that sets the scale of the near-horizon regime, and the limit $L_p \to 0$ is non-singular for both energy and temperature.

$L_p \to 0$ with T_H, Q fixed

We may want to consider the near-horizon limit where both Q and T_H are fixed to a non-zero constant. The latter implies that the horizons do not coincide exactly. We can show that the near-horizon geometry is independent of T_H at the classical level, and is still described by a patch of AdS_2 spacetime. For a non-zero temperature $T_H = \frac{r_+ - r_-}{4\pi r_+^2}$, the near-horizon geometry ($r \sim QL_p$) is determined by:

$$f(r) = \frac{(r-r_+)(r-r_-)}{r^2} = \frac{(r-r_+)(r-r_+ + 4\pi r_+^2 T_H)}{r^2} \sim \frac{Q^2L_p^2}{z^2}(1+4\pi z T_H). \tag{1.25}$$

Explicitly, the line element becomes for $dr = -\frac{Q^2L_p^2}{z^2}dz$:

$$ds^2 = \frac{Q^2L_p^2}{z^2}\left(-(1+4\pi z T_H)dt^2 + \frac{dz^2}{1+4\pi z T_H}\right) + Q^2L_p^2 d\Omega_2^2. \tag{1.26}$$

By a coordinate transformation[3], this metric can be brought to the Poincaré patch of AdS_2 in the transformed coordinates, independent of T_H. Therefore, we have the same qualitative features of the extremal $T_H = 0$ regime. From the energy-temperature relation Eq 1.23 with fixed Q and T_H, the near-horizon geometry pushes the energy excitations to zero, while the energy gap Eq 1.24 is pushed towards infinity.

$L_p \to 0$, $Q \to \infty$ with E, T_H fixed

A different approach is to keep the energy and temperature fixed in the near-extremal energy-temperature relation Eq 1.23, while scaling the charge with the Planck length $Q \sim L_p^{-1/3}$. In the near-horizon limit $L_p \to 0$, the charge consequently diverges $Q \to \infty$. It was argued in [55] that this limit is equivalent to a free supergravity limit of the AdS_2/CFT_1 correspondence. Therefore all backreaction on matter is again suppressed.

The limits above demonstrate that pure AdS_2, as the near-horizon limit of near-extremal black holes, does not allow for finite energy excitations. On the CFT_1-side, we have seen that the only consistent description is in terms of ground states, or extremal states. This is one of the headaches in formulating an interesting AdS_2/CFT_1 correspondence. Resolving the tensions around it has been one of the key successes of JT gravity, which I will introduce in the following.

[3] The explicit coordinate transformation is $t' \pm z' = \tanh\left(\pi T_H\left(t \pm \frac{1}{4\pi T_H}\ln\frac{1}{1+4\pi T_H z}\right)\right)$ [12].

1.3 2d dilaton-gravity models of AdS_2-backreaction

We look at a dimensional reduction of the Einstein-Maxwell action in $D = 4$ in terms of a generic 2d dilaton-gravity model. This will allow us to consider the general backreaction of matter in terms of the dynamics of a dilaton field. This ultimately leads to the conclusion that the presence of non-zero matter sources destroys the asymptotic AdS_2-geometry. This discussion is an explicit derivation of the argumentation in [41].

1.3.1 Dimensional reduction of the magnetically charged EM action

The Einstein-Maxwell (EM) action can be expressed in terms of the dimensional factor L_p (with $G_N^{(4)} = L_p^2$) as:

$$S_{EM} = \frac{1}{16\pi L_p^2} \int d^4x \sqrt{-g} \left(R - \frac{L_p^2}{4} F_{\mu\nu} F^{\mu\nu} \right). \tag{1.27}$$

Of course, this action has various families of variational solutions. We can restrict to a specific class of static, spherically symmetric solutions with magnetic charge Q, by imposing a spherically symmetric ansatz on the metric tensor:

$$ds^2 = h_{mn}(r,t) dx^m dx^n + e^{2\psi(r,t)} d\Omega_2^2 \quad (i,j \in \{x^1 = t, x^2 = r\}), \qquad F = Q \sin\theta d\phi \wedge d\theta. \tag{1.28}$$

$d\Omega_2^2 = d\theta^2 + \sin^2\theta \; d\phi^2$ is the line element of the two-sphere. Inserting this ansatz in the EM action yields an action that is specified completely in terms of the dynamical variables $h_{ij}(r,t)$ and $\psi(r,t)$. The latter will be related to the dilaton field. h_{ij} can be interpreted as the induced metric on the two-dimensional manifold spanning t and r.
In general, for any warped product space $ds^2 = ds_{(1)}^2 + e^{2\psi} ds_{(2)}^2$, where we denote $x_{(1)} = x_m$ and $x_{(2)} = x_\mu$, the total Ricci tensor R can be decomposed in terms of the Ricci tensors on both product spaces $R_{(1)}$ and $R_{(2)}$ [56]:

$$R = R_{(1)} + e^{-2\psi(r,t)} R_{(2)} - 2k \nabla_{(1)}^2 \psi(r,t) - k(k+1) g^{mn} \nabla_{(1)_m} \psi(r,t) \nabla_{(1)_n} \psi(r,t). \tag{1.29}$$

k is the dimension of the $ds_{(2)}$-space. In the case of Eq 1.28, this number is $k = 2$, and the Ricci curvature of the two-sphere is known to be $R_{(2)} = 2$.
Exploiting the fact that the electromagnetic two-form only has non-vanishing angular components, leads to a simple parameterization of the Maxwell term in the total action:

$$-\frac{L_p^2}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{L_p^2}{4} F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} = -\frac{L_p^2}{2} Q^2 \sin^2\theta \frac{e^{-4\psi}}{\sin^2\theta} = -\frac{L_p^2}{2} Q^2 e^{-4\psi}. \tag{1.30}$$

Thus, the dependence on the angular variables is trivial in the action. The metric determinant is decomposed into $\sqrt{-g} = \sqrt{-h} e^{2\psi} \sin\theta$. Plugging Eqs 1.29 and 1.30 into the EM action Eq 1.27, and integrating over the angular part yields:

$$
\begin{aligned}
S_{EM} &= \frac{1}{16\pi L_p^2}\int d^4x\sqrt{-g}\left(R - \frac{L_p^2}{4}F_{\mu\nu}F^{\mu\nu}\right) \\
&= \frac{4\pi}{16\pi L_p^2}\int d^2x\sqrt{-h}e^{2\psi}\left(R_{(1)} + 2e^{-2\psi} - 4\nabla^2_{(1)}\psi - 6h^{mn}\nabla_{(1)_m}\psi\nabla_{(1)_n}\psi - \frac{L_p^2}{2}Q^2e^{-4\psi}\right).
\end{aligned}
$$

Next, we partially integrate the Laplacian term and use the metric postulate $\nabla_{(1)}h_{mn} = 0$, to obtain:

$$
\begin{aligned}
S_{EM} &= \frac{1}{4L_p^2}\int d^2x\sqrt{-h}e^{2\psi}\left(R_{(1)} + 2e^{-2\psi} + 8h^{mn}\nabla_{(1)_m}\psi\nabla_{(1)_n}\psi - 6h^{mn}\nabla_{(1)_n}\psi\nabla_{(1)_m}\psi - \frac{L_p^2}{2}Q^2e^{-4\psi}\right) \\
&= \frac{1}{4L_p^2}\int d^2x\sqrt{-h}e^{2\psi}\left(R_{(1)} + 2e^{-2\psi} + 2h^{mn}\nabla_{(1)_m}\psi\nabla_{(1)_n}\psi - \frac{L_p^2}{2}Q^2e^{-4\psi}\right).
\end{aligned}
\tag{1.31}
$$

We have assumed that the corresponding boundary terms at flat spacelike infinity vanish. Note that this calculation was simplified for the magnetically charged solution $F = Q\sin\theta d\phi\wedge d\theta$, where the quadratic form $F_{\mu\nu}F^{\mu\nu}$ could be expressed completely in terms of the spherical part of the metric. For electrically charged solutions, with non-vanishing $F_{rt} = \frac{Q}{r^2}$, the quadratic form has a non-trivial dependence on the induced metric h_{ij}.

Replacing covariant derivatives on a scalar field with ordinary derivatives, and defining a new field $\Phi \equiv e^{2\psi}$, we obtain the following two-dimensional theory:

$$
\begin{aligned}
S &= \frac{1}{4L_p^2}\int dtdr\sqrt{-h}\left(e^{2\psi}(R_{(1)} + 2h^{mn}\partial_m\psi\partial_n\psi) + 2 - \frac{L_p^2}{2}Q^2e^{-2\psi}\right) \\
&= \frac{1}{4G_N^{(4)}}\int dtdr\sqrt{-h}\left(\Phi R_{(1)} + \frac{h^{mn}\partial_m\Phi\partial_n\Phi}{2\Phi} + 2 - \frac{L_p^2}{2\Phi}Q^2\right) \qquad (1.32) \\
&= \frac{1}{16\pi G_N^{(2)}}\int dtdr\sqrt{-h}\left(\Phi R_{(1)} + \frac{h^{mn}\partial_m\Phi\partial_n\Phi}{2\Phi} + 2 - \frac{L_p^2}{2\Phi}Q^2\right). \qquad (1.33)
\end{aligned}
$$

In the second line, I have rewritten the explicit dependence on the Newton's constant $G_N^{(4)} = L_p^2$. In the four-dimensional theory, this has dimensions of Length^2. We can introduce the derived two-dimensional Newton's constant $G_N^{(4)} = 4\pi G_N^{(2)}$. In order for the action to be dimensionless, the dilaton field Φ has dimension of Length^2.

This action is a special instance of a dimensionally reduced dilaton-gravity model

$$
S = \frac{1}{16\pi G_N^{(2)}}\int dtdr\sqrt{-h}\left(\Phi R_{(1)} + \frac{h^{mn}\partial_m\Phi\partial_n\Phi}{2\Phi} - U(\Phi)\right)
\tag{1.34}
$$

, paramterized in terms of a dimensionless dilaton potential $U(\Phi) = -2 + \frac{L_p^2}{2\Phi}Q^2$. The classical solution with constant value of the dilaton describes a geometry with constant sphere area according to the ansatz Eq 1.28. Its classical solution is the rigid $AdS_2 \otimes S_2$ geometry. Allowing the dilaton to fluctuate represents the deviations away from this extremal regime. Therefore, the dilaton model captures the near-horizon regime of near-extremal black holes discussed earlier.

1.3.2 General dilaton-gravity models

General dilaton-gravity models are parameterized by a set of dilaton potentials $U_i(\Phi)$ $(i = 1, 2, 3)$:

$$S = \frac{1}{16\pi G_N^{(2)}} \int d^2x\sqrt{-h}\,(U_1(\Phi)R + U_2(\Phi)h^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - U_3(\Phi))\,. \tag{1.35}$$

$h_{\mu\nu}$ is the two-dimensional metric on t, and r. We can redefine the dilaton field in order to remove the dependence on $U_1(\Phi)$: $\Phi \to U_1(\Phi)^{-1}$. We have assumed $U_1'(\Phi) \neq 0$ for every value of Φ, such that this function is indeed invertible.
In string theory, the dilaton field is often present in the action as a prefactor of the Ricci scalar. In dimensions greater than two, a local Weyl rescaling $g_{\mu\nu} \to g'_{\mu\nu} = e^{2\omega}g_{\mu\nu}$ for some scalar field ω, allows one in general to transition from the string frame to the Einstein frame, where this prefactor drops in the action $\Phi R \to R$.
In 2d however, the ΦR term is an invariant combination under Weyl rescalings. Instead, this combination transforms as [25]:

$$\sqrt{-h}R \to \sqrt{-h}(R - 2\nabla^2\omega). \tag{1.36}$$

The Laplace-Beltrami operator ∇^2 is the covariantized Laplacian and can be rewritten by using the identity $\nabla_\mu V^\mu = \frac{1}{\sqrt{-h}}\partial_\mu(\sqrt{-h}V^\mu)$ on any vector field[4] V^μ. Using this identity, the Laplace-Beltrami operator is compactly written as:

$$\nabla^2\omega = \nabla_\mu\nabla^\mu\omega = \nabla_\mu\partial^\mu\omega = \frac{1}{\sqrt{-h}}\partial_\mu(\sqrt{-h}h^{\mu\nu}\partial_\nu\omega).$$

To remove the redundancy of $U_2(\Phi)$, we can specify

$$\omega(x) = -\frac{1}{2}\int^{\Phi(x)} U_2(\Phi')d\Phi'.$$

Under the Weyl rescaling $g \to e^{2\omega}g_{\mu\nu}$, the transformation law Eq 1.36 is explicitly:

$$\begin{aligned}\sqrt{-h}\Phi R \;&\to \sqrt{-h}\Phi R - 2\sqrt{-h}\Phi\nabla^2\omega = \sqrt{-h}\Phi R - 2\Phi\partial_\mu(h^{\mu\nu}\sqrt{-h}\partial_\nu\omega)\\ &= \sqrt{-h}\Phi R + \Phi\partial_\mu(h^{\mu\nu}\sqrt{-h}U_2(\Phi)\partial_\nu\Phi)\\ &\simeq \sqrt{-h}\Phi R - \sqrt{-h}U_2(\Phi)h^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\end{aligned}$$

In the last line, we have performed a partial integration in the second term under the integral. This term exactly cancels the kinetic term of the dilaton field in Eq 1.35. In the same way, we can set the kinetic term in the dimensionally reduced EM action Eq 1.34 to zero. Note that we are neglecting a possible contribution of a Weyl anomaly, appropriate in the classical treatment.

[4]This equality follows from a combination of the definition $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\alpha}V^\alpha$, together with $\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-h}}\left(\frac{1}{2}\sqrt{-h}h^{\mu\alpha}\partial_\nu h_{\mu\alpha}\right) = \frac{1}{\sqrt{-h}}\partial_\nu\sqrt{-h}$. The last identity follows from the derivative on the metric determinant $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\partial_\nu g_{\alpha\beta}$.

The most general 2d dilaton-gravity model is therefore:

$$S = \frac{1}{16\pi G_N^{(2)}} \int d^2x\sqrt{-h}\,(\Phi R - U(\Phi))\,. \tag{1.37}$$

In accordance with the tradition of string theory, the scalar field multiplying the Einstein-Hilbert term is called the *dilaton field*.

1.3.3 Holographic renormalization

We have argued that the dimensional reduction of the EM action leads to the class of dilaton-gravity models Eq 1.34, where we can remove the kinetic term associated with the dilaton field without loss of generality

$$S = \frac{1}{16\pi G_N^{(2)}} \int d^2x\sqrt{-h}\,(\Phi R - U(\Phi))\,, \qquad U(\Phi) = -2 + \frac{L_p^2}{2\Phi}Q^2. \tag{1.38}$$

It turns out that there exists a large degree of universality, whereby the near-horizon near-extremal geometry of a wide array of gravity theories with more general matter content are described by general 2d dilaton-gravity models with a unique dilaton potential, see e.g. [57][58].
We can couple this action to some a priori specified matter action S_m, and describe the general backreaction of the AdS_2 geometry on the presence of these sources in terms of the dynamics of the dilaton field. We have seen earlier that taking the near-horizon limit of nearly extremal RN black holes is incompatible with the presence of finite energy-excitations. We can argue that the presence of finite energy excitations destroys the asymptotic AdS_2-geometry in terms of the blowing-up of the dilaton.
First of all, it is convenient to parameterize the 2d metric in the conformal gauge:

$$ds^2 = -e^{2\omega(u,v)}dudv \tag{1.39}$$

, where $u = t + z$ and $v = t - z$ are the ingoing and outgoing null coordinates. It is always possible to locally impose this gauge on every 2d metric, since we can always find local coordinate transformations of the two spacetime coordinates that remove two of the three independent metric coordinates. We will see that the equations of motion of the dilaton field, obtained by varying the dilaton-gravity action Eq 1.38 in the presence some additional matter action S_m with respect to the uu-metric component g^{uu}, yields (c.f. Eq 1.63):

$$-e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi) \propto T^m_{uu}. \tag{1.40}$$

T^m_{uu} is the usual definition of the matter stress tensor $T^m_{uu} = -\frac{2}{\sqrt{g}}\frac{\delta S_m}{\delta g^{uu}}$. This equation is independent of the specific form of the dilaton potential $U(\Phi)$.
We can embed this model in the class of asymptotic AdS_2-geometries describing the near-horizon geometry of the near-extremal black holes. This implies that near the asymptotic boundaries, the metric should take the form of the global AdS_2-spacetime Eq 1.22; $e^{2\omega} \to \frac{1}{\sin^2(u-v)}$. There are two boundaries where the metric diverges; at $u = v$ ($z = 0$) and at $u = v + \pi$ ($z = \pi/2$). We can specify the u direction and integrate along the

line $v = 0$. The metric should therefore take the asymptotic form:

$$e^{2\omega} \to \frac{1}{u^2} \quad (u \to 0), \qquad e^{2\omega} \to \frac{1}{(u-\pi)^2} \quad (u \to \pi).$$

Integrating the equation of motion Eq 1.40 along $v = 0$ yields:

$$\int_0^\pi du e^{-2\omega}\, T^+_{uu} \propto -\, e^{-2\omega}\partial_u \Phi|_{u\to 0} + e^{-2\omega}\partial_u \Phi|_{u\to\pi}.$$

Since the LHS is non-zero for non-zero matter sources, and the metric takes the above asymptotic behaviour, it follows that the dilaton field should diverge at least linearly near at least one of the two boundaries: $\Phi|_{u\to 0} \sim \frac{1}{u}$ and $\Phi|_{u\to\pi} \sim \frac{1}{u-\pi}$. Therefore, the geometry cannot be asymptotic to pure AdS_2 when T_{uu} is non-zero. Presence of matter sources destroy the asymptotic regions by the diverging dilaton profile.

1.3.4 Universal description of nearly extremal black holes

We have argued that the dynamics of a large class of near-extremal black holes are captured by dilaton-gravity models Eq 1.37;

$$S = \frac{1}{16\pi G_N^{(2)}} \int d^2x\sqrt{-h}\left(\Phi R - U(\Phi)\right). \tag{1.41}$$

For constant sphere area, the dilaton is constant $\Phi(x) \equiv \Phi_0$, as can be seen from the metric ansatz Eq 1.28. To consider excitations above extremality, we take this constant to be large and consider small fluctuations around it: $\Phi(x) = \Phi_0 + \tilde{\Phi}(x)$, with $\tilde{\Phi} \ll \Phi_0$. Here, $\tilde{\Phi}$ represents the deviations away from the extremal limit into the near-extremal regime. From the previous section, we expect the deformations to blow up like $\tilde{\Phi} \sim \frac{1}{z}$ in the presence of matter as we approach the boundary $z \to 0$. We therefore consider a cutoff region at $z = \epsilon \to 0$ where $\tilde{\Phi}(x) = \frac{\phi_r(x)}{\epsilon}$ is large, but still bounded $\frac{\phi_r(x)}{\epsilon} \ll \Phi_0$ to remain in the near-extremal region. Expanding this action to first order in $\tilde{\Phi}$ yields:

$$S = \frac{1}{16\pi G_N^{(2)}} \left(\int d^2x\sqrt{-h}(\Phi_0 R - U(\Phi_0)) + \int d^2x\sqrt{-h}\tilde{\Phi}(R - U'(\Phi_0) \right) + \dots \tag{1.42}$$

The first term contains an IR divergent integral (divergence for large distances) over the total 2d spacetime volume

$$-\frac{U(\Phi_0)}{16\pi G_N^{(2)}} \int d^2x\sqrt{-h}.$$

We regulate this integral by discarding it entirely. Demanding the spacetime to be AdS_2 away from the boundary, independent of the value of $\tilde{\Phi}(x)$, we require that variation of $\delta\tilde{\Phi}$ yields:

$$\begin{aligned} R - U'(\Phi_0) &= -\frac{2}{L^2} - U'(\Phi_0) \equiv 0 \\ \iff \quad U'(\Phi_0) &= -\frac{2}{L^2}. \end{aligned}$$

The geometry is set to describe patches of AdS_2 by setting the Ricci tensor equal to $R_{AdS_D} = -\frac{-(D-1)D}{L^2}$ [59], which in $D = 2$ is indeed $R = -\frac{2}{L^2}$. L is the characteristic curvature scale of the AdS solutions. In this case, the first derivative of the dilaton potential takes the role of a cosmological constant term in the action. Of course, one requires $U'(\Phi_0) < 0$ in order to describe negatively curved solutions, which is the case for e.g. the nearly extremal RN solution Eq 1.34, whose dilaton potential has negative derivatives throughout $U'(\Phi_0) = -\frac{L_p^2}{2\Phi_0^2}Q^2$. In this case, the AdS curvature scale is related to the parent space quantities as $L = \frac{2\Phi_0}{L_p Q}$. In the following, we will set this curvature scale to unity $L \equiv 1$.

To leading order, the universal dynamics of near-horizon near-extremal black holes inside the cutoff AdS_2 manifold are therefore described by:

$$S = \frac{\Phi_0}{16\pi G_N^{(2)}} \int d^2x \sqrt{-h} R + \frac{1}{16\pi G_N^{(2)}} \int d^2x \sqrt{-h} \tilde{\Phi}(R+2) + \dots \tag{1.43}$$

The first term is associated to the topological Einstein-Hilbert term Eq 1.6, which in Euclidean signature reduces to the Euler characteristic χ. This term describes the near-horizon dynamics of the leading extremal black hole regime. The second term describes the deviations away from this extremal limit.
The total area of the near-extremal black hole in the coordinates Eq 1.28 is related to $\Phi(x) = \Phi_0 + \tilde{\Phi}(x)$, which takes the role of the radius squared. In the regime $\tilde{\Phi}(x) \ll \Phi_0$, we consider the extremal Bekenstein-Hawking entropy associated to Φ_0. In the parent four-dimensional spacetime with Newton's constant $G_N^{(4)}$, we use the general description Eq D.20:

$$S_0 = \frac{A}{4G_N^{(4)}} = \frac{4\pi\Phi_0}{4G_N^{(4)}}. \tag{1.44}$$

In terms of the reduced 2d Newton's constant $G_N^{(4)} = 4\pi G_N^{(2)}$, this can be rewritten as:

$$S_0 = \frac{\Phi_0}{4G_N^{(2)}}. \tag{1.45}$$

Therefore, from the Bekenstein-Hawking prescription, Φ_0 is interpreted as the area of the extremal surface in the 2d perspective. This is the prefactor that appears in front of the Euler characteristic of the total action, which weights the different topologies of the 2d spacetime.
The second term in the action describing the deviations away from extremality, is called the Jackiw-Teitelboim action and captures the *universal* dynamics of the near-horizon region of a large class of near-extremal black holes inside the boundary cutoff region.

1.4 Classical equations of motion of JT gravity

The most general model of 2d dilaton-gravity is

$$S = \frac{1}{16\pi G_N} \int d^2x \sqrt{-g} \left(\Phi R - U(\Phi)\right) \tag{1.46}$$

, where $g_{\mu\nu}$ is the metric on the the 1+1d spacetime. The dilaton coupling to the 2d Ricci tensor parameterizes a spacetime dependent effective Newton's coupling constant

$$G_{\text{eff}}(x) = \frac{G_N}{\Phi(x)}. \tag{1.47}$$

This is in general a dimensionless quantity, as it should be for a 2d pure gravity theory. In most applications to 2d quantum gravity, we are not a priori interested in the parent higher-dimensional theory which sets the scale of the Newton's constant G_N and the dilaton Φ to Length^2. Therefore, one often rescales both G_N and Φ to dimensionless numbers, while keeping the total action dimensionless. Unlike higher dimensions, the effective coupling constant is dimensionless, and therefore does not set the scale of new physics.

The **Jackiw-Teitelboim** gravity (JT) model is a model of 2d dilaton-gravity described by a specific choice of the dilaton potential: $U(\Phi) = -2\Phi$.

$$\boxed{S_{JT}[\Phi, g] = \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\sqrt{-g}\Phi\,(R+2) + 2\oint_{\partial\mathcal{M}} \sqrt{-h}\Phi_b(K-1)\right]}. \tag{1.48}$$

This model was first written down by Jackiw [13] and Teitelboim [14] in the '80s, and later used as a toy model in 2015 [15] to model AdS_2 backreaction. The latter noted that the UV geometry regulates the backreaction on matter and allows for finite energy states.
The second term is the GHY boundary term, described by the extrinsic curvature K and induced metric h along the boundary $\partial\mathcal{M}$ of the spacetime. The total action is often equipped with an additional matter term $S_m[\phi, g]$, and a pure gravity topological term, represented by a constant shift in the dilaton field $\Phi \to \Phi_0 + \Phi$. All matter fields are collectively written as ϕ, where we assume that the matter action does not explicitly depend on the dilaton. This will regularize the bulk description in terms of pure AdS_2, regardless of the presence of matter. The total action, including the boundary contributions, is then:

$$S[\Phi, \phi, g] \;=\; \frac{\Phi_0}{16\pi G_N}\left[\int_{\mathcal{M}} \sqrt{-g}R + 2\oint_{\partial\mathcal{M}} \sqrt{-h}K\right] + S_{JT}[\Phi, g] + S_m[\phi, g]. \tag{1.49}$$

This is exactly the action that describes the leading order corrections away from AdS_2 in the near-horizon geometry of nearly extremal black holes. Note that the prefactor in front of the Euler characteristic Eq 1.6 in the (Euclidean) Einstein-Hilbert term is exactly the extremal entropy S_0 Eq 1.45.
In the following, we will look at the classical solutions, and the subsequent boundary dynamics.

1.4.1 Variation of the dilaton field

The classical solutions with respect to variation of the dilaton, fixes the manifold to a patch of AdS_2:

$$\boxed{\delta_\Phi S_{JT}[\Phi, g] = 0 \quad \Longleftrightarrow \quad R = -2}. \tag{1.50}$$

This specifies the metric completely since all manifolds in 1+1d are completely symmetric and are fixed in terms of a single Ricci scalar. This does not allow for dynamical gravitons. Resorting to light-cone coordinates $u = t + z$, $v = t - z$, we have argued previously that any metric can locally be brought to the conformal gauge $ds^2 = -e^{2\omega(u,v)}dudv$. The function $\omega(u, v)$ parameterizes the Ricci tensor as $R = 8e^{-2\omega}\partial_u\partial_v\omega$, as can readily be checked. The dilaton equation of motion Eq 1.50 thus reduces to the Liouville equation for the field ω:

$$4\partial_u\partial_v\omega + e^{2\omega} = 0. \tag{1.51}$$

In Poincaré coordinates, its solution is:

$$e^{2\omega} = \frac{1}{z^2} = \frac{4}{(u-v)^2} \qquad \Longleftrightarrow \qquad ds^2 = -\frac{4dudv}{(u-v)^2}. \tag{1.52}$$

Since $z > 0$ in generic AdS_2, the patch is limit to $u > v$. This is the near-horizon geometry of extremal black holes (c.f. Eq 1.10), which have a horizon at the end of an infinite throat that lies at an infinite proper distance as elaborated around Eq 1.11. This horizon is located at the null surface where the metric vanishes, which happens at $u - v \to \infty$ ($u \to +\infty$, or $v \to -\infty$). Compactifying the ranges in u and v, the diagonal boundaries of the Penrose diagram describe the extremal black holes horizons (see figure 1.2). The boundary where the metric blows up is conveniently located at $z = 0$.

The most general solution to Eq 1.51 is a conformal transformation of the Poincaré patch Eq 1.52 in terms of chiral functions $U(u)$ and $V(v)$ that preserve the conformal gauge:

$$ds^2 = -\frac{4\partial_u U(u)\partial_v V(v)dudv}{(U(u) - V(v))^2} = -\frac{4dUdV}{(U-V)^2}. \tag{1.53}$$

This allows to reach the global AdS_2 frame defined in Eq 1.22:

$$U(u) = \tan(u), \qquad V(v) = \tan(v) \tag{1.54}$$

, which becomes

$$ds^2 = -\frac{4}{\sin^2(u-v)}dudv. \tag{1.55}$$

These coordinate transformations are chosen such that the regions where the metric blows up are the timelike boundaries at $z = 0$ and $z = \pi/2$.

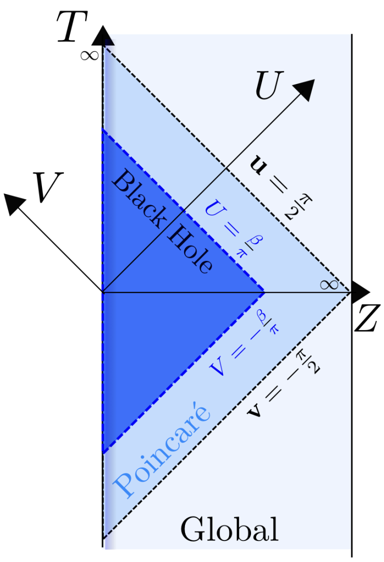


Figure 1.2: Different coordinate frames of AdS_2 and their relation with the Poincaré patch. The bold face coordinates represent the global AdS patch, while the capital coordinates represent the black hole frame. Figure taken from [25].

Analogously to how the Rindler transformations Eqs D.12 define a thermal patch in flat Minkowksi space, we can find a thermal frame in AdS_2, defined by the coordinate transformations

$$U(u) = \frac{\beta}{\pi}\tanh\left(\frac{\pi}{\beta}u\right), \qquad V(v) = \frac{\beta}{\pi}\tanh\left(\frac{\pi}{\beta}v\right) \tag{1.56}$$

, leading to the metric of the black hole patch:

$$ds^2 = -\frac{\pi^2}{\beta^2}\frac{4}{\sinh^2(\frac{\pi}{\beta}(u-v))}dudv. \tag{1.57}$$

Since $|\tanh|< 1$, the black hole patch is contained within the Poincaré patch (that stretches to infinity). The horizons are situated at the locus where the metric vanishes at $u - v \to \infty$. This now happens at a finite proper distance $\int^{+\infty} \frac{dz}{\sinh \frac{2\pi}{\beta} z} < \infty$. We may wish to bring the horizons to finite coordinate distance by compactifying the $z = (u - v)/2$ coordinate. Thereto, one introduces a radial coordinate

$$r = r_h \coth(r_h z) \tag{1.58}$$

with $r_h = \frac{2\pi}{\beta}$. Using $(r^2 - r_h^2) = r_h^2(\coth^2 \frac{2\pi}{\beta} z - 1) = \frac{r_h^2}{\sinh^2 \frac{2\pi}{\beta} z}$ and $dr = \frac{r_h^2 dz}{\sinh^2 \frac{2\pi}{\beta} z}$, this allows us to rewrite the metric as:

$$ds^2 = -(r^2 - r_h^2)dt^2 + \frac{dr^2}{r^2 - r_h^2}. \tag{1.59}$$

Note that this is exactly the same metric that we have encountered in the near-horizon region of a nearly extremal black hole Eq 1.9. In particular, we can repeat the argument given around Eq 1.13, to conclude that the temperature can be determined by demanding regularity of the solutions near the horizon:

$$T = \frac{r_h}{2\pi}. \tag{1.60}$$

This ensures that $\beta = 2\pi/r_h$ can indeed be identified with an inverse temperature. The same discussion allows us to identify this temperature with the temperature of the parent nearly-extremal black hole in the near-horizon limit.

By rewriting the black hole patch in terms of the proper distance[5] $d\rho = \frac{2\pi}{\beta}\frac{dz}{\sinh^2(\frac{\pi}{\beta}(u-v))}$, we obtain the Rindler patch:

$$ds^2 = -\frac{4\pi^2}{\beta^2}\sinh^2 \rho \, dt^2 + d\rho^2. \tag{1.61}$$

A convenient summary of the different coordinate patches and their embeddings within the Poincaré patch is given in figure 1.2 [25].

1.4.2 Variation of the metric

Variation of the total bulk JT action Eq 1.48 with respect to the metric leads to the equations of motion governing the backreaction of the dilaton in response to the matter sources. Let us do this calculation explicitly. We use the identity[6] $\delta\sqrt{-g} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}$, and the Palatini identity[7] $\delta R^\lambda_{\ \mu\sigma\nu} = \nabla_\sigma(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\sigma})$. Summing over $\lambda \equiv \sigma$, the variation of the Ricci tensor $R_{\mu\nu} = R^\rho_{\ \mu\rho\nu}$ is conveniently $\delta R_{\mu\nu} = \nabla_\sigma\delta\Gamma^\sigma_{\mu\nu} - \nabla_\nu\Gamma^\sigma_{\sigma\mu}$. Varia-

[5] Integration yields $\rho = \text{Arcsinh}\left(1/\sinh \frac{2\pi z}{\beta}\right)$

[6] This is an application of the matrix identity $\text{Tr}\log M = \log\det M$, which can be realized in the eigenbasis of M. Variation yields $\text{Tr}(M^{-1}\delta M) = \frac{\delta \det M}{\det M}$. Using the chain rule for the square root in $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g$, together with the variation of $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$ yields the desired identity.

[7] This is easiest to deduce via variation of the definition $[\nabla_\sigma, \nabla_\nu]V^\lambda = R^\lambda_{\ \mu\sigma\nu}V^\mu$. Using $\delta\nabla_\nu V^\lambda = \delta\Gamma^\lambda_{\nu\alpha}V^\alpha$ and the fact that the variation of the torsion-free connection is a vector field with $[\nabla_\sigma, \delta\nabla_\nu] = \nabla_\sigma(\delta\nabla_\nu)$, yields the desired Palatini identity.

tion of the Ricci scalar $R = R_{\mu\nu}g^{\mu\nu}$ is therefore $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu})$. To proceed, we use the key identity $\delta\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\nabla_\nu\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\nu\lambda} - \nabla_\lambda\delta g_{\mu\nu})$. The derivation is straightforward, but rather long so I skip it for brevity, see e.g. [60] [59]. Variation of the Ricci tensor is therefore explicitly:

$$\begin{aligned}\delta R_{\mu\nu}g^{\mu\nu} &= \nabla_\rho\left[g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\nu_{\nu\mu}\right]\\ &= \nabla_\rho\left[g^{\mu\nu}\frac{g^{\rho\lambda}}{2}\left(\nabla_\nu\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\nu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right) - g^{\mu\rho}\frac{g^{\nu\lambda}}{2}\left(\nabla_\nu\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\nu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right)\right]\\ &= \nabla_\rho\left[g^{\mu\nu}g^{\rho\lambda}\nabla_\mu\delta g_{\nu\lambda} - g^{\mu\nu}g^{\rho\lambda}\nabla_\lambda\delta g_{\mu\nu}\right].\end{aligned}$$

Plugged into the bulk JT action, we obtain the following variation:

$$\begin{aligned}\delta_g S_{JT} &= \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\ \Phi\left[\sqrt{-g}\left(\delta R_{\mu\nu}g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}\right) + \delta\sqrt{-g}\left(R+2\right)\right]\\ &= \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[\Phi\nabla_\rho\left(g^{\mu\nu}g^{\rho\lambda}\nabla_\mu\delta g_{\nu\lambda} - g^{\mu\nu}g^{\rho\lambda}\nabla_\lambda\delta g_{\mu\nu}\right) + \Phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\right)\delta g^{\mu\nu}\right].\end{aligned}$$

We have argued that for any 2d metric, the vacuum Einstein equations are trivially satisfied. Therefore we set $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \equiv 0$. Additionally performing a double partial integration in the first term, and using the metric postulate $\nabla_\mu g_{\nu\lambda} = 0$ yields

$$\delta_g S_{JT} = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[\delta g_{\nu\lambda}\nabla^\nu\nabla^\lambda\Phi - \delta g_{\mu\nu}g^{\mu\nu}\nabla^2\Phi - \Phi g_{\mu\nu}\delta g^{\mu\nu}\right] + \text{boundary terms}.$$

Of course, Stokes theorem produces boundary terms that do not automatically vanish for a non-constant metric and a non-compact spacetime. A thorough analysis in appendix A of [36] deals with these boundary terms, and shows explicitly that they are precisely compensated by variation of the GHY boundary term in the total action Eq 1.48. Relabeling indices, and using[8] $g_{\mu\alpha}\delta g^{\alpha\nu} = -g^{\alpha\nu}\delta g_{\mu\alpha}$, yields:

$$\delta_g S_{JT} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi + \Phi g_{\mu\nu}\right]\delta g^{\mu\nu} + \cancel{\text{boundary terms}}.$$

Adding a possible matter sector $S_m[\phi, g]$ that does not explicitly couple to the dilaton, incorporates a matter stress tensor in the total variation $T^m_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$. The equations of motion of the dilaton in terms of the non-vanishing matter sources are:

$$\boxed{\delta(S_{JT}[\Phi, g] + S_m[\phi, g]) = 0 \quad\Longleftrightarrow\quad \nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi + \Phi g_{\mu\nu} = -8\pi G_N T^m_{\mu\nu}}. \tag{1.62}$$

These equations can be written more tractable in the conformal gauge $ds^2 = -e^{2\omega(u,v)}dudv$. Straightforward calculations show that the only non-vanishing Chritoffel connections are $\Gamma^u_{uu} = 2\partial_u\omega(u,v)$ and $\Gamma^v_{vv} = 2\partial_v\omega(u,v)$.
The uu (resp. vv)-components are readily worked out:

$$-8\pi G_N T^m_{uu} = \nabla_u\partial_u\Phi = \partial_u\partial_u\Phi - \Gamma^u_{uu}\partial_u\Phi = \partial_u\partial_u\Phi - 2\partial_u\omega\partial_u\Phi = e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi).$$

[8] Taking the variation of $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$.

The uv-component involves the non-vanishing metric contributions:

$$\begin{aligned}-8\pi G_N T^m_{uv} &= \nabla_u\partial_v\Phi - g_{uv}g^{\alpha\beta}\nabla_\alpha\partial_\beta\Phi + g_{uv}\Phi = \partial_u\partial_v\Phi - 2g_{uv}g^{uv}\partial_u\partial_v\Phi + g_{uv}\Phi \\ &= -\partial_u\partial_v\Phi - \frac{1}{2}e^{2\omega}\Phi.\end{aligned}$$

Summarizing, the non-vanishing dilaton equations of motion in the conformal gauge are

$$-e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi) \;=\; 8\pi G_N T^m_{uu}, \tag{1.63}$$

$$-e^{2\omega}\partial_v(e^{-2\omega}\partial_v\Phi) \;=\; 8\pi G_N T^m_{vv}, \tag{1.64}$$

$$2\partial_u\partial_v\Phi + e^{2\omega}\Phi \;=\; 16\pi G_N T^m_{uv}. \tag{1.65}$$

Therefore, we note that all backreaction on the presence of matter is contained within the dilaton equations of motion, and not in the spacetime geometry itself.
We conclude that the model describes pure AdS_2 in the bulk, while it gets adjusted in the UV by a non-trivial dilaton profile. This in turn regulates the gravitational backreaction. Note that although the dilaton and metric are now dynamical, they still have no local excitations and are completely determined by the equations of motion in terms of an influx of matter.

Vacuum solutions

Let us start by solving the above equations of motion without matter sources: $T_{uu} = T_{vv} = T_{uv} \equiv 0$. In the Poincaré patch, the metric solution is given by Eq 1.52:

$$e^{2\omega} = \frac{1}{z^2} = \frac{4}{(u-v)^2}.$$

We can integrate the constraints Eqs 1.63, 1.64, to obtain an ansatz that is imposed on the equation of motion Eq 1.65. Integrating the former yields directly:

$$\begin{aligned}\partial_u(e^{-2\omega}\partial_u\Phi) = \partial_u\left(\frac{(u-v)^2}{4}\partial_u\Phi\right) = 0 \quad &\rightarrow \quad \Phi(u,v) = \frac{c_1(v)}{u-v} + c_2(v) = \frac{c_1'(v) + c_2'(v)u}{u-v}, \\ \partial_v(e^{-2\omega}\partial_v\Phi) = \partial_v\left(\frac{(u-v)^2}{4}\partial_v\Phi\right) = 0 \quad &\rightarrow \quad \Phi(u,v) = \frac{c_3(u)}{u-v} + c_4(u) = \frac{c_3'(u) + c_4'(u)v}{u-v},\end{aligned}$$

in terms of chiral functions $c_1'(v), c_2'(v), c_3'(u), c_4'(u)$, playing the role of integration constants. We note that a proper ansatz for $\Phi(u,v)$ is $\Phi(u,v) = \frac{M(u,v)}{u-v}$, where $M(u,v)$ is at most linear in both u and v. In other words,

$$\Phi(u,v) = \frac{a + b\,u + c\,v + d\,uv}{u-v}.$$

Plugging this ansatz into the equation of motion Eq 1.65, we obtain the consistency requirement:

$$0 \;\equiv 2\partial_u\partial_v\Phi + e^{2\omega}\Phi = 2\frac{b-c}{(u-v)^2} \qquad \Longleftrightarrow \;\; b = c.$$

The most general Poincaré vacuum solution can conveniently be described in terms of three integration constants:

$$\Phi(u,v) = \frac{a + b(u+v) - \mu uv}{u-v}. \tag{1.66}$$

To obtain the correct dimensions for the dilaton, b is dimensionless, while a has dimensions of length, and μ has dimensions of energy[9]. Since the Poincaré vacuum Eq 1.52 is PSL$(2,\mathbb{R})$ invariant, we can perform such a transformation and set $b \equiv 0$. This brings the Poincaré vacuum solution to its canonical form:

$$\Phi(u,v) = \frac{a - \mu uv}{u-v}. \tag{1.67}$$

Using the chain rule in the dilaton equations of motion, the most general metric solution is again a conformal transformation of the Poincaré frame in terms of chiral functions $U(u)$, $V(v)$ Eq 1.53. Therefore, the most general solutions to the equations of motion are the conformal transformations of Eq 1.66:

$$\boxed{e^{2\omega} = \frac{4\partial_u U(u)\partial_v V(v)}{(U(u)-V(v))^2}, \qquad \Phi(u,v) = \frac{a - \mu\, U(u)V(v)}{U(u)-V(v)}}. \tag{1.68}$$

Note that u, v are the proper coordinates, while U, V are the Poincaré embedding coordinates. For $\mu = 0$, the entire AdS_2 manifold is covered by transitioning to the global frame via Eq 1.54:

$$e^{2\omega} = \frac{4}{\sin^2(u-v)}, \qquad \Phi = a\frac{\cos u \cos v}{\sin(u-v)}. \tag{1.69}$$

Solutions with matter

Since the dilaton potential for JT gravity is linear in Φ, the equations of motion remain tractable in the presence of non-zero matter sources. We will mainly focus on conformal matter, which is characterised by the vanishing trace condition $T^{\mu}_{\mu} = 0$. In the conformal gauge, the only non-vanishing metric components are of mixed signature, which therefore implies[10] $T_{uv} = 0$. Energy conservation $\nabla_\mu T^{\mu\nu} = 0$ restricts these stress tensor components further to chiral functions[11] $T_{uu}(u)$ and $T_{vv}(v)$. Writing $\Phi = \frac{M(u,v)}{U-V}$, and using the general AdS_2 Poincaré coordinates $e^{2\omega} = 4/(U-V)^2$, it can be checked that the UU (resp VV)-constraint equations Eq 1.63 (1.64) can be written as:

$$\begin{aligned} \partial_U\partial_U M(U,V) &= -(U-V)8\pi G_N T_{UU}, \\ \partial_V\partial_V M(U,V) &= -(U-V)8\pi G_N T_{VV}. \end{aligned} \tag{1.70}$$

When $T_{UU} = T_{VV} = 0$, we know that the general solution is a bilinear polynomial in U and V; $M_0(U,V) = a + bU + cV + d\, UV$, where we can set $b = c = 0$ by means of a PSL$(2,\mathbb{R})$ transformation. The general solution can be written as [15]:

$$M(U,V) = M_0(U,V) - I_U(U,V) - I_V(U,V) \tag{1.71}$$

[9] We will see how this constant can indeed be interpreted as an energy variable.

[10] $T^{\mu}_{\mu} = 0 \iff T^u_u + T^v_v = 0 \iff 2g^{uv}T_{uv} = 0 \iff T_{uv} = 0$

[11] $\nabla_\mu T^{\mu\nu} = 0$ yields for the u (resp. v) component $\nabla_u T^{uu} = 0 \iff \nabla_u T_{vv} = 0 \iff \partial_u T_{vv} = 0$. Therefore $T_{vv}(u,v) \equiv T_{vv}(v)$.

, where

$$
\begin{aligned}
I_U(U,V) &= 8\pi G_N \int_U^{+\infty} ds(s-V)(s-U)T_{UU}(s),\\
I_V(U,V) &= 8\pi G_N \int_{-\infty}^{V} ds(s-V)(s-U)T_{VV}(s).
\end{aligned} \tag{1.72}
$$

We can check explicitly that these integral equations indeed satisfy Eq 1.70:

$$
\begin{aligned}
\partial_U\partial_U I_U(U,V) &= 8\pi G_N \partial_U \left[-(U-V)(U-U)T_{UU}(U) + \int_U^{+\infty} ds(s-V)(-\partial_U(U))T_{UU}(s)\right]\\
&= 8\pi G_N (U-V)T_{UU}(U),\\
\partial_V\partial_V I_V(U,V) &= 8\pi G_N \partial_V \left[(V-V)(V-U)T_{VV}(V) + \int_{-\infty}^{V} ds(s-U)(-\partial_V(V))T_{VV}(s)\right]\\
&= 8\pi G_N (U-V)T_{VV}(V).
\end{aligned}
$$

Furthermore, the cross-term vanishes

$$
\partial_U\partial_U I_V(U,V) = -8\pi G_N \partial_U \int_{-\infty}^{V} ds(s-V)T_{VV}(s) = 0
$$

, and the solution indeed satisfies Eq 1.70:

$$
\partial_U\partial_U M(u,v) = -8\pi G_N (U-V)T_{UU}.
$$

The general solution of the dilaton field in the presence of matter is therefore

$$
\boxed{\Phi(u,v) = \frac{a}{U-V}\left(1 - \frac{\mu}{a}UV - \frac{1}{a}(I_+(u,v) + I_-(u,v))\right)}. \tag{1.73}
$$

If we start with the Poincaré vacuum solution with $\mu = 0$, the vacuum solution Eq 1.67 interpolates between pure AdS_2 at large z and a UV modification near the boundary $z \sim 0$, where the dilaton diverges. The physical interpretation is more transparent when looking at the parent higher-dimensional theory. We have seen how the near-horizon geometry of a large class of near-extremal black holes is described by the JT gravity model, by shifting $\Phi \to \Phi_0 + \tilde{\Phi}$. Since the near-horizon region is AdS_2, there are no curvature singularities. On the other hand, taking this $\Phi_0 \equiv 1$, the geometry has a singularity where the total dilaton field $\Phi = 1 + \tilde{\Phi}$ vanishes. Since the dilaton describes the transverse area of the higher-dimensional black hole, this is a curvature singularity of the parent higher-dimensional spacetime, where the Ricci tensor of the two-sphere blows up $R = 1/\Phi \to \infty$. In terms of the 2d target space, this is manifested in the blowing up of the effective Newton's coupling constant Eq 1.47.
In the case under consideration ($\mu = 0$), the dilaton

$$
\Phi(u,v) = 1 + \frac{a}{u-v}
$$

goes to zero in the complementary Poincaré patch, which is the mirroring image $z \to -\tilde{z}$. Indeed, the locus of $\Phi = 1 - a/2\tilde{z} \equiv 0$ has a non-vanishing solution for $\tilde{z} > 0$. The singularity is displayed as a red line in figure

1.3, together with the Poincaré patch in green.

We can throw in a pulse of energy from the boundary along the U-null line $T_{VV}(V) = E\delta(V)$. By application of Eq 1.73, the only non-vanishing component is $I_V(V) = 8\pi G_N\, EUV\theta(V)$, and the solution is:

$$\Phi(U,V) = 1 + \frac{a - 8\pi G_N E\theta(V)UV}{U-V} \equiv 1 + \frac{a - \mu\theta(V)UV}{U-V} \tag{1.74}$$

, where $\mu \equiv 8\pi G_N E$. Starting with $\mu = 0$ at times $V < 0$, the influx of matter from the boundary creates a vacuum solution with non-vanishing $\mu = 8\pi G_N E$. In Poincaré coordinates, a singularity is found at the zeros of the dilaton $(U + \frac{1}{\mu})(V - \frac{1}{\mu}) = \frac{1}{\mu}(a - \frac{1}{\mu})$. The subsequent computations were performed explicitly in [15].
Qualitatively, the singularity remains timelike when $\mu < 1/a$ and a second singularity appears at the boundary $z = 0$. For $\mu > 1/a$, the singularity becomes spacelike. In terms of the energy, there is a timelike singularity for energies below some critical value $E < E_c = 1/8\pi G_N a$, and a spacelike singularity for $E > E_c = 1/8\pi G_N a$.
We can interpret the influx of matter from the boundary as the creation of a black hole, which produces a naked singularity in the Poincaré patch for $E > E_c$ at $t = \sqrt{a/\mu}$ (c.f. Eq 1.77). This singularity is always shielded in the black hole frame $U(u) = \sqrt{\frac{a}{\mu}}\tanh\sqrt{\frac{\mu}{a}}u,\ \ V(v) = \sqrt{\frac{a}{\mu}}\tanh\sqrt{\frac{\mu}{a}}v$, which explicitly yields:

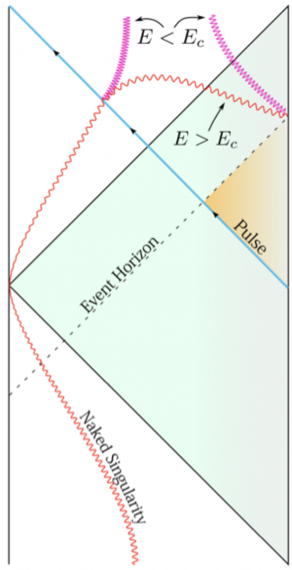


Figure 1.3: Embedding of the Poincaré patch and black hole patch after creation of black hole, where z increases to the left. After the influx of the energy pulse, a naked singularity emerges in the Poincaré patch. Taken from [15].

$$\begin{aligned}\Phi(u,v) = 1 + \frac{a - \mu U(u)V(v)}{U(u) - V(v)} &= 1 + \frac{a - a\tanh\sqrt{\mu/a}u\tanh\sqrt{\mu/a}v}{\sqrt{a/\mu}(\tanh\sqrt{\mu/a}u - \tanh\sqrt{\mu/a}v)} \\ &= 1 + \sqrt{a\mu}\frac{\coth\sqrt{\mu/a}v\coth\sqrt{\mu/a}u - 1}{\coth\sqrt{\mu/a}v - \coth\sqrt{\mu/a}u} \\ &= 1 + \sqrt{a\mu}\coth\left(\sqrt{\mu/a}u - \sqrt{\mu/a}v\right) = 1 + \sqrt{a\mu}\coth(\sqrt{\mu/a}2z) \\ &= 1 + \frac{a}{2}r\end{aligned}$$

, with r defined in Eq 1.58. Within the Poincaré patch, these coordinates are restricted to $U, V < \sqrt{a/\mu}$. In retrospect, this is the physical frame that describes the exterior of the black hole created after the pulse, and shields the naked singularity. The Hawking temperature near the horizon is given by Eq 1.60, yielding:

$$\frac{\beta}{\pi} = \sqrt{\frac{a}{\mu}} \iff T = \frac{1}{\pi}\sqrt{\frac{\mu}{a}} = \frac{1}{\pi}\sqrt{\frac{8\pi G_N E}{a}}. \tag{1.75}$$

Black hole shockwave

We can generalize this black hole solution to include the effect of an additional infalling matter pulse, where we start from a black hole solution with mass $M = E$ at time $t = t_1$, and send in another massless particle with $\delta E = \hbar\omega_1$. This increases the black hole mass to $\tilde{E} = E + \delta\omega_1$. Writing $\sqrt{a/\mu} = \sqrt{2C/E}$ with $C := \frac{a}{16\pi G_N}$, this modifies the dilaton profile and the black hole patch coordinates $u \to \tilde{u}$. At time $t = t_1$, there should be a continuous transition of the latter:

$$\sqrt{\frac{2C}{E}}\tanh\left(\sqrt{\frac{E}{2C}}(u - t_1)\right) = \sqrt{\frac{2C}{\tilde{E}}}\tanh\left(\sqrt{\frac{\tilde{E}}{2C}}(\tilde{u} - t_1)\right). \tag{1.76}$$

This will cover a smaller triangular black hole area in the original black hole frame (see figure 1.4), where the shift $u - \tilde{u}$ at late times is equal to [16]:

$$u - \tilde{u} = \frac{\hbar\omega_1}{8}\sqrt{\frac{2C}{E^3}}e^{2\sqrt{E/2C}(u-t_1)}.$$

This represents a classical black hole shockwave since the outgoing signals in the original black hole frame along the null line $u = t_2$ ($t_2 > t_1$), reach the boundary at a later coordinate time $\tilde{u} = \tilde{t}_2$ given by $\delta t_2 = \frac{\hbar\omega_1}{8}\sqrt{\frac{2C}{E^3}}e^{2\sqrt{E/2C}(t_2-t_1)}$, with a maximal Lyapunov exponent $\lambda_L = \sqrt{2E/C} = 2\pi T$, saturating the bound on chaos [61].

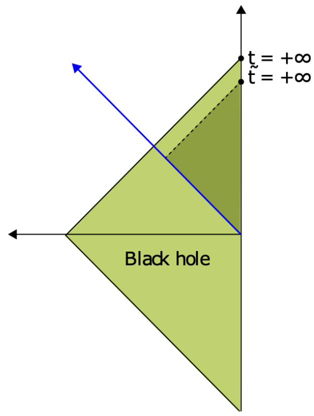


Figure 1.4: Creation of a new black hole frame in an original black hole after the injection of a pulse with energy $\delta E = \hbar\omega_1$. Figure taken from [16]

Backreaction in AdS_2

The general vacuum solution with matter

$$\Phi = 1 + \frac{a - \mu\, uv}{u - v}$$

has a strong coupling singularity for $\Phi = 0$ at

$$t = \pm\sqrt{\frac{a + 2z + \mu z^2}{\mu}} \tag{1.77}$$

, which reaches the boundary $z = 0$ at $t = \pm\sqrt{a/\mu}$. Thus the singularity remains shielded in the black hole frame $U(u) = \sqrt{\frac{a}{\mu}}\tanh\sqrt{\frac{\mu}{a}}u$ for which $|U(u)| \leqslant \sqrt{\frac{a}{\mu}}$ (respectively $|V(v)| \leqslant \sqrt{\frac{a}{\mu}}$). On the other hand, when $a = 0$, no region on the boundary remains and the past and future singularities meet at $t = 0$.

Similarly, for any pulse thrown into the $a = 0, \mu = 0$ solution, there is an instant divergence in the dilaton profile at the boundary, where now suddenly $\mu > 0$ and $\Phi \to \infty$. We should therefore always keep $a > 0$ and allow for solutions where the dilaton always diverges at the boundary to avoid a runaway backreaction in the dilaton profile. This is again a manifestation that AdS_2 does not allow for non-zero energy excitations, and one needs to consider UV modifications near the boundary by introducing a diverging dilaton profile.

1.5 Conformal symmetry breaking in nearly AdS_2

We have seen pure AdS_2 to be inconsistent with the existence of finite energy excitations above the AdS_2 vacuum. Furthermore, pure gravity in $D = 2$ is topological and does not allow for dynamical gravitons. In the putative dual CFT_1, the backreaction problem is translated to the vanishing of the trace, which for a one-component tensor implies the vanishing of the Hamiltonian. In a 1d CFT, the conformal symmetries are all the reparametrizations at the timelike asymptotic boundary $t \to \tilde{t}(t)$. This reparametrization symmetry is *spontaneously* broken by the presence of the asymptotic boundary down to PSL$(2, \mathbb{R})$. The latter is the isometry group of the AdS_2 metric, which we can treat as a gauge redundancy. The Nambu-Goldstone modes of this spontaneous symmetry breaking are all the Fourier modes of this reparametrization symmetry and describe the zero modes of the associated spontaneous symmetry breaking pattern. Since the bulk manifold is topological away from the boundary, all remaining dynamics are encoded by the boundary reparametrization modes, which we can interpret as boundary gravitons.

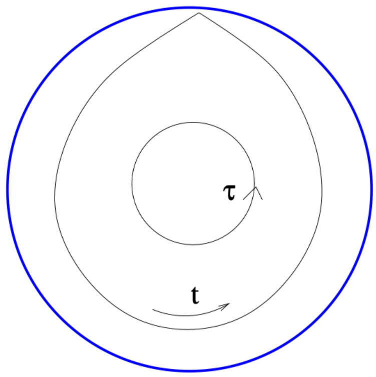


Figure 1.5: The Euclidean disk, parameterized in terms of a radial coordinate and the Poincaré time (denoted here by t) -or the black hole time τ coordinates. Figure taken from [17].

In section 1.3.4, we have argued that the leading order corrections away from the extremal pure AdS_2-regime are described by the presence of a dilaton field, which blows up near the boundary. This leading order correction deforms the manifold at the boundary from pure AdS_2 to nearly (N)AdS_2. In a sense, all backreaction to matter is encoded by the dynamics of the dilaton field, and the different spacetimes are characterized by different renormalized dilaton fields. The backreaction of the dilaton also controls the shape of the boundary curve completely. This correction breaks the conformal symmetry at the boundary *explicitly* down to the PSL$(2, \mathbb{R})$ subgroup. The boundary reparametrization modes are weighted by an action under this explicit symmetry breaking, which is still invariant under the PSL$(2, \mathbb{R})$ isometry subgroup. The simplest candidate that is PSL$(2, \mathbb{R})$ invariant and contains the least amount of derivatives, is the 0+1d Schwarzian boundary action. This section argues how this action turns up directly from the total Euclidean JT action [17] and at the level of the equations of motion [16].

1.5.1 Spontaneous symmetry breaking in pure AdS_2

We will henceforth denote the Poincaré lightcone coordinates with capital letters $U(u)$, $V(v)$, where $u = t + z$ and $v = t - z$ denote the proper coordinates. In terms of the temporal F and spatial Z coordinate, they are defined as $U = F + Z$, $V = F - Z$. In what follows, we will consider the Euclidean variant of the AdS_2

Figure 1.6: The hyperbolic disc, and an arbitrary cutoff geometry near the boundary. All these geometries have the same topology and share the same leading order Einstein-Hilbert action. Figure taken from [17].

spacetime, by Wick rotating the Lorentzian time coordinate $F_{Lorentz} \to -iF_{Euclid}$. The Euclidean version of AdS_2 is the Poincaré upper half plane, which can be compactified to the 2d hyperbolic disk by adding a point at infinity, see figure 1.5. Both the Poincaré and black hole coordinates describe the same patch in Euclidean signature:

$$ds^2 = \frac{dF^2 + dZ^2}{Z^2}, \quad ds^2 = d\rho^2 + \frac{4\pi^2}{\beta^2}\sinh^2\rho d\tau^2 \tag{1.78}$$

, where now the black hole time coordinate τ is periodic in β, while the Euclidean Poincaré coordinate covers $-\infty < F < +\infty$.
The low-energy contribution of the Wick rotated total action Eq 1.49 is the Euclidean Einstein-Hilbert action, which describes pure AdS_2:

$$I[g] = -\frac{\Phi_0}{16\pi G_N}\left[\int_{\mathcal{M}} \sqrt{-g}R + 2\oint_{\partial\mathcal{M}} \sqrt{-h}K\right]. \tag{1.79}$$

In Euclidean signature, the Gauss-Bonnet theorem dictates that this action can be identified with the topological Euler characteristic $I[g] = -S_0\chi$. The prefactor $S_0 = \frac{\Phi_0}{4G}$ describes the leading extremal entropy of the ambient higher-dimensional black hole (c.f. Eq 1.45). Since the action depends only on the topology of the manifold $\mathcal{M}$, there is a huge degree of degeneracy. In particular, the lowest energy states have a disk-shaped topology (with $\chi = 2 - 2g - n = 1$). Therefore, all deformations of the disk share the same action.

We imagine that we cut off the hyperbolic disk along some boundary curve $(F(\tau), Z(\tau))$, below which the IR theory is described by AdS_2. τ is the β-periodic Euclidean boundary time coordinate that is well-defined in the UV and parameterizes the asymptotic boundary curve. In order to compare the different geometries, we need to gauge-fix their asymptotic behaviour. To proceed, we move the boundary slightly inwards $z = \frac{u-v}{2} \sim \epsilon$, and choose to describe the different cutoff geometries by fixing the effective metric along the boundary curve $g_{bdry} = g_{\tau\tau} = \frac{1}{\epsilon^2}$. This implies that the boundary curves have large proper length as $\epsilon \to 0$:

$$\int ds = \int_0^\beta \frac{d\tau}{\epsilon} = \frac{\beta}{\epsilon} \to \infty.$$

This fixes the asymptotic length of the boundary curves, but leaves the freedom for reparametrizations with

$$\frac{1}{\epsilon^2} = g_{\tau\tau} = \frac{F'(\tau)^2 + Z'(\tau)^2}{Z^2(\tau)}. \tag{1.80}$$

This in turn controls the shape of the boundary curve $Z \equiv \epsilon\sqrt{F'(\tau)^2 + Z'(\tau)^2} \sim \epsilon F'(\tau) + \mathcal{O}(\epsilon)$. To first order, all boundary curves are described by a single function $F(t)$, whose Fourier modes are the boundary gravitons, which are generated by the asymptotic symmetries of AdS_2: $\zeta^F \partial_F = \varepsilon(\tau)$, $\zeta^Z \partial_Z = Z\varepsilon'(\tau)$, corresponding to shifts in $F(\tau) = \tau + \varepsilon(\tau)$. Since the Euler characteristic only depends on the topology of the disk, all cutoff spaces of the hyperbolic disk share the same action. The infinite asymptotic reparametrization modes are all the zero-modes of this degeneracy.
There is still a redundancy in the description since the Euclidean AdS_2-manifold is invariant under the isometry group $\mathrm{SO}(2,1) \simeq \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \equiv \mathrm{PSL}(2,\mathbb{R})$. This is the same isometry group of Lorentzian AdS_2. It is a projective group that relates the representation matrices with minus itself, and acts on the reparametrization modes along the boundary as a Möbius transformation:

$$F(\tau) \to \frac{aF(\tau) + b}{cF(\tau) + d}, \qquad ad - bc = 1. \tag{1.81}$$

We can see this by writing the Euclidean Poincaré plane in terms of the complex coordinate $w = F + iZ$ and its conjugate:

$$ds^2 = \frac{dw d\overline{w}}{(\Im w)^2} = -4\frac{dw d\overline{w}}{(w - \overline{w})^2}$$

, where the $\mathrm{PSL}(2,\mathbb{R})$ symmetry acts as[12] $w \to \frac{aw+b}{cw+d}$, with $ad - bc = 1$. The transformation rule for F along the boundary curve follows directly by noting that Z is subleading in ϵ. The different cutoff spaces are simply translated and rotated in the hyperbolic geometry under this subgroup, and do not correspond to new cutoff geometries. We should therefore simply mod the group of all reparametrization symmetries by this underlying isometry subgroup. Thus, the physical asymptotic symmetries of the Einstein-Hilbert term are spontaneously broken down to equivalence classes under $\mathrm{PSL}(2,\mathbb{R})$.

1.5.2 Explicit symmetry breaking in $NAdS_2$

The Einstein-Hilbert action is only capable of describing the extremal and ground state entropy. Its topological nature is translated in a full conformal reparametrization symmetry at the asymptotic boundary.
The leading-order correction in the universal action Eq 1.49 breaks this reparametrization symmetry explicitly and associates with every cutoff disk a unique action up to an identification of isometry related $\mathrm{PSL}(2,\mathbb{R})$ cutoff disk geometries. The true vacuum symmetry is broken explicitly down to $\mathrm{PSL}(2,\mathbb{R})$, and is no longer degenerate. We have a situation in mind where the very near-horizon region of the parent extremal black hole enjoys a full reparametrization symmetry, which is broken explicitly by moving away from the horizon.
The action weighting the different cutoff surfaces should effectively be one-dimensional and should depend only on the asymptotic reparametrization modes of the boundary curve. Furthermore, it should still be invariant

[12] Explicitly, we work out $dw \to \frac{a\,dw}{cw+d} - \frac{aw+b}{(cw+d)^2}c dw = \frac{dw}{(cw+d)^2}$ since $ad - bc = 1$, and analogously for $\overline{w}$. Furthermore, it acts on $\Im w$ as $(w - \overline{w}) \to \frac{aw+b}{cw+d} - \frac{a\overline{w}+b}{c\overline{w}+d} = \frac{w-\overline{w}}{(cw+d)(c\overline{w}+d)}$. This leaves $ds^2 = -4\frac{dw d\overline{w}}{(w-\overline{w})^2}$ invariant.

under the $\mathrm{PSL}(2,\mathbb{R})$ isometry subgroup. The simplest such action is the Schwarzian boundary action:

$$S \sim \int d\tau \{F(\tau),\tau\}, \qquad \text{with} \qquad \{F,\tau\} \equiv \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \left(\frac{F''}{F'}\right)' - \frac{1}{2}\left(\frac{F''}{F'}\right)^2. \tag{1.82}$$

To see how this effective action emerges from the Euclidean Jackiw-Teitelboim model, we need to fix the boundary conditions of the dilaton and the metric. We again choose to regularize the asymptotic boundary curves to a fixed length $g_{\tau\tau} = \frac{1}{\epsilon^2}$. We have also seen that the metric equations of motion impose the dilaton to blow up near the boundary as $\Phi \sim \frac{1}{z}$. This regulates the backreaction of matter completely in terms of a diverging dilaton profile. We choose to parameterize the asymptotic behaviour of dilaton near $\epsilon \sim 0$ in terms of a regularized dilaton field $\phi_r(\tau)$:

$$\Phi = \frac{\phi_r(\tau)}{2\epsilon}. \tag{1.83}$$

This regularized dilaton field completely determines the different UV modifications to pure AdS_2. Note that we cut off the space at $\frac{1}{\epsilon} \ll \Phi_0$, in order to remain in the near-extremal regime.
The Euclidean Jackiw-Teitelboim action, equipped with a GHY boundary term is:

$$I_{JT}[\Phi, g] = -\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\sqrt{g}\Phi\,(R+2) + 2\oint_{\partial\mathcal{M}} \sqrt{h}\Phi(K-1)\right]. \tag{1.84}$$

The equation of motion of the dilaton immediately sets the on-shell bulk term to zero ($R+2=0$). This is a clear manifestation that the bulk is topological, and all dynamics are encoded at the asymptotic boundary. Using the asymptotic behaviour of the metric and dilaton field reduces the total JT action to an effective boundary description:

$$I_{JT}[\Phi, g] \to -\frac{1}{16\pi G_N}\oint_{\partial\mathcal{M}} \frac{d\tau}{\epsilon^2}\phi_r(\tau)(K-1). \tag{1.85}$$

To calculate the extrinsic curvature, we parameterize the tangent and normal vectors along the boundary curve $(F(\tau), Z(\tau))$ as $T^\mu = (F'(\tau), Z'(\tau))$, respectively $n_\mu = \frac{1}{Z(\tau)\sqrt{(F'(\tau))^2+(Z'(\tau))^2}}(Z'(\tau), -F'(\tau))$. Using the hyperbolic metric, these are indeed seen to be orthogonal $n_\mu T^\mu = 0$. Furthermore, n_μ is properly normalized $n_\mu n^\mu = 1$. The extrinsic curvature along the boundary curve is defined as [42]:

$$K = \frac{T^\mu T^\nu \nabla_\mu n_\nu}{g_{\mu\nu}T^\mu T^\nu} \tag{1.86}$$

, where $g_{\mu\nu}$ is the metric of the hyperbolic disk and T^μ act as pullback vectors along the boundary curve. Note that the standard definition of the extrinsic curvature $K = g^{\mu\nu}\nabla_\mu n_\nu$ reduces to the latter by replacing the metric by projectors on directions orthogonal to n^μ:

$$K = g^{\mu\nu}\nabla_\mu n_\nu = (g^{\mu\nu} - n^\mu n^\nu)\nabla_\mu n_\nu \equiv \frac{T^\mu T^\nu}{T^2}\nabla_\mu n_\nu.$$

This follows from the normalization constraint $n^\mu n_\mu = 1$. The non-vanishing Christoffel connections of the hyperbolic metric are $\Gamma^F_{FZ} = \Gamma^Z_{ZZ} = -\frac{1}{Z}$ and $\Gamma^Z_{FF} = \frac{1}{Z}$. The normalization $g_{\mu\nu}T^\mu T^\nu$ is simply:

$$g_{\mu\nu}T^\mu T^\nu = \frac{F'^2 + Z'^2}{Z^2}. \tag{1.87}$$

The derivative along the asymptotic time coordinate τ is taken from the Poincaré coordinates via the chain rule; $\frac{df(\tau)}{dF} = \frac{f'(\tau)}{F'(\tau)}$, $\frac{df(\tau)}{dZ} = \frac{f'(\tau)}{Z'(\tau)}$. Using the definition of the extrinsic curvature, we eventually obtain [42] [25]:

$$K = \frac{F'(F'^2 + Z'^2 + ZZ'') - ZZ'F''}{(F'^2 + Z'^2)^{3/2}}. \tag{1.88}$$

Expanding the extrinsic curvature along the curve $(F(\tau), \epsilon F'(\tau))$, the leading order correction is captured precisely by the Schwarzian derivative defined in Eq 1.82:

$$K = 1 + \epsilon^2\{F, \tau\} + \mathcal{O}(\epsilon^4). \tag{1.89}$$

Inserted in Eq 1.85, the JT gravity model indeed reduces to an effective one-dimensional boundary action of the reparametrization modes:

$$\boxed{I_{Schw}[\Phi, g] = -\frac{1}{16\pi G_N}\int d\tau \phi_r(\tau)\{F, \tau\}} \tag{1.90}$$

This is the *Schwarzian boundary action* that captures entirely the holographic description of bulk JT gravity. Unlike the novel example of AdS/CFT concerning an equivalence between a type IIB superstring theory on $AdS_5 \otimes S_5$ and an $SU(N)$ SUSY Yang-Mills theory in $D = 4$, this is an example of holography where the dual description is obtained by integrating out the bulk degrees of freedom, leading to a dual theory that *actually* describes the remaining degrees of freedom that live on the boundary.
It is a generic statement that for holographic dualities where this property holds, one obtains a boundary action in terms of effective boundary fields, by integrating out the bulk fields. These effective boundary fields are constrained only by a predefined choice of asymptotic boundary conditions. Inside the path integral, the integral over the bulk fields can be seen as preparing an operator insertion in the remaining boundary path theory. In all other examples of holography, most notably Maldacena's version of AdS/CFT [4], the dual theory should instead be considered everywhere, all at once.

The Schwarzian action explicitly breaks the degenerate vacua and associates with the Nambu Goldstone modes a non-zero action. Of course, the solutions associated to this action should correspond to the solutions determined from the bulk JT action. In particular, the equations of motion for the dilaton obtained by variation of the metric, should be recovered from the boundary action by variation with respect to the reparametrization modes; hence motivating the nomenclature *boundary gravitons*. Variation of the Schwarzian derivative yields:

$$\delta\{F(\tau), \tau\} = \frac{(\delta F)'''}{F'} - \frac{F'''}{(F')^2}(\delta F)' - 3\frac{F''}{F'}\left(\frac{(\delta F)''}{F'} - \frac{F''}{(F')^2}(\delta F)'\right).$$

Successive partial integration yields:

$$\begin{aligned}\phi_r(u)\delta\{F, \tau\} &\sim \left[-\left(\frac{\phi_r(\tau)}{F'}\right)''' + \left(\phi_r(\tau)\frac{F'''}{(F')^2}\right)' - 3\left(\phi_r(\tau)\frac{F''}{(F')^2}\right)'' - 3\left(\phi_r(\tau)\frac{(F'')^2}{(F')^3}\right)'\right]\delta F \\ &= -\left(\frac{1}{F'}\left(\frac{(F'\phi_r)'}{F'}\right)'\right)'\delta F.\end{aligned}$$

This is readily integrated in terms of three integration constants a, b, μ:

$$\phi_r(\tau) = \frac{a + bF(\tau) - \mu F(\tau)^2}{F'(\tau)}. \tag{1.91}$$

Dividing both sides by ϵ, this is the analogue of the dilaton solution Eq 1.66 at the boundary curve $(F(\tau), Z(\tau) = \epsilon F'(\tau))$ to leading order:

$$\Phi(\tau) \sim \frac{\phi_r(\tau)}{\epsilon} = \frac{a + b(U+V) - \mu UV}{Z(\tau)} \tag{1.92}$$

, with $(U+V)/2 = F$, $(U-V)/2 = Z$, and dilaton profile $\Phi \sim \phi_r/\epsilon$. To leading order, we also have $U(\tau) = V(\tau) \equiv F(\tau)$.

It might seem odd at first to have an interpretation of gravitons in terms of reparametrization modes, when general relativity is fundamentally a theory of diffeomorphism invariance. While it is true that GR is locally diffeomorphism invariant, it is not invariant under diffeomorphisms that reach the boundary. The local diffs can be interpreted as gauge redundancies that do not have associated conserved charges, while symmetries that reach the boundary are true symmetries of the system[13]. In the case of the latter, it is understood that states are mapped to different states by symmetries that reach the boundary. This is already present in classical GR, where the bulk Hamiltonian vanishes and the true energy is encoded completely in terms of an ADM Hamiltonian at the boundary [62].
In most practical applications, we take the renormalized field to be constant $\phi_r(\tau) \equiv a$. The constant a now determines the different UV completions of the theory. The Schwarzian boundary action Eq 1.90 becomes:

$$\boxed{I_{JT}[\Phi, g] = -C \int d\tau \{F, \tau\}}, \qquad C \equiv \frac{a}{16\pi G_N}. \tag{1.93}$$

The parameter $C \equiv \frac{a}{16\pi G_N}$ is the effective 1d coupling constant. Note that unlike the bulk JT Newton's constant, this effective coupling constant has the proper dimension of Length. From dimensional analysis, this signals quantum effects might become important in the UV, although we will see that most quantum effects are tamed beyond one-loop order. The classical solutions are simplified to

$$\begin{aligned}\delta\{F, \tau\} &\sim -\left(\frac{1}{F'}\left(\frac{F''}{F'}\right)'\right)' \delta F \\ &= -\frac{\{F, t\}'}{F'} \delta F.\end{aligned} \tag{1.94}$$

At the classical saddle, the Schwarzian equation of motion becomes:

$$\frac{d}{dt}\{F, \tau\} \equiv 0. \tag{1.95}$$

We will see how this can be related to the notion of energy conservation in the next section.

[13]Diffs on a scalar density along a vector field ζ^μ act with a Lie derivative. On the gravitational Lagrangian $\delta_\zeta(\sqrt{g}\mathscr{L}) = \mathcal{L}_\zeta(\sqrt{g}\mathscr{L}) = \partial_\mu(\mathscr{L}\zeta^\mu)$ [62]. Inserted in the action, the diffs only act on the boundary $\int_{\mathcal{M}} \delta_\zeta(\sqrt{g}\mathscr{L}) = \int_{\partial\mathcal{M}} dx^\mu \zeta_\mu \mathscr{L}$.

Symmetries of the Schwarzian action

The Schwarzian action has the desired properties of the symmetry breaking pattern. More precisely, it is $\mathrm{PSL}(2,\mathbb{R})$ invariant; that is two functions F and G related by a Möbius transformation $G = \frac{aF+b}{cF+d}$ have the same associated action $\{F,\tau\} = \{G,\tau\}$. Conversely, any two functions with the same Schwarzian action are associated by a $\mathrm{PSL}(2,\mathbb{R})$ transformation. This is easily seen from the composition law [25]:

$$\{F(G(\tau)),\tau\} = \{G(\tau),\tau\} + G'(\tau)^2\{F(G),G\} \tag{1.96}$$

, which tells us that $\{F,G\} = 0$ when $\{F,\tau\} = \{G,\tau\}$. The solution to the latter is in general a Möbius transformation, since we know that $F(G) = \frac{aG+b}{cG+d}$ with $ad - bc = 1$ is a solution. Since $\{F(G),G\} = 0$ is a third order differential equation which requires three integration constants, this is indeed the most general solution. This implies that two cutoff geometries have the same associated action iff they are related by a hyperbolic isometry $\mathrm{PSL}(2,\mathbb{R})$.

To interpret the symmetries of the action in terms of conserved charges, we need to look at the real-time Lorentzian rotation. This is obtained by Wick-rotating the Euclidean geometry back to Lorentzian signature $t_{Euclid} \to it_{Lorentz}$, $F_{Euclid} \to iF_{Lorentz}$. The Schwarzian transforms as $\mathscr{L}_{Euclid} \to -\mathscr{L}_{Lorentz}$, while the action transforms as $I \to -iS$. This is the correct interpretation from the general discussion on the Euclidean path integral in section D.4. From the chain rule, it indeed follows that the Euclidean Schwarzian derivative transforms to the Lorentzian Schwarzian derivative as $\{F,\tau\} \to -\{F,t\}$. Including the measure factor $d\tau$, the Lorentzian Schwarzian action has the same form as its Euclidean counterpart,

$$S = iI = -C\int dt\{F(t),t\}. \tag{1.97}$$

The $\mathrm{PSL}(2,\mathbb{R})$ transformations act on the Lorentzian Poincaré time as:

$$F(t) \to \frac{aF(t)+b}{cF(t)+d}, \qquad ad - bc = 1. \tag{1.98}$$

This is a global symmetry of the Lorentzian action. The infinitesimal transformations determine the Noether charges. More precisely, $\mathrm{PSL}(2,\mathbb{R})$ is generated by three zero modes $l_n = -F^{n+1}\partial_F$ for $n = -1, 0, +1$ [52], that act as

$$F(t) \to F(t) + \epsilon_n l_n F(t). \tag{1.99}$$

They are found to span the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra $[l_m, l_n] = (m-n)l_{n+m}$, and act on the configuration space as respectively $F \to F - \epsilon_{-1}$, $F \to (1-\epsilon_0)F$, and $F \to F - \epsilon_2 F^2$. These transformations combine to a finite Möbius transformation.

The $\mathrm{PSL}(2,\mathbb{R})$ transformations act infinitesimally on the Schwarzian action as $\delta S \simeq C\int \frac{\{F,t\}'}{F'}\delta F$. For each generator, there exists a conserved charge determined by the action on the configuration space that leaves the action invariant. This is the content of the Noether theorem:

$$0 \equiv \int \frac{\{F,t\}'}{F'}\delta_n F \sim \int \frac{d}{dt}Q_n. \tag{1.100}$$

Taking $\delta_- F = \epsilon, \quad \delta_0 F = \epsilon F, \quad \delta_+ F = \epsilon F^2$, it is straightforward to check that the time derivative of the following charges satisfies the requirement Eq 1.100:

$$\begin{aligned} Q_- &= C\left(\frac{F'''}{F'^2} - \frac{F''^2}{F'^3}\right), \\ Q_0 &= C\left(\frac{F'''F}{F'^2} - \frac{FF''^2}{F'^3} - \frac{F''}{F'}\right), \\ Q_+ &= C\left(\frac{F'''F^2}{F'^2} - \frac{F^2F''^2}{F'^3} - 2\frac{FF''}{F'} + 2F'\right). \end{aligned} \tag{1.101}$$

These $\mathrm{PSL}(2,\mathbb{R})$ diffeomorphisms do not generate a new cutoff geometry. As elaborated earlier, they move the same geometry around in AdS_2 space. We should therefore not include them as pseudo-Goldstone modes, but treat them as a gauge symmetry instead. Gauge symmetries have in general zero associated Noether charges [63], and we should impose this constraint on the charges above. However, the coordinate transformation $F = e^{\omega}$ (appropriate for Lorentzian finite temperature computations) shows that [17]:

$$\begin{aligned} Q_- &= Ce^{-\omega}\left(\frac{\omega'''}{\omega'^2} - \frac{\omega''^2}{\omega'^3} + \frac{\omega''}{\omega'}\right), \\ Q_0 &= C\left(\frac{\omega'''}{\omega'^2} - \frac{\omega''^2}{\omega'^3} - \omega'\right), \\ Q_+ &= Ce^{\omega}\left(\frac{\omega'''}{\omega'^2} - \frac{\omega''^2}{\omega'^3} - \frac{\omega''}{\omega'}\right). \end{aligned} \tag{1.102}$$

As a consequence, a solution with non-zero ω' cannot have zero charges. However, real-time AdS_2 describes a two sided thermofield double (TFD) state. This is a maximally entangled state between two causally disconnected universes. The $\mathrm{PSL}(2,\mathbb{R})$ isometry acts on the entire thermofield double state, such that the total charge across the two patches should only be zero. Focusing on one state, the charge can be any predefined value. The only constraint is that the charges on either side should be equal and opposite: $Q^a_L = -Q^a_R$. Treating $\mathrm{PSL}(2,\mathbb{R})$ as a gauge symmetry implies that the pseudo-Goldstone modes associated to these charges cannot be further excited. Their excitation is set by amount determined by their predefined conserved value.

The Hamiltonian associated to the Schwarzian action is determined by the Noether charge corresponding to time translations $t \to t + a$ of the Lagrangian; $dL = \frac{dL}{dt}dt \approx \frac{dL}{dt}a \simeq \frac{d(-C\{F,t\})}{dt}a$. This vanishes on-shell due to Eq 1.95, $dL \equiv 0$. We can therefore associate the Schwarzian derivative with the Hamiltonian of the system:

$$H \equiv -C\{F,t\} = \frac{1}{2C}\left[Q_0^2 - \frac{1}{2}\{Q_+, Q_-\}\right]. \tag{1.103}$$

Inserting the explicit forms of the Noether charges Eqs 1.101, one shows that this is the quadratic Casimir associated to representations of $\mathfrak{sl}(2,\mathbb{R})$ [25]. The identification of the total energy present in the system with the Schwarzian Lagrangian itself $E = -C\{F,t\}$, agrees with the ADM energy given in terms of the boundary values of the fields. This was already shown at a classical level in [15].
An important note is that while the conformal reparametrization mode of the Poincaré time coordinate $F(t)$ is broken down to $\mathrm{PSL}(2,\mathbb{R})$, the Schwarzian action breaks the conformal symmetry associated to the physical

boundary t coordinate down to $U(1)$, where it is easy to see that the Schwarzian $\{F(t), t\}$ is only invariant under constant shifts $t \to t + a$. This symmetry is protected by the Killing vector $\zeta^\mu = \epsilon^{\mu\nu}\partial_\nu \Phi$, which is always a Killing vector[14] from the classical equations of motion Eq 1.62.

1.5.3 Finite-temperature solutions

The most natural frame to describe finite-temperature solutions in a Euclidean setting is obtained by transforming the Euclidean Poincaré time coordinate to a β-periodic time variable $f(\tau)$:

$$F(\tau) := \tan \frac{\pi}{\beta} f(\tau), \qquad f'(\tau) \geqslant 0. \tag{1.104}$$

The solution depends on the temperature through the condition that $f(\tau)$ winds around once along the thermal circle for translations in $\tau \to \tau + \beta$:

$$f(\tau + \beta) \equiv f(\tau) + \beta.$$

This effectively determines a map between the zero-temperature frame ($-\infty < F < +\infty$), and a finite time interval $-\beta/2 < f < \beta/2$ (c.f. figure 1.5). The pseudo-Goldstone modes of the finite-temperature solutions are the reparametrization modes of the thermal boundary circle $f \in \text{diff}(S_1)$. The pattern of explicit symmetry breaking of the Schwarzian theory is therefore:

$$\text{diff}(S^1) \to \text{PSL}(2, \mathbb{R}).$$

Considering another preserved symmetry group leads to the insertion of operational defects in the disk partition function [37]. The composition law Eq 1.96 allows us to write the Schwarzian in terms of the f-reparametrization modes:

$$\begin{aligned} I = -C \int d\tau \{F(\tau), \tau\} &= -C \int d\tau \left[\{f(\tau), \tau\} + f'(\tau)^2 \left\{ \tan\left(\frac{\pi}{\beta} f\right), f \right\} \right] \\ &= -C \int d\tau \left[\{f(\tau), \tau\} + \frac{2\pi^2}{\beta^2} f'(\tau)^2 \right]. \end{aligned} \tag{1.105}$$

The classical saddles have a constant Schwarzian derivative: $\{F(\tau), \tau\}' = 0$. The interesting saddles are the linear reparametrization modes $f(\tau) \equiv \tau$. Indeed, for this solution, the Schwarzian $\{f(\tau), \tau\}$ vanishes and $\frac{2\pi^2}{\beta^2} f'(\tau)^2$ remains constant.

Transforming back to real-time coordinates $\tau \to it$, the corresponding Poincaré modes Eq 1.104 are precisely the black hole solutions Eq 1.56, up to an unobservable proportionality factor that is part of the $\text{PSL}(2, \mathbb{R})$ subgroup. In the saddle-point approximation, we already know how to deduce the associated ground state entropy and energy. The Euclidean action is associated with the thermal partition function Eq D.31: $Z(\beta) = e^{-I}$. The on-shell solutions determine the action:

$$\log Z = -I = \frac{2\pi^2 C}{\beta^2} \int_0^\beta d\tau = \frac{2\pi^2 C}{\beta} = 2\pi^2 CT. \tag{1.106}$$

[14] This is a quick check from $\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0$.

The classical thermodynamic relations Eqs D.27 D.28 lead to a linear growth of the entropy with temperature:

$$S = (1-\beta\partial_\beta)\log Z = 4\pi^2 CT, \qquad E = -\partial_\beta \log Z = 2\pi^2 CT^2. \tag{1.107}$$

The extremal entropy contribution Eq 1.45 is implemented by adding a purely topological term to the action:

$$-I_0 = \frac{\Phi_0}{8\pi G_N}\int d\tau \frac{2\pi}{\beta} f'(\tau) = \frac{\Phi_0}{4G_N}.$$

These thermodynamic relations can also be deduced by working directly in the real-time black hole frame Eq 1.56. In this context, the temperature of the black hole was found by periodically identifying the time coordinate in the near-horizon region, which resulted in $T = \frac{r_h}{2\pi}$. In the discussion on the dilaton equations of motion, this was related to (c.f. 1.75) $T = \frac{1}{\pi}\sqrt{\frac{\mu}{a}}$. Solving the energy for the black hole solution $F(t) = \tanh \frac{\pi}{\beta} t = \tanh\sqrt{\frac{\mu}{a}}t$ leads to:

$$E(t) = -C\{F, t\} = \frac{\mu}{8\pi G_N}.$$

Identifying with the temperature $T = \frac{1}{\pi}\sqrt{\frac{\mu}{8\pi G_N}\frac{1}{2C}}$ leads to the same relation between the energy and temperature:

$$E(T) = 2\pi^2 CT^2. \tag{1.108}$$

From $\frac{\partial S}{\partial E} = \frac{1}{T}$, the entropy also agrees with Eq 1.107:

$$S(E) = S_0 + 2\pi\sqrt{2CE} = S_0 + 4\pi^2 CT. \tag{1.109}$$

This insightful argument demonstrates yet again that the bulk JT action leads to the same quantitative conclusions as those obtained from the Schwarzian boundary perspective.

1.6 Real-time derivation of the Schwarzian boundary action

Starting from the Euclidean JT action is the most geometrically intuitive method to derive the Schwarzian boundary action. It is also the most convenient way to generalize the setup to higher genus surfaces, and to use it directly in the Euclidean path integral. However, as with many applications in Euclidean settings, it obscures many of the physical real-time properties of the conformal symmetry breaking pattern. Using the same boundary conditions, we should be able to derive this description directly from the equations of motion [16].

In parallel to the geometric derivation in the previous section, we fix the boundary asymptotics to describe a cutoff of AdS_2. We will relate the proper real-time coordinates of the Poincaré patch ($u = t+z, v = t-z$)

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} = -4\frac{dudv}{(u-v)^2} \tag{1.110}$$

to different parametrizations in terms of chiral functions $U(u)$ and $V(v)$. To leading order in $z \to 0$ near the boundary $u = v \equiv t$, these should be identified $U(t) = V(t) \equiv F(t)$ in order to preserve the asymptotic form of the metric Eq 1.110.
To regularize the effective boundary dynamics, we again introduce an infinitesimal regulator ϵ and move the boundary slightly inwards $z \to \epsilon$. To leading order in ϵ, the reparametrizations near the boundary can be expanded to:

$$\frac{1}{2}\left(U(t+\epsilon) + V(t-\epsilon)\right) = F(t), \qquad \frac{1}{2}\left(U(t+\epsilon) - V(t-\epsilon)\right) = \epsilon F'(t). \tag{1.111}$$

These are indeed the conformal reparametrization degrees of freedom that preserve the asymptotic form of the induced metric Eq 1.80. The boundary condition of the dilaton field is again a diverging profile near the boundary $\Phi(r) = \frac{a}{2\epsilon}$.
The boundary curves define a cutoff of the pure AdS_2 geometry consistent with the asymptotic symmetries. The 1d conformal invariance of pure AdS_2 relates these cutoff surfaces. In nearly "$NAdS_2$", the presence of the dilaton field regulates the reparametrization of the boundary curve in terms of the backreaction on the matter sources. This breaks explicitly the conformal invariance near the boundary, and deforms it into a nearly conformal field theory "$NCFT$".
We directly see from the equations of motion of the dilaton that equating Φ in Eq 1.73 with the asymptotic boundary condition $\Phi = \frac{a}{2\epsilon}$ leads to:

$$\Phi(t) = \frac{a}{2\epsilon F'(z)}\left(1 - \frac{1}{a}(I_+[F(t)] + I_-[F(t)])\right) \equiv \frac{a}{2\epsilon}.$$

This leads to an integro-differential equation that determines the shape of the boundary curve $(F(t), \epsilon F'(t))$ in terms of the influx of matter:

$$\boxed{F'(t) = 1 - \frac{1}{a}(I_+[F(t)] + I_-[F(t)])}. \tag{1.112}$$

$I_+[F(t)]$ and $I_-[F(t)]$ are understood to be functionals of the shape of the boundary curve to leading order in ϵ:

$$I_+[F(t)] = 8\pi G_N \int_{F(t)}^{+\infty} ds(s - F(t))(s - F(t))T_{UU}(s), \tag{1.113}$$

$$I_-[F(t)] = 8\pi G_N \int_{-\infty}^{F(t)} ds(s - F(t))(s - F(t))T_{VV}(s). \tag{1.114}$$

Since the shape of the boundary curve is fixed by the equations of motion, this procedure manifestly breaks the conformal invariance. We can apply the same series of derivatives as Eq 1.94 to Eq 1.112, to obtain the equation of motion regulating the boundary reparametrization mode $F(t)$ [25]:

$$-C\frac{d\{F,t\}}{dt} = \left(T_{VV}(t) - T_{UU}(t)\right)F'^2|_{\partial\mathcal{M}} = T_{vv}(t) - T_{uu}|_{\partial\mathcal{M}}. \tag{1.115}$$

Using the expression of the energy Eq 1.103 $E = -C\{F,t\}$, we can interpret this as an energy conservation law, in terms of the influx T_{vv} and outflux T_{uu} of matter from the asymptotic boundary. Again using $\int dt\ \delta\{F,t\} = -\int dt\ \frac{\{F,t\}'}{F'}\delta F$, these are the equations of motion determined from the Schwarzian boundary action with an additional matter contribution [64]:

$$S = -C\int dt\{F,t\} + \int dF dZ \mathscr{L}_m(\phi, \partial_F\phi) \tag{1.116}$$

$$\simeq -C\int dt\{F,t\} + \int dt dZ F' \mathscr{L}_m(\phi, \partial_F\phi). \tag{1.117}$$

The matter term is minimally coupled to the boundary graviton modes F. Variation of the matter action with respect to the pseudo Goldstone mode yields (using $\partial_F\phi = \frac{\partial_t\phi}{F'}$):

$$\delta\int dt dZ F'\mathscr{L}_m(\phi,\partial_F\phi) = \int dt dZ \delta F' \mathscr{L}_m + \int dt dZ F' \frac{\partial \mathscr{L}_m}{\partial\partial_F\phi}\left(-\frac{1}{F'^2}\delta F'\right)\partial_t\phi$$
$$= \int dt dZ \frac{d}{dt}\left(\partial_F\phi\frac{\partial\mathscr{L}_m}{\partial\partial_F\phi} - \mathscr{L}_m\right)\delta F$$

, where we recognise the general form of the Hamiltonian $H_m = \int dZ \, \left(\partial_F\phi\frac{\partial\mathscr{L}_m}{\partial\partial_F\phi} - \mathscr{L}_m\right)$. Combined with the variation of the Schwarzian leads to:

$$-C\frac{d}{dt}\{F,t\} = F'\frac{dH_m}{dt} = F'^2\frac{dH_m}{dF} \equiv (T_{VV} - T_{UU})F'^2|_{\partial\mathcal{M}}. \tag{1.118}$$

The last identity simply expresses energy conservation of the matter sector $\frac{dH_m}{dF} = T_{VV} - T_{UU}|_{\partial\mathcal{M}}$.
This indeed demonstrates that the Schwarzian perspective can be derived directly from the equations of motion.

1.6.1 Hamiltonian formulation

Since the Schwarzian derivative itself is third order in derivatives, solving the Schwarzian equation of motion $\frac{d}{dt}\{F(t),t\} = 0$ requires in general four integration constants. An alternative formulation involves a first order derivative action, yielding two second order derivative equations of motion [16][21].
Consider the Schwarzian action without matter $S = -C\int dt\{F,t\}$. We define a new dynamical variable φ by identifying $e^\varphi = F'(t)$. This is always possible since $F' > 0$. The Schwarzian derivative can be rewritten as:

$$\{F,t\} = -\frac{1}{2}(\partial_t\varphi)^2 + \partial_t^2\varphi, \qquad \varphi \equiv \log F'. \tag{1.119}$$

Plugged into the action, the total derivative term vanishes and we can write its contribution in the Schwarzian action as a free massless boson:

$$S = \int dt\frac{C}{2}(\partial_t\varphi)^2 - \lambda e^\varphi + \lambda F'. \tag{1.120}$$

The additional dynamical field λ is introduced as a Lagrange multiplier, imposing the constraint $e^\varphi = F'$ directly in the action. The action of the φ-field corresponds to the 1D Liouville action. Its equations of motion are:

$$C\partial_t^2\varphi + \lambda e^\varphi = 0, \qquad e^\varphi = F'. \tag{1.121}$$

This is a manipulated form of the original equation of motion Eq 1.112, obtained by differentiating twice with respect to F:

$$\frac{1}{2}\frac{d^2F'}{dF^2} + \frac{8\pi G_N}{a}(P_+ - P_-) = 0, \quad \Longleftrightarrow \quad C\frac{d^2F'}{dF^2} + (P_+ - P_-) = 0$$

, with $P_+ := \int_F^{+\infty} ds T_{UU}(s)$ and $P_- := -\int_{-\infty}^{F} ds T_{VV}(s)$. Further using the chain rule $\frac{d}{dF} = \frac{1}{F'}\frac{d}{dt}$, we can relate this to:

$$C\frac{1}{F'}\frac{d}{dt}\left(\frac{1}{F'}\frac{d}{dt}\left(\frac{dF}{dt}\right)\right) + (P_+ - P_-) = 0$$
$$\Longleftrightarrow \quad C\frac{d}{dt}\left(\frac{d}{dt}\log F'\right) + (P_+ - P_-)F' = 0.$$

Identifying $e^{\varphi} = F'$ and $(P_+ - P_-) = \lambda$ trivially leads to Eq 1.121.

To transition to a Hamiltonian formulation, we write the first order Lagrangian in terms of four canonical phase space variables (φ, π_φ) and (F, π_F):

$$L = \pi_\varphi \partial_t \varphi + \pi_F \partial_t F - H \tag{1.122}$$

, where the Hamiltonian is specified as

$$H \equiv \frac{1}{2C}\pi_\varphi^2 + e^{\varphi}\pi_F. \tag{1.123}$$

Integrating out π_φ via its equation of motion indeed yields the Lagrangian Eq 1.120, where we identify $\pi_F \equiv \lambda$:

$$L = \frac{C}{2}(\partial_t\varphi)^2 - \pi_F e^{\varphi} + \pi_F F'.$$

Integrating out π_F leads to the Schwarzian, while integrating out F yields the 1d Liouville action $L = \frac{C}{2}\varphi'^2 - \pi_F e^{\varphi}$.

The first order Hamiltonian equations of motion corresponding to the first order Lagrangian 1.122 are readily deduced ($\dot{q} = \frac{dH}{\partial \pi_q}$, $\dot{\pi}_q = -\frac{dH}{dq}$):

$$\partial_t\varphi = \frac{1}{C}\pi_\varphi, \quad \partial_t F = e^{\varphi}, \quad \partial_t\pi_\varphi = -e^{\varphi}\pi_F, \quad \partial_t\pi_F = 0. \tag{1.124}$$

These equations reproduce the equations of motion Eqs 1.121 upon plugging the first into the third equation

$$\partial_t\pi_\varphi = C\partial_t^2\varphi = -e^{\varphi}\pi_F$$

, and setting again $\pi_F = \lambda$.

The usual commutation relations are $[\varphi, \pi_\varphi] = i$ and $[F, \pi_F] = i$. Using these commutation relations, the set of generators

$$l_{-1} = \pi_F, \qquad l_0 = F\pi_F + \pi_\varphi, \qquad l_1 = F^2\pi_F + 2F\pi_\varphi - 2Ce^{\varphi} \tag{1.125}$$

are found to satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra [16]:

$$[l_0, l_{\pm 1}] = \mp i l_{\pm 1}, \qquad [l_1, l_{-1}] = 2il_0. \tag{1.126}$$

Canonically quantizing $\pi_q = -i\partial_q$ identifies the action on F with Eq 1.99.
Using these charges, one can verify that the first order Hamiltonian Eq 1.123 is just the definition of the $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir Eq 1.103 [16]:

$$H = \frac{1}{2C}\left(l_0^2 - \frac{1}{2}\{l_{+1}, l_{-1}\}\right). \tag{1.127}$$

1.7 Gravitational path integral

Having discussed the finite-temperature black hole solutions of JT gravity in terms of the classical saddle points of the Schwarzian boundary action in section 1.5.3, the next step would be to include the quantum corrections on this spectrum by doing the full path integral. In section D.4.2, the gravitational path integral is defined along the lines of the original Gibbons-Hawking prescription in [65]. Performing the path integral in Euclidean signature over all metric- and matter fields with asymptotic boundary conditions on the thermal circle is postulated to result in the thermal partition function. However, more often than not, defining a proper measure for a fluctuating metric is not readily obvious. Furthermore, there is the omnipresent issue of the unstable conformal mode, making the action unbound from below and the path integral to diverge. We will see how both issues are resolved for the path integral over the JT action. A spectacular property is that the full path integral can be solved exactly to all orders in perturbation theory. This was first observed in a paper by Stanford and Witten [20]. By using a fermionic localization argument, they were able to prove that the path integral is in fact one-loop exact. Coupling the theory to matter, the matter bilocal operators in the Schwarzian holographic perspective can likewise be solved exactly to all orders [21][22]. This makes JT gravity more than an attractive toy model to study the effects of AdS_2-backreaction along the lines of [15], but makes it one of the most valuable laboratories to study the effects of quantum gravity as a whole.
The thermal disk partition function is the gravitational path integral Eq D.29 of the full Euclidean JT action over all metric and dilaton field configurations, subject to the prescribed disk topology and boundary condition of fixed inverse temperature β:

$$Z(\beta) = e^{S_0} \int \mathcal{D}g\mathcal{D}\Phi \; e^{\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} \Phi(R+2) + 2\int_{\partial\mathcal{M}} \Phi_b (K-1)\right]}. \tag{1.128}$$

Note that in most of this thesis, we will assume a topologically trivial disk manifold, with one boundary and no holes ($\chi = 1$).
On a classical level, we have seen that the equations of motion of the dilaton field fix the bulk geometry to patches of pure AdS_2. This in turn reduces the bulk JT action to a holographic Schwarzian boundary description. In the full path integral, one integrates over all off-shell values of the dilaton, and we cannot a priori use the on-shell equivalence between the boundary description. However, since the dilaton appears linear in the action, it acts as a Lagrange multiplier in the path integral, fixing the constraint $R(x) + 2 = 0$ also off-shell. This is readily seen by noting that the linear dilaton term in the action is of Gaussian type. Performing the full Gaussian path integral is, in general, equivalent to inserting its on-shell value. A more direct way to realize this, is to note that the path integral over the dilaton is the functional counterpart of the

integral representation of the Dirac delta function:

$$\delta(x) = \int \frac{d^D x}{(2\pi)^D}\, e^{ik\cdot x} \qquad \Longleftrightarrow \qquad \delta[f] = \int \mathcal{D}\lambda\, \exp(i \int d^D x \lambda(x) f(x))$$

, the latter imposing the constraint[15] $f(x) \equiv 0$.
By integrating along an imaginary contour for the dilaton field, the bulk JT action vanishes off-shell, and the discussion in section 1.5.2 continues to hold. In particular, the bulk dynamics are fixed to describe patches of AdS_2. The remaining degrees of freedom are the pseudo-Goldstone reparametrization modes of the thermal boundary circle $f \in$ diff(S_1). These describe the cutoff geometries of the Euclidean disk with fixed asymptotic boundary length β/ϵ. The action that weights these reparametrization modes stems again from the GHY boundary term. The latter reduces to the Schwarzian boundary action by imposing fixed length boundary conditions on the metric, and blowing up boundary conditions on the dilaton $\Phi_b = a/2\epsilon$. a is an inverse energy scale, labeling the different asymptotic nearly-$(N)AdS_2$ regimes. The resulting action is invariant under the (P)SL$(2,\mathbb{R}) \subset$ diff(S_1) reparametrization subgroup[16]. This is the isometry group of the $NAdS_2$ space and does not generate new geometries. These parametrization modes can be thought of as gauge redundancies, and should not be included in the path integral, in accordance with the Faddeev-Popov path integral procedure over gauge fields. The natural integration space is therefore diff$(S_1)/$SL$(2,\mathbb{R})$, where one mods out SL$(2,\mathbb{R})$ into equivalence classes of reparametrization modes. The Euclidean JT path integral Eq 1.128 reduces to a path integral over the reparametrization modes, weighted by the Schwarzian action:

$$\begin{aligned} Z(\beta) &= e^{S_0} \int \mathcal{D}g\mathcal{D}\Phi\; e^{\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} \Phi(R+2) + 2\int_{\partial\mathcal{M}} \Phi_b (K-1)\right]} \\ &= e^{S_0} \int_{\text{diff}(S_1)/SL(2,\mathbb{R})} [\mathcal{D}f]\; e^{C\int_0^\beta d\tau \left\{\tan\frac{\pi f(\tau)}{\beta}, \tau\right\}} \end{aligned} \tag{1.129}$$

I have indicated the measure factor over the Schwarzian reparameterization modes in brackets $[\mathcal{D}f]$ since the measure is not a priori clear. I note again that this a particular example of quantum holography where the dual path integral is obtained by integrating out all bulk fields, leaving a path integral over boundary fields compatible with the prescribed boundary conditions. Dual operators are obtained in the same way, by integrating over the bulk fields in the path integral insertion. This procedure of holography is general for theories where the dual description actually lives on the boundary.
The constant $C = \frac{a}{16\pi G_N}$ is an emergent dimensionful scale in the boundary description, quantifying the coupling strength of the Schwarzian modes.

1.7.1 Evaluation of the path integral to one-loop order

To actually do gravitational path integral, we need an appropriate measure factor. In most applications of quantum gravity, this is not readily available, and we are restricted to the on-shell approximation. However, in the case of JT quantum gravity, the measure is the natural volume form over the integration space diff$(S_1)/$SL$(2,\mathbb{R})$ of the dual Schwarzian description. The Schwarzian derivative is probably most familiar

[15] The proper normalization factors are absorbed in the abstract measure $\mathcal{D}\lambda$.
[16] In the following, I will often simply refer to PSL$(2,\mathbb{R})$ as SL$(2,\mathbb{R})$.

from 2d CFT, where it describes the anomalous transformation property of the Virasoro stress tensor. The latter is a coadjoint vector of the Virasoro algebra. It turns out that the right quotient space diff$(S_1)/\mathrm{SL}(2,\mathbb{R})$ is actually a coadjoint orbit of an identity element under the action of the Virasoro group. Unlike pure diff(S_1), this is also a symplectic manifold and possesses a natural symplectic integration measure. This measure is then related to the volume form in the path integral by exponentiation. Since these facts are not readily obvious, I have written an extensive appendix C on coadjoint orbits of the Virasoro group and the symplectic measure, combining the perspectives of [66] [20] and [67]. I had the ambition to be as pedagogical as possible and work out the derivations in as much detail as possible. At the end of the appendix, we note that the Schwarzian action is actually the generator of a $U(1)$-subgroup. By the Duistermaat-Heckman theorem [68], this implies that the first order correction to the classical saddle is in fact the final answer, quantifying all quantum corrections [20]. This is a strong statement, since a priori, one would expect the partition function to be dominated by large fluctuations around the saddle for large gravitational coupling G_N (small C). In the main text, we will now simply use the result for the symplectic measure, and evaluate the Schwarzian integral to first order.

It turns out that the symplectic measure associated to the Schwarzian reparametrization modes f is given by Eq C.16:

$$\omega = \int_0^{2\pi} d\tau \, \left[\frac{df'(\tau) \wedge df''(\tau)}{f'(\tau)^2} - df(\tau) \wedge df'(\tau) \right] . \tag{1.130}$$

Here, d is an abstract exterior derivative that works only on the fields $f(\tau)$ and formally commutes with ∂_τ. Since exterior derivatives anticommute in the wedge product, we can consider df as fermionic variables. We show in the appendix that this is the natural measure inherited on the Virasoro coadjoint orbits. It is seen to be both closed and antisymmetric. It is also non-degenerate on the integration space diff$(S_1)/\mathrm{SL}(2,\mathbb{R})$ (c.f. C.20). This verifies that the two-form ω is indeed a symplectic measure and confirms the integration space to be a symplectic manifold. The symplectic measure defines the volume form in the path integral Eq 1.129, in terms of the Pfaffian of the former Eq C.21:

$$[\mathcal{D}f] = \mathrm{Pf}(\omega) \; \mathcal{D}f \tag{1.131}$$

, with $\mathrm{Pf}(\omega) = \sqrt{\det(\omega)}$ defined in Eq C.22. It is convenient to write the Pfaffian associated with Eq C.16 as a Gaussian integral over Grassmann fields $\psi(\tau) = df(\tau)$ (c.f. Eq C.23):

$$\mathrm{Pf}(\omega) = \int \mathcal{D}\psi \; \exp\left[\frac{1}{2} \int_0^{2\pi} d\tau \, \left(\frac{\psi'\psi''}{f'^2} - \psi\psi' \right) \right] . \tag{1.132}$$

Once we know the appropriate measure associated with the reparametrization modes over diff$(S_1)/\mathrm{SL}(2,\mathbb{R})$, the evaluation of the path integral[17] Eq 1.129 to first order in perturbation theory is trivial. Using the canonical

[17] Imagine neglecting the extremal contribution e^{S_0} for now.

action Eq C.2 yields

$$\begin{aligned}
Z(\beta) &= \int \mathcal{D}g\mathcal{D}\Phi \; e^{\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}}\Phi(R+2)+2\int_{\partial\mathcal{M}}\phi_b(K-1)\right]} \\
&= \int_{\text{diff}(S_1)/SL(2,\mathbb{R})} [\mathcal{D}f] \; e^{\frac{2\pi C}{\beta}\int_0^{2\pi} d\tau \{\tan\frac{f}{2},\tau\}} \\
&= \int_{\text{diff}(S_1)/SL(2,\mathbb{R})} \mathcal{D}f \; \text{Pf}(\omega) \; e^{\frac{2\pi C}{\beta}\int_0^{2\pi} d\tau \{\tan\frac{f}{2},\tau\}} \\
&= \int_{\text{diff}(S_1)/SL(2,\mathbb{R})} \mathcal{D}f\mathcal{D}\psi \; \exp\left[\int_0^{2\pi} d\tau \; \left(\frac{2\pi C}{\beta}\left\{\tan\frac{f}{2},\tau\right\} + \frac{1}{2}\left(\frac{\psi'\psi''}{f'^2} - \psi\psi'\right)\right)\right].
\end{aligned} \tag{1.133}$$

In the last line, I have rewritten the natural symplectic measure Eq C.16 in terms of a Gaussian path integral over these fermionic variables. We first expand the reparametrization mode around its saddle: $f(\tau) = \tau + \epsilon(\tau)$, and expand the Schwarzian derivative to first order in ϵ:

$$\begin{aligned}
\left\{\tan\frac{f}{2},\tau\right\} &= \{f,\tau\} + \frac{1}{2}f'^2 = \frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2} + \frac{1}{2}f'^2 \\
&= \frac{\epsilon'''}{1+\epsilon'} - \frac{3}{2}\frac{\epsilon''^2}{(1+\epsilon')^2} + \frac{1}{2}(1+\epsilon')^2 \\
&\simeq \epsilon''' - \epsilon'''\epsilon' - \frac{3}{2}\epsilon''^2(1-2\epsilon') + \frac{1}{2} + \epsilon' + \frac{\epsilon'^2}{2} \\
&= \frac{1}{2} + (\epsilon' + \epsilon''') + \left(\frac{1}{2}\epsilon'^2 - \frac{1}{2}\epsilon''^2 - (\epsilon''\epsilon')'\right) + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{1.134}$$

Note that in order for $f(\tau+2\pi) = f(\tau) + 2\pi$, we should have $\epsilon(\tau+2\pi) = \epsilon(\tau)$, and all orbits of ϵ close after a rotation in 2π. The first factor of $\left\{\tan\frac{f}{2},\tau\right\} = \frac{1}{2}$ is associated with the classical saddle, which from Eq C.2 has an associated action $-I = \frac{2\pi C}{\beta}\int_0^{2\pi}\frac{d\tau}{2} = \frac{2\pi^2 C}{\beta}$. This coincides with the finite-temperature solution in section 1.5.3. Further expanding the action associated with the Pfaffian, leads to:

$$\frac{\psi'\psi''}{f'^2} - \psi\psi' = \psi'\psi'' - 2\psi'\psi''\epsilon' - \psi\psi' + \mathcal{O}(\epsilon^2). \tag{1.135}$$

It is convenient to perform a redefinition $\epsilon \to \left(\frac{\beta}{2\pi C}\right)^{1/2}\epsilon$, and obtain an action to first order in $\frac{\beta}{2\pi C}$. This factor is associated with the gravitational coupling strength, and determines an effective coupling of the boundary gravitons. Dropping all total derivatives, the action then becomes

$$-I = \frac{2\pi^2 C}{\beta} - \frac{1}{2}\int_0^{2\pi} d\tau \left[(\epsilon''^2 - \epsilon'^2) - (\psi'\psi'' - \psi\psi') + \mathcal{O}\left(\frac{\beta}{2\pi C}\right)^{1/2}\right]. \tag{1.136}$$

The quadratic action in ϵ and ψ is associated with the one-loop propagators, and is independent of the effective coupling strength. It can be exactly integrated with the rules of Gaussian path integrals [69], and yields a $\left(\frac{\beta}{2\pi C}\right)^{1/2}$-independent prefactor. However, the redefinition $\epsilon \to \left(\frac{\beta}{2\pi C}\right)^{1/2}\epsilon$ has an associated redefinition in the measure of the path integral: $\Pi_\tau d\epsilon(\tau) \to \Pi_\tau \left(\frac{\beta}{2\pi C}\right)^{1/2} d\epsilon(\tau)$. Therefore, the path integral has an infinite product over $\left(\frac{\beta}{2\pi C}\right)^{1/2}$-dependent factors. In any regularization scheme, this dependence should drop out, and the product over all modes n should yield a β-independent factor $\prod_{n\in\mathbb{Z}}\left(\frac{\beta}{2\pi C}\right)^{1/2} \to 1$. However, since we are gauge-fixing 3 modes associated with the $\mathrm{SL}(2,\mathbb{R})$ zero-modes, we should divide this infinite product by $\left(\frac{\beta}{2\pi C}\right)^{3/2}$, and obtain:

$$Z(\beta) \propto \left(\frac{2\pi C}{\beta}\right)^{3/2} \exp\left(\frac{2\pi^2 C}{\beta}\right) \tag{1.137}$$

The exponential factor is associated with the classical saddle, while the β-dependent prefactor is associated with the one-loop determinant correction on this saddle point. Reinstating the extremal contribution, and using zeta function regularization of the infinite product, yields the explicit prefactor [25] [30]:

$$\boxed{Z(\beta) \; = \; \frac{e^{S_0}}{4\pi^2}\left(\frac{2\pi C}{\beta}\right)^{3/2} \exp\left(\frac{2\pi^2 C}{\beta}\right)}. \tag{1.138}$$

Note that if we had restricted the stabilizer subgroup determining the coadjoint orbits to $\mathrm{U}(1)$ instead of $\mathrm{SL}(2,\mathbb{R})$, there would have only been one zero mode, and the one loop determinant would have had an exponential factor of $1/2$ instead. This is exactly what happens in some modifications of the Schwarzian theory that will become important later on.

As already noted before, [20] demonstrated that the one-loop answer is in fact the full answer to all orders in perturbation theory. Section C.3 reviews this argument starting from the fact that the integration space is a symplectic manifold that is $\mathrm{U}(1)$-invariant, and that the Schwarzian action is the generator of particular $\mathrm{U}(1)$ transformations. Using fermionic localization arguments, we can add a term to the action with an arbitrary large prefactor, without changing the integral. This additional term has the effect of rendering all higher order corrections in the perturbative expansion arbitrary small, leaving only the one-loop determinant.

1.7.2 Holographic interpretation of the partition function

In the context of holography, the partition function should be recovered in the dual field theory as a partial trace of the modular Hamiltonian H over the states of a Hilbert space spanning the black hole microstates $\mathcal{H}_{BH}$;

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}_{BH}}[e^{-\beta H}] = \int \rho(E) e^{-\beta E} dE. \tag{1.139}$$

In the last equality, this is rewritten in terms of an integral over the density of states $\rho(E)$. Using the exact expression of the thermal partition function, we obtain an expression of the density of states using an inverse Laplace transform [25];

$$\boxed{\rho(E) = \frac{C}{2\pi^2} e^{S_0} \sinh(2\pi\sqrt{2CE})}. \tag{1.140}$$

This is the famous density of states corresponding to bosonic JT gravity. An immediate issue that arises is that the latter does not correspond to a sum of delta functions $\rho(E) = \sum_n \delta(E - E_n)$. A continuous spectrum in quantum mechanics is usually associated with an infinite spatial volume. However a 0+1d boundary theory has no associated spatial direction, and a continuous spectrum implies that the entropy of the micro-canonical ensemble is actually infinite. This has the immediate implication that information can be lost within a black hole. The tension can be resolved by considering the boundary theory as a random matrix ensemble [29] [30]. A qualitative feature of Eq 1.140 is that for large energies $E \gg 1/C$, we can approximate the hyperbolic sine by $\rho(E) \propto e^{S_0} e^{2\pi\sqrt{2CE}}$. Using the classical energy-temperature relation Eq 1.108 $E = 2\pi^2 CT^2$, this exactly coincides with the density of states deduced from the classical saddle entropy Eq 1.109 $S = 4\pi^2 CT$. Quantum effects on this saddle only become important at low energies $E \ll 1/C$, where now $\rho \propto e^{S_0}\sqrt{2CE}$, due to the one-loop determinant factor. Unlike the classical presumption that $\rho \to e^{S_0}$ as $E \to 0$, quantum effects lead to a vanishing ground state degeneracy as $E \to 0$. In this sense, it is actually wrong to think of e^{S_0} as an extremal entropy once quantum effects are turned on.

1.8 Quantum JT gravity coupled to matter

When coupling JT gravity to matter, the Gibbons-Hawking prescription of section D.4.2 instructs us to perform a path integral over the gravitational and matter fields (collectively denoted by resp. g, and ϕ) of an extended action $I[g,\phi]$, describing the matter fields coupled to gravity;

$$Z(\beta) = \int \mathcal{D}g\mathcal{D}\phi \; e^{-I[g,\phi]} \tag{1.141}$$

The boundary conditions should describe an asymptotic AdS_2-universe with periodic time identification. As a sufficiently simple example, most approaches consider a free scalar field $\phi(t,z)$ of mass m, propagating in the background of a Euclidean JT black hole geometry. The Euclidean matter action is given by the Klein-Gordon action, minimally coupled to gravity:

$$I_m[\phi, g] = -\int d^2x\, \sqrt{g}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right]. \tag{1.142}$$

The holographic dictionary, elaborated in the appendix D.5, argues that the boundary values of fields propagating in the gravitational bulk act as sources in the generating functional for the dual operators $\mathcal{O}(t)$. Concretely, we recall that the full fledged quantum gravitational path integral over all bulk fields subject to the boundary condition $\phi(t,z)|_{z=0} = \phi_b(t)$, acts as a generating functional for the dual boundary operators (c.f. Eq D.33):

$$\langle e^{\int dt\; \phi_b(t)\mathcal{O}(t)} \rangle = Z_{grav}\left[\phi(t,z)|_{z=0} = \phi_b(t)\right]. \tag{1.143}$$

Calculating the boundary matter correlators involves taking functional derivatives of this generating functional with respect to these sources and evaluating them to zero at the end:

$$\begin{aligned} \mathrm{Tr}[e^{-\beta H}\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\ldots\mathcal{O}_n(t_n)] &= \langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\ldots\mathcal{O}_n(t_n)\rangle \\ &= \frac{\delta^n}{\delta\phi_b^1\,\delta\phi_b^2\ldots\delta\phi_b^n} Z_{grav}\left[\phi(t,z)|_{z=0} = \phi_b(t)\right]|_{\phi_b^i=0}. \end{aligned} \tag{1.144}$$

In the case of a scalar field propagating in the gravitational bulk, we have argued that the correct boundary condition is given by (c.f. Eq D.41 with[18] $d = 1$):

$$\phi(t,z)|_{z=0} = z^{1-\Delta}\tilde{\phi}_b(t), \qquad z \to 0. \tag{1.145}$$

Δ corresponds to the conformal scaling dimension of the dual quasi-primary operator $\mathcal{O}$, which was argued to be related to the mass m^2 (c.f. Eq D.39):

$$\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}. \tag{1.146}$$

The Jackiw-Teitelboim gravitational path integral coupled to a scalar field is obtained by adding the minimally coupled matter action to the JT action, and additionally integrating over all matter fields consistent with the above boundary conditions,

$$Z(\beta,\phi_b) = e^{S_0}\int \mathcal{D}g\mathcal{D}\Phi\; e^{\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}}\Phi(R+2)+2\int_{\partial\mathcal{M}}\Phi_b(K-1)\right]}\int \mathcal{D}\phi\; e^{-I_m[\phi,g]}. \tag{1.147}$$

Since the matter action is considered to be independent of the dilaton, the latter still acts as a Lagrange multiplier enforcing the gravitational degrees of freedom to a patch of AdS_2. We eventually obtain a holographic matter path integral coupled to the Schwarzian reparameterization degrees of freedom:

$$Z(\beta,\phi_b) = e^{S_0}\int_{\text{diff}(S_1)/SL(2,\mathbb{R})} \mathcal{D}f\; e^{C\int_0^\beta d\tau\,\left\{\tan\frac{\pi f(\tau)}{\beta},\tau\right\}}\int \mathcal{D}\phi\; e^{-I_m[\phi,f]}. \tag{1.148}$$

1.8.1 Free field generating functional

To rewrite the minimally coupled scalar field action $I_m[g,\phi]$ in a holographic context, we work out the free field generating functional. This is a generic exercise of the holographic dictionary, and is often the starting point in much of the literature on quantum JT. However, let me work this out explicitly along the lines of [70], and motivate why this is appropriate in the JT context.

In most semi-classical approximations, the gravitational background of the matter fields is fixed to the saddle point value of the Einstein-Hilbert action. In many applications of AdS_D/CFT_d, this bulk geometry is not pure AdS_D but contains a non-trivial black hole geometry in its bulk, although it should asymptote to AdS near the boundary. The isometry subgroup is therefore smaller than $SO(D-1,2)$ (Lorentzian) or $SO(D,1)$ (Euclidean), and usually contains only the subgroup of rotations and translations. In case the saddle corresponds to pure AdS, the geometry is additionally invariant under inversions. The dual field theory of the latter is fully conformally invariant, and retains the symmetry under inversions. The generic two-point correlation functions between two quasi-primary operators $\mathcal{O}_\Delta(x)$ of scaling dimension Δ should therefore be proportional to:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle \propto \frac{1}{(x-y)^{2\Delta}}. \tag{1.149}$$

[18] Conform the convention in the appendix, I denote D as the dimension of the bulk theory, and $d = D-1$ as the dimension of the dual boundary theory.

In the case of JT gravity, this analysis is exact quantum mechanically since the bulk contains no dynamical gravitational degrees of freedom anyway, and the geometry is fixed to AdS_2. To find the generating functional corresponding to the generalized free fields, we consider the matter action Eq 1.142 in the Poincaré patch of AdS_2, and place a cutoff at $z = \epsilon$. The corresponding equations of motion obtained by varying the action, take the following form (c.f. Eq D.37 with $D = 2$):

$$\partial_z^2\phi + \partial_t^2\phi = \frac{1}{z^2}m^2\phi \tag{1.150}$$

, subject to the boundary condition $\phi(\epsilon, t) = \epsilon^{1-\Delta}\tilde{\phi}_b$. We can integrate the bulk action $I_m[\phi, g]$ by parts, and write the action in terms of a bulk term and a non-trivial lower boundary term at $z = \epsilon$:

$$I_m[\phi, g] = \frac{1}{2}\int d^2x\sqrt{g}\phi(z,t)\left[\Box - m^2\right]\phi(z,t) - \frac{1}{2}\int dt\ \phi(\epsilon,t)[\partial_z\phi(z,t)]|_{z=\epsilon}. \tag{1.151}$$

$\Box$ is the covariantized Klein-Gordon operator $\Box = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$. In the following, we consider the Poincaré patch of AdS_2. Since the bulk action is quadratic in the fields, its contribution in the Gaussian path integral over ϕ is exact. Integrating it out yields the one-loop determinant, which we can reabsorb in an overall normalization. The remaining boundary term has no dynamical degrees of freedom, and is fixed completely by the boundary conditions of the fields at $z = \epsilon$. To find the solution, we transform to momentum space [59]

$$\phi(z,t) = \int \frac{dp}{2\pi}\ e^{ipt}\phi(z,p) \tag{1.152}$$

to rewrite the equation of motion Eq D.37 as:

$$\left[z^2\partial_z^2 - (p^2z^2 + m^2)\right]\phi(z,p) = 0, \qquad \phi(\epsilon,p) = \epsilon^{1-\Delta}\tilde{\phi}_b(p). \tag{1.153}$$

The unique exponentially damped solution in the bulk corresponds to the Bessel function $z^{1/2}K_\nu(pz)$, where $\nu = \Delta - 1/2$. We choose the normalized solution

$$\phi(z,p) = \frac{z^{1/2}K_\nu(pz)}{\epsilon^{1/2}K_\nu(p\epsilon)}\epsilon^{1-\Delta}\tilde{\phi}_b(p) \tag{1.154}$$

, to obtain the correct boundary value in Eq 1.153. The boundary action in momentum space becomes

$$\begin{aligned} I_m &= -\frac{1}{2}\int\frac{dp}{2\pi}\int\frac{dq}{2\pi}\int dt\ e^{i(p+q)t}\phi(\epsilon,q)[\partial_z\phi(z,p)]|_{z=\epsilon} = -\frac{1}{2}\int\frac{dp}{2\pi}\int\frac{dq}{2\pi}\ 2\pi\delta(p+q)\phi(\epsilon,q)[\partial_z\phi(z,p)]|_{z=\epsilon} \\ &= -\frac{1}{2}\int\frac{dp}{2\pi}\phi(\epsilon,-p)[\partial_z\phi(z,p)]|_{z=\epsilon} = -\frac{\epsilon^{2(1-\Delta)}}{2}\int\frac{dp}{2\pi}\tilde{\phi}_b(-p)\frac{d}{d\epsilon}\log\left(\epsilon^{1/2}K_\nu(p\epsilon)\right)\tilde{\phi}_b(p). \end{aligned}$$

To investigate the limit $\epsilon \to 0$, we need the asymptotic properties of the Bessel functions [71]. For non-integer ν, these are:

$$K_\nu(u) = u^{-\nu}(a_0 + a_1u^2 + \dots) + u^\nu(b_0 + b_2u^2 + \dots), \quad a_0 = 2^{\nu-1}\Gamma(\nu), \quad b_0 = -2^{-\nu-1}\Gamma(1-\nu)/\nu.$$

Computing the logarithmic derivative yields:

$$\frac{d}{d\epsilon}\log\left(\epsilon^{1/2}K_\nu(p\epsilon)\right)=\frac{1}{\epsilon}\left[\frac{1}{2}-\nu(1+c_2(\epsilon p)^2+\dots)+\frac{2\nu b_0}{a_0}(\epsilon p)^{2\nu}(1+d_2(\epsilon p)^2+\dots)\right]. \tag{1.155}$$

Returning to position space, the first bracket is a series in integer powers of p^2. This produces singular contact terms $\delta(x-y)$, $\Box\delta(x-y)$, $\dots$. We renormalize these singular terms by neglecting them entirely. The physically relevant term corresponds to the first non-integer power in p. The others are subleading in ϵ as $\epsilon\to 0$. Using $\nu=\Delta-1/2$, and the explicit coefficients of $a_0,\ b_0$, this can be written as:

$$\frac{d}{d\epsilon}\log\left(\epsilon^{1/2}K_\nu(p\epsilon)\right)=\epsilon^{2(\Delta-1)}\frac{2\nu b_0}{a_0}p^{2\nu}=-\epsilon^{2(\Delta-1)}(2\nu)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{p}{2}\right)^{2\nu} \tag{1.156}$$

Inserted in the action, we take the inverse Fourier transform of $\tilde{\phi}_b(-p),\ \tilde{\phi}_b(p)$:

$$I_m[\tilde{\phi}_b,g]=\frac{2\nu\Gamma(1-\nu)}{2\Gamma(1+\nu)}\int dt_1\,\tilde{\phi}_b(t_1)\int dt_2\,\tilde{\phi}_b(t_2)\int\frac{dp}{2\pi}\left(\frac{p}{2}\right)^{2\nu}e^{ip(t_1-t_2)}. \tag{1.157}$$

We need the basic transform of a non-integer power law [59]:

$$\frac{1}{2\pi}\int dp\,e^{ipx}p^{2\nu}=\frac{2^{2\nu}}{\pi^{1/2}}\frac{\Gamma(\Delta)}{\Gamma(-\nu)}\frac{1}{x^{2\Delta}} \tag{1.158}$$

, where again $\nu=\Delta-1/2$ is used. Finally,

$$I_m[\tilde{\phi}_b,g]=\frac{\nu\Gamma(1-\nu)\Gamma(\Delta)}{\sqrt{\pi}\Gamma(-\nu)\Gamma(1+\nu)}\int dt_1\int dt_2\,\tilde{\phi}_b(t_1)\frac{1}{(t_1-t_2)^{2\Delta}}\tilde{\phi}_b(t_2). \tag{1.159}$$

Using $\Gamma(1+x)=x\Gamma(x)$,

$$\boxed{I_m[\tilde{\phi}_b,g]=-\frac{(\Delta-1/2)\Gamma(\Delta)}{\sqrt{\pi}\Gamma(\Delta-1/2)}\int dt_1\int dt_2\,\tilde{\phi}_b(t_1)\frac{1}{(t_1-t_2)^{2\Delta}}\tilde{\phi}_b(t_2)}. \tag{1.160}$$

This indeed has the correct scaling behaviour expected from both the inversion symmetry of the AdS_2-manifold and the conformal symmetry of dual correlation functions.

To turn on the effects of quantum gravity, we should couple the matter fields to the reparametrization modes. An effective way to do this, is to modify the renormalized scalar field $\tilde{\phi}_b$, by incorporating the wiggle boundary curve in Poincaré coordinates $(F(\tau),Z=\epsilon F'(\tau))$:

$$\phi(\epsilon,F)=Z^{1-\Delta}\tilde{\phi}(F)_b=\epsilon^{1-\Delta}F'^{1-\Delta}(\tau)\tilde{\phi}_b(F)\equiv\epsilon^{1-\Delta}\phi_b(\tau) \tag{1.161}$$

, where $\phi_b(\tau)=\phi_b(F(\tau))\equiv F'^{1-\Delta}(\tau)\tilde{\phi}_b(F(\tau))$. The above generating functional can therefore be rewritten in terms of $\phi_b(\tau)$ [72]:

$$\begin{aligned} I_m[\phi_b, f] =& - D\int dF_1 \int dF_2 \frac{1}{(F_1-F_2)^{2\Delta}}\, \tilde{\phi}_b(F_1)\tilde{\phi}_b(F_2) \\ &= -D\int d\tau_1 \int d\tau_2 \left(\frac{F'(\tau_1)F'(\tau_2)}{(F(\tau_1)-F(\tau_2))^2}\right)^{\Delta} \phi_b(\tau_1)\phi_b(\tau_2) \end{aligned} \tag{1.162}$$

, where $D = \frac{(\Delta-1/2)\Gamma(\Delta)}{\sqrt{\pi}\Gamma(\Delta-1/2)}$. It is often more natural to work in terms of the thermal reparametrization mode $f(\tau)$, via $F(\tau) = \tan\frac{\pi}{\beta}f(\tau)$:

$$I_m[\phi_b, f] = -D\int d\tau_1 \int d\tau_2 \left(\frac{f'(\tau_1)f'(\tau_2)}{\left(\frac{\beta}{\pi}\sin\left[\frac{\pi}{\beta}(f(\tau_1)-f(\tau_2))\right]\right)^2}\right)^{\Delta} \phi_b(\tau_1)\phi_b(\tau_2). \tag{1.163}$$

The effects of quantum gravity are now encoded in the fluctuations of the Schwarzian boundary mode f. Note that the scalar fields, although originating from a free field theory, interact with each other due to the coupling to gravity. Indeed, they create a non-trivial dilaton profile which gives rise to non-trivial interactions due to their backreaction on the wiggly boundary curve. It is remarkable how simple this coupling to quantum gravity is accounted for in this model. From now on, it will be convenient to absorb the constant prefactor into a redefinition of the boundary fields $\phi_b(\tau) \to \phi_b(\tau)/\sqrt{2D}$.
The total generating functional of JT gravity coupled to free scalar fields is therefore:

$$\boxed{Z(\beta,\phi_b) = e^{S_0}\int \mathcal{D}f\, e^{C\int_0^\beta d\tau\, \left\{\tan\frac{\pi f(\tau)}{\beta},\tau\right\}}\, e^{\frac{1}{2}\int d\tau_1 \int d\tau_2 \left(\frac{f'(\tau_1)f'(\tau_2)}{\left(\frac{\beta}{\pi}\sin\left[\frac{\pi}{\beta}(f(\tau_1)-f(\tau_2))\right]\right)^2}\right)^{\Delta} \phi_b(\tau_1)\phi_b(\tau_2)}}. \tag{1.164}$$

1.8.2 Correlation functions

Correlation functions are obtained by functionally differentiating the above generating functional with respect to the boundary matter sources, and evaluating them to zero (see Eq 1.144). This leads to the two-point function:

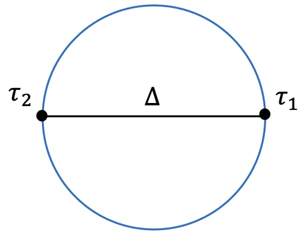
$$\begin{aligned} \langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle &= \frac{\delta}{\delta\phi_b(\tau_1)}\frac{\delta}{\delta\phi_b(\tau_2)} Z(\beta,\phi_b)|_{\phi_b\to 0} \\ &= e^{S_0}\int \mathcal{D}f\, e^{C\int_0^\beta d\tau\, \left\{\tan\frac{\pi f(\tau)}{\beta},\tau\right\}} \left(\frac{f'(\tau_1)f'(\tau_2)}{\left(\frac{\beta}{\pi}\sin\left[\frac{\pi}{\beta}(f(\tau_1)-f(\tau_2))\right]\right)^2}\right)^{\Delta}. \end{aligned} \tag{1.165}$$

We may interpret the quantity within brackets as a bilocal operator in the Schwarzian theory

$$\mathcal{O}_\Delta(\tau_1,\tau_2) = \left(\frac{f'(\tau_1)f'(\tau_2)}{\left(\frac{\beta}{\pi}\sin\left[\frac{\pi}{\beta}(f(\tau_1)-f(\tau_2))\right]\right)^2}\right)^{\Delta} \tag{1.166}$$

, which leads to the two point correlation function:

$$\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle = \langle \mathcal{O}_\Delta(\tau_1,\tau_2)\rangle \quad = \tag{1.167}$$



We can consider this operator as the two-point function of some 1d matter CFT at finite temperature coupled to the Schwarzian theory, or as the boundary-to-boundary propagator of a bulk matter field coupled to the 2d dilaton-gravity theory in a classical black hole background, as indicated by the above Witten diagram. Note that this operator is invariant under the $\mathrm{SL}(2,\mathbb{R})$-isometries of the AdS_2-manifold that act on the reparametrization mode $F(\tau)$ as a Möbius transformation Eq 1.81. This implies that the bilocal operator $\mathcal{O}_\Delta(\tau_1,\tau_2)$ commutes with the Hamiltonian, which we argued is identified to the quadratic Casimir Eq 1.103;

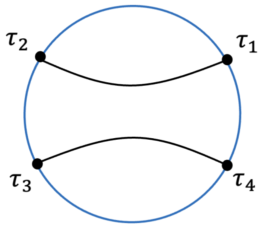
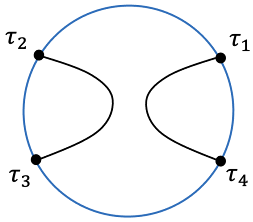
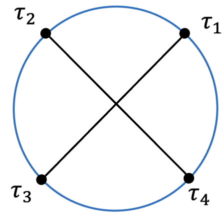
$$[H, \mathcal{O}_\Delta(\tau_1,\tau_2)] = 0. \tag{1.168}$$

This implies that the bilocal operators are diagonal between energy eigenstates, and the energy is conserved at each node at the boundary. Using Euclidean propagation in the Heisenberg picture $\mathcal{O}(\tau) = e^{\tau H}\mathcal{O}(0)e^{-\tau H}$, one expects to find an expansion of this amplitude of the form

$$\begin{aligned}\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle &= \mathrm{Tr}_{\mathcal{H}_{BH}}\left[e^{-\beta H}\mathcal{O}(\tau_2)\mathcal{O}(\tau_1)\right] = \mathrm{Tr}_{\mathcal{H}_{BH}}\left[e^{-\beta H}e^{\tau_2 H}\mathcal{O}e^{-\tau_2 H}e^{\tau_1 H}\mathcal{O}e^{-\tau_1 H}\right] \\ &= \int dE_1\rho(E_1)\int dE_2\rho(E_2)e^{-\tau E_2}e^{-(\beta-\tau)E_1}|\langle E_1 \mid \mathcal{O} \mid E_2\rangle|^2\end{aligned} \tag{1.169}$$

, where $\tau = \tau_2 - \tau_1$. Since the presence of matter does not affect the density of states to leading order, we can take $\rho(E)$ to be the result for pure gravity Eq 1.140. The factor $e^{-\tau E}$ represents a propagation of energy E along a boundary slice of length τ. Its complement is a propagation over $\beta - \tau$. The operator-insertion $|\langle E_1 \mid \mathcal{O} \mid E_2\rangle|$ is the matrix element of the bilocal operator at the boundary. Exact expressions for this vertex operator have been obtained using various techniques in the literature, e.g. using a suitable double-scaling limit of a 2d Virasoro CFT [21], using a free-particle approach [73], or a geometric interpretation of JT gravity in its first order BF formulation [22] [23].
One can take higher order derivatives to obtain n-point functions. Since the generating functional is quadratic in the sources, Wick's theorem [69] is appropriate in this context, and the correlation function is obtained by summing over all possible pairings between the different boundary points. For example, assuming the cyclic ordening $\tau_1 < \tau_2 < \tau_3 < \tau_4$, the four-point function can be written as:

$$\begin{aligned}\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\mathcal{O}(\tau_3)\mathcal{O}(\tau_4)\rangle =& \langle \mathcal{O}_\Delta(\tau_1,\tau_2)\mathcal{O}_\Delta(\tau_3,\tau_4)\rangle + \langle \mathcal{O}_\Delta(\tau_1,\tau_4)\mathcal{O}_\Delta(\tau_2,\tau_3)\rangle \\ &+ \langle \mathcal{O}_\Delta(\tau_1,\tau_3)\mathcal{O}_\Delta(\tau_2,\tau_4)\rangle\,.\end{aligned} \tag{1.170}$$

$=$  $+$  $+$ 

Since the bilocal operators commute with the Hamiltonian, the same energy eigenstate appears in each sector of the bulk separated by the bilocal operators. E.g. the region between $\tau_2 \leftrightarrow \tau_3$ and $\tau_1 \leftrightarrow \tau_4$ in the first diagram, and $\tau_2 \leftrightarrow \tau_1$ and $\tau_3 \leftrightarrow \tau_4$ in the second diagram. Finally, the third diagram has no opposite region of equal energies, since the two bilocal operators cross in the bulk, and we refer to it as an out-of-time-ordered (OTO) 4-point function. Here, one has to take into account the effect of the two crossing legs in the diagram, which leads to scattering amplitudes of particles. These describe gravitational shockwaves in the AdS_2 black hole background of the bulk description [21].

A first approach to obtain the disk amplitudes with matter is obtained by Schwarzian perturbation theory, by expanding the reparameterization modes around the classical saddle: $f(\tau) = \tau + \epsilon(\tau)$. This is sufficient to investigate the effect of quantum chaos. In particular, one finds that it saturates the maximal bound on chaos set in [61]. The connection between JT gravity and quantum chaos is one of the main applications and motivations to study this model more thoroughly.

In the next chapters however, we will solve the disk amplitudes directly in the bulk theory via the gauge theoretical perspective [22]. This comes to the core of this thesis, where we will obtain the exact quantum amplitudes of end-of-the-world (EOW) branes in various topologies using this perspective.

Chapter 2

First order quantization of JT gravity

"In the middle of difficulty lies opportunity."

Einstein, Albert

The tractability of quantum gravitational calculations in lower dimensions stems from the fact that they are perturbatively equivalent to topological gauge theories. The gravitons in the corresponding gravity theory have no propagating degrees of freedom. A particular example has been known since the results of Witten for 2+1d gravity, which in the presence of a negative cosmological constant can be rewritten in terms of a Chern-Simons gauge theory based on an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ gauge group, see [74] [75] [76].
In this chapter, I will review the bulk quantization of JT gravity in its first order formalism. This will provide an efficient tool to gain geometric intuition on quantum-gravitational amplitudes involving EOW branes later. Specifically, one rewrites the path integral of the second-order bulk JT action in terms of an integration over a gauge triplet $\mathbf{A}$ involving frame fields e^a and spin connections $\omega^a{}_b$, and an auxiliary $\mathbf{B}$-triplet involving the scalar fields associated with the dilaton field. These can be packaged together in terms of an $\mathfrak{sl}(2,\mathbb{R})$ topological BF gauge theory. This approach was used in [22] and [23] to obtain the same diagrammatic expressions for boundary correlation functions of [21]. However, both studies considered a different global structure of the gauge group and of the precise form of the boundary action. Although the on-shell gauge algebra corresponds in both cases to $\mathfrak{sl}(2,\mathbb{R})$, the global structure needed to capture JT gravity at a quantum level is not readily obvious. Indeed, the density of states corresponding to pure $\mathrm{SL}(2,\mathbb{R})$ does not match with the Schwarzian density of states, derived in Eq 1.140. Matching results have been found by using the semigroup $\mathrm{SL}^+(2,\mathbb{R})$ in [22], while [23] used an extension of the universal covering group $\tilde{\mathrm{SL}}(2,\mathbb{R})$ by $\mathbb{R}$. Over the years, more evidence has been provided in favour of the former [64]. Among others, this choice naturally excludes singular geometries in the path integral.
Furthermore, the bulk BF theory yields a topologically trivial theory, consistent with Dirichlet boundary conditions on the frame fields and spin connections. Therefore, the role of the boundary action is to yield interesting dynamics that reduces to the boundary Schwarzian theory in its second order formalism. The former uses the natural boundary action obtained by dimensionally reducing 3D Chern-Simons theory. The latter uses a

boundary-changing defect to switch the natural Dirichlet boundary conditions to those needed to reproduce the Schwarzian dynamics. Both approaches use boundary-anchored Wilson lines to describe bilocal operators in the Schwarzian theory. Correlation functions of these Wilson operators in the gauge theory match with those of the bilocal operators obtained in the Schwarzian theory. It was noted in [23] that these Wilson lines are equivalent to propagating probe particles between two distinct points on the boundary.

In this chapter, I will largely follow the approach taken by [22], after explaining the on-shell equivalence between the bulk JT action and a topological BF theory. This approach will then be used in the subsequent chapter to provide an alternative perspective on the quantum amplitudes involving the end-of-the-world (EOW) brane amplitudes of [34] and [36]. Consequently, the BF framework will then be extended to describe $\mathcal{N}=1$ JT supergravity amplitudes along the lines of [40] in the chapter thereafter. This thesis then culminates near the end of that chapter by giving an effective description of EOW branes in superspace using the same techniques. As far as we know, these amplitudes have not yet been considered in the literature before, demonstrating the fruitfulness of this approach.

2.1 First-order formulation of general relativity

Frame field

[1] In D-dimensional GR, the metric tensor $g_{\mu\nu}$ has Lorentzian signature $(-,+,\dots,+)$. Linear algebra provides a tool to diagonalize the metric $g_{\mu\nu}$ by an orthogonal transformation $(O^{-1})_\mu{}^a = O^a_\mu$ [59]

$$g_{\mu\nu}(x) = O^a_{\ \mu}(x)\Lambda_{ab}(x)O^b_{\ \nu}(x)$$

, where $\Lambda_{ab} = \text{diag}(-\lambda^0, \lambda^1, \dots, \lambda^{D-1})$ contains the positive eigenvalues $\lambda^a > 0$. This property holds since the metric is non-degenerate throughout. Furthermore, the sign of the eigenvalues is preserved under general coordinate transformations. We may therefore define frame fields $e^a_\mu(x) \equiv \sqrt{\lambda^a(x)}O^a{}_\mu(x)$ that determine completely the non-trivial spacetime dependence of the metric tensor

$$\boxed{g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x)}. \tag{2.1}$$

η_{ab} is the flat Minkowksi metric. By definition of the Minkowksi metric, this identity is preserved under the group of proper Lorentz transformations $SO(D-1,1)$ that leave the Minkowksi line element invariant $\Lambda^a{}_c\eta_{ab}\Lambda^b{}_d \equiv \eta_{cd}$. Therefore, all geometric quantities related by equivalent frame field configurations $e'^a_\mu(x) = \Lambda^a{}_b(x)e^b_\mu(x)$ should be identified. These local Lorentz-transformations are allowed to differ at each point x, and act only on the local Lorentz indices a. In contrast, the frame fields transform under general coordinate transformations as a covariant vector $e'^a_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}e^a_\nu(x)$. Since the metric is invertible, the frame fields are also invertible, and one can define the inverse frame field $e^\mu_a(x)$, which satisfies $e^a_\mu e^\mu_b = \delta^a_b$ and $e^\mu_a e^a_\nu = \delta^\mu_\nu$. Any contravariant vector field $V^\mu(x)$ can be expanded in the local basis $e^\mu_a(x)$ as $V^\mu(x) = V^a(x)e^\mu_a(x)$, where $V^a(x) = V^\mu(x)e^a_\mu(x)$ are the local Lorentz components with respect to e^a_μ. The latter are invariant under general coordinate transformations, and transform under local Lorentz transformations as $V^{a\prime}(x) = \Lambda^a{}_b(x)V^b(x)$. Likewise, covariant tensors $W_\mu(x)$ may be expanded in the $e^a_\mu(x)$ basis as $W_\mu(x) = W_a(x)e^a_\mu(x)$.

[1] This section is largely based on [59]. See also e.g. [60].

The frame fields thereby define a transition from the basis of one-forms on the spacetime manifold to a basis of one-forms on the local Lorentz manifold

$$e^a \equiv e^a_\mu \, dx^\mu \tag{2.2}$$

This transition is dual to the change of basisvectors on the tangent space: $E_a \equiv e^\mu_a(x)\frac{\partial}{\partial x^\mu}$. It acts on the Levi-Civita tensor densities [60] as:

$$\epsilon_{\mu_1\mu_2\ldots\mu_D} \equiv e^{-1}\epsilon_{a_1a_2\ldots a_D}\, e^{a_1}_{\mu_1}e^{a_2}_{\mu_2}\ldots e^{a_D}_{\mu_D}, \qquad \epsilon^{\mu_1\mu_1\ldots\mu_D} \equiv e\epsilon^{a_1a_2\ldots a_D}e^{\mu_1}_{a_1}e^{\mu_2}_{a_2}\ldots e^{\mu_D}_{a_D} \tag{2.3}$$

, by definition of the determinant $e \equiv \det(e^a_\mu)$. Using $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, the latter is related to the determinant of the metric by $e = \sqrt{-g}$. $\epsilon_{a_1a_2\ldots a_D}$ is the completely antisymmetric Levi-Civita symbol in local Lorentz coordinates, and is naturally invariant under general coordinate transformations.
This defines a natural volume form on the spacetime manifold that reduces to the definition of the second order volume form $dV = \sqrt{-g}\, d^Dx$:

$$\begin{aligned} dV &\equiv e^0 \wedge e^1 \wedge \cdots \wedge e^{D-1} \\ &= \frac{1}{D!}\epsilon_{a_1\ldots a_D}e^{a_1} \wedge \cdots \wedge e^{a_D} = \frac{1}{D!}e\epsilon_{\mu_1\ldots\mu_D}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} \\ &= \sqrt{-g}\, d^Dx. \end{aligned} \tag{2.4}$$

Spin-connection

We have seen that geometric quantities are covariant under local Lorentz transformations of the frame field. However, this symmetry is locally anomalous since the derived two-form de^a transforms non-covariantly

$$de'^a = d(\Lambda^a{}_b e^b) = \Lambda^a{}_b de^b + d\Lambda^a{}_b \wedge e^b.$$

The situation is resolved by gauging the local Lorentz symmetry, and introducing an anti-symmetric one-form gauge connection $\omega^{ab} = \omega^{ab}_\mu \, dx^\mu = -\omega^{ba}_\mu dx^\mu$. This cures local Lorentz covariance of the torsion vector T^a, defined as

$$\boxed{de^a + \omega^a{}_b \wedge e^b \equiv T^a} \tag{2.5}$$

, when ω^{ab} transforms as an $SO(D-1,1)$ Yang-Mills gauge field

$$\omega'^a_b = \Lambda^a{}_c \, d\Lambda^{-1c}{}_b + \Lambda^a{}_c \, \omega^c{}_d \, \Lambda^{-1d}{}_b.$$

Equation 2.5 is known as the first Cartan structure equation, and defines the torsion two-form T^a in terms of the spin-connection ω^{ab}. The torsion is often taken to be zero in most application of general relativity. A non-zero torsion tensor ultimately leads to a non-symmetric Christoffel connection, which often appears in supergravity theories coupled to matter fields. For now, we simply define $T^a \equiv 0$. ω_{ab} is called the spin-connection of the first-order formalism, in resemblance to its application in supergravity. In local coordinates, we write the coordinate transformation as $\omega'^a_\mu{}_b = \Lambda^a{}_c\partial_\mu\Lambda^{-1c}{}_b + \Lambda^a{}_c\omega^c_\mu{}_d\Lambda^{-1d}{}_b$. This allows us to introduce covariant derivatives on Lorentz-vectors and -covectors

$$D_\mu V^a = \partial_\mu V^a + \omega^a_\mu{}_b V^b, \qquad D_\mu W_a = \partial_\mu W_a - W_b \, \omega^b_\mu{}_a. \tag{2.6}$$

The combinations $D_\mu V^a$ and $D_\mu V_a$ now transform covariantly under local Lorentz transformations:

$$V^{a\prime}(x) = \Lambda^a{}_b(x)V^b(x), \qquad V'_a(x) = \Lambda_a{}^b(x)V_b(x) = V_b(x)(\Lambda^{-1})^b{}_a. \tag{2.7}$$

Applied to the Minkowksi metric, the metric postulate reads: $D_\mu \eta_{ab} = -\eta_{cb}\omega^c_{\mu\ a} - \eta_{ac}\omega^c_{\mu\ b} = -\omega_{\mu,ba} - \omega_{\mu,ab} \equiv 0$. The spin-connection is consistently an antisymmetric Lorentz tensor $\omega^{ab} = -\omega^{ba}$ such that η_{ab} indeed satisfies the metric postulate.
The conventional covariant derivative on a spacetime manifold is defined as $\nabla_\mu V^\nu \equiv e^\nu_a D_\mu V^a$. Written out explicitly;

$$\nabla_\mu V^\nu = e^\nu_a D_\mu V^a = e^\nu_a D_\mu(e^a_\rho V^\rho) = \partial_\mu V^\nu + e^\nu_a(\partial_\mu e^a_\rho + \omega_\mu{}^a{}_b e^b_\rho)V^\rho.$$

The affine connection derived from it is $\Gamma^\rho_{\mu\nu} = e^\rho_a(\partial_\mu e^a_\nu + \omega^a_{\mu\ b} e^b_\nu)$. Using the definition of the torsion-free spin-connection, this is seen to transform anomalously under general coordinate transformations, and reduces to the definition of the Christoffel connection in terms of the metric [59]. We can take it as the definition of the Christoffel connection, and rewrite it as the *frame field postulate*:

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^a_{\mu\ b} e^b_\nu - \Gamma^\sigma_{\mu\nu} e^a_\sigma = 0. \tag{2.8}$$

Curvature two-form

A corresponding field strength is associated with any Yang-Mills gauge connection. In the case of the $SO(D-1,1)$ local Lorentz symmetry, the field strength $R_{\mu\nu ab}$ is constructed from the commutator of two Lorentz-covariant derivatives:

$$[D_\mu, D_\nu]V^a = R_{\mu\nu}{}^a{}_b V^b. \tag{2.9}$$

Written in terms of the spin-connections;

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b. \tag{2.10}$$

From the antisymmetry in $[\mu\nu]$, we can define the curvature two-form

$$\rho^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab}(x)\, dx^\mu \wedge dx^\nu \tag{2.11}$$

, to rewrite the field strength defined above in terms of the second Cartan structure equation

$$\boxed{d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab}}. \tag{2.12}$$

The curvature two-form is related to the Riemann tensor $R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu ab}\, e^{a\rho} e^b_\sigma$, such that

$$[\nabla_\mu, \nabla_\nu]V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma. \tag{2.13}$$

This can readily be checked using the frame field postulate Eq 2.8, and the definition of the covariant derivative $\nabla_\mu V^\nu \equiv e^\nu_a D_\mu V^a$;

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu(e^\rho_a D_\nu V^a) - \nabla_\nu(e^\rho_a D_\mu V^a) = e^\rho_a[D_\mu, D_\nu]V^a \equiv e^\rho_a R_{\mu\nu}{}^a{}_c V^c = R_{\mu\nu}{}^a{}_c e^\rho_a e^c_\lambda V^\lambda.$$

In the second identity, I have used the symmetry of the torsionless Christoffel symbols. Equating both definitions yields the desired equivalence between the Riemann tensor and the curvature two-form.

2.2 BF formulation of JT gravity

We can now proceed to formulate 2d JT gravity in its first order formalism using the geometric structures above. In a Euclidean setting, we define frame fields $e^a = e^a_\mu dx^\mu$ that diagonalize the metric in terms of the Euclidean metric δ_{ab}:

$$g_{\mu\nu}(x) = e^a_\mu(x)\delta_{ab}e^b_\nu(x). \tag{2.14}$$

Note that in flat 2d Euclidean space, frame indices are raised and lowered by the flat Euclidean metric δ_{ab}, so there is no material difference between up and down in this context.
As a consequence of the antisymmetry of the spin connection of the internal Euclidean $SO(2) \simeq U(1)$ symmetry, they are specified by only one independent component. We write this in terms of the 2d Levi-Civita symbol ${\omega^a}_b = {\epsilon^a}_b\omega$, with ${\epsilon^0}_1 \equiv 1$. This, in turn, leads to a simplification of the curvature two-form in the second Cartan structure equation Eq 2.12:

$${\rho^a}_b = d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b = d{\omega^a}_b.$$

The last identity holds in 2d from the commutation of $[\omega_\mu, \omega_\nu] = 0$, leading to ${\omega^a}_c \wedge {\omega^c}_b = -\delta^a_b\omega \wedge \omega = 0$. We can relate this to the Riemann tensor via Eq 2.11. We also recall from Eq 1.2 that 2d spaces are maximally symmetric $R_{\mu\nu\rho\sigma} = \frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$, such that:

$$\begin{aligned} d{\omega^a}_b = d\omega{\epsilon^a}_b &= \frac{1}{2}e^a_\rho e^\sigma_b {R_{\mu\nu}{}^\rho}_\sigma(x)\, dx^\mu \wedge dx^\nu = \frac{R}{4}e^a_\rho e^\sigma_b(\delta^\rho_\mu g_{\nu\sigma} - g_{\mu\sigma}\delta^\rho_\nu)dx^\mu \wedge dx^\nu \\ &= \frac{R}{4}(e^a \wedge e_b - e_b \wedge e^a) = \frac{1}{2}R\, e^a \wedge e_b = \frac{1}{2}R\, e^0 \wedge e^1\, {\epsilon^a}_b. \end{aligned} \tag{2.15}$$

Furthermore, we still have the appropriate volume form Eq 2.4;

$$e^0 \wedge e^1 = \sqrt{g}d^2x. \tag{2.16}$$

This allows us to identify $d\omega = \frac{1}{2}R\sqrt{g}d^2x$, and to rewrite the first-order bulk JT action over a compact $(\partial\mathcal{M} = 0)$ manifold $\mathcal{M}$ (c.f. Eq 1.48) entirely in terms of frame fields and spin connections;

$$\boxed{\frac{1}{4}\int_\mathcal{M} d^2x\sqrt{g}\phi(R+2) \;\simeq\; \frac{1}{2}\int_\mathcal{M}[\phi(d\omega + e^0 \wedge e^1) - \phi_a(de^a + {\epsilon^a}_b\omega \wedge e^b)]}. \tag{2.17}$$

The last terms are sums over $\mathfrak{so}(2) \simeq \mathfrak{u}(1)$ Lorentz indices $a = 0, 1$, where the scalar fields ϕ_a act as Lagrange multipliers enforcing the no-torsion constraint. We can combine the scalar fields ϕ_a, ϕ, and the gauge fields e^a, ω into $\mathfrak{sl}(2, \mathbb{R})$ triplets according to:

$$B_I = (-\phi_a, \phi), \qquad A^I = (e^a, \omega), \qquad \mathbf{B} = B^I P_I, \qquad \mathbf{A} = A^I P_I \tag{2.18}$$

, where I use boldface letters throughout to distinguish between the gauge field components, and a vector in the algebra. Note that the indices of B_I in the definition are lowered, while those of A^I are raised. P_0, P_1, P_2 are the generators satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra;

$$[P_I, P_J] = \epsilon_{IJK} P^K, \qquad \mathrm{Tr}(P_I P_J) = \frac{1}{2}\eta_{IJ}. \tag{2.19}$$

, where $\epsilon_{012} \equiv -1$, and $I, J, K \in \{0,1,2\}$. $\eta_{IJ} = \mathrm{diag}(1,1,-1)$ is the Cartan-Killing metric deduced from the normalization of the generators P_I. The algebra indices are raised and lowered with respect to this metric. Explicitly, the algebra reads:

$$[P_0, P_1] = P_2, \qquad [P_0, P_2] = P_1, \qquad [P_1, P_2] = -P_0. \tag{2.20}$$

One can relate this basis to the more familiar Cartan-Weyl basis of the $\mathfrak{sl}(2,\mathbb{R})$ algebra for generators H, E^-, E^+

$$P_0 = -H, \qquad P_1 = \frac{1}{2}(E^- + E^+), \qquad P_2 = \frac{1}{2}(E^- - E^+). \tag{2.21}$$

, satisfying the known $\mathfrak{sl}(2,\mathbb{R})$ algebra [77]:

$$[H, E^\pm] = \pm E^\pm, \qquad [E^+, E^-] = 2H. \tag{2.22}$$

A particular realization of the algebra above is by the traceless 2×2-matrices:

$$P_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.23}$$

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.24}$$

We readily check that they satisfy both the algebra and the normalization condition. We will interpret the one-form $\mathbf{A}$ triplet as the $\mathfrak{sl}(2,\mathbb{R})$ gauge connection, and the $\mathbf{B}$ zero-form as the triplet containing the auxiliary fields, along the lines of [78] [79]. With any gauge connection, we associate a field strength according to $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$. Written out in components using the algebra commutation relations yields:

$$\begin{aligned} F^C P_C &= dA^C P_C + \frac{1}{2} A^A \wedge A^B [P_A, P_B] = dA^C P_C + \frac{1}{2} A^A \wedge A^B \epsilon_{ABC} P^C \\ \leftrightarrow \quad F^C &= dA^C + \frac{1}{2} A^A \wedge A^B \epsilon_{ABD} \eta^{DC}. \end{aligned} \tag{2.25}$$

Explicitly distinguishing the a and 2-components (taking in mind $\epsilon_{ab2} = -\epsilon_{ab}$ from the definition $\epsilon_{012} = -1$) yields:

$$\begin{aligned} F^a &= dA^a + \frac{1}{2}\epsilon_{AB}{}^{a} A^A \wedge A^B = de^a + \frac{1}{2}\epsilon_{2b}{}^{a}\omega \wedge e^b + \frac{1}{2}\epsilon_{b2}{}^{a} e^b \wedge \omega = de^a + \epsilon^a{}_b \omega \wedge e^b, \\ F^2 &= dA^2 - \frac{1}{2}\epsilon_{AB2} A^A \wedge A^B = d\omega + \frac{1}{2}\epsilon_{ab} e^a \wedge e^b = d\omega + e^0 \wedge e^1. \end{aligned} \tag{2.26}$$

Using the normalization of the generators, we take the inner product between $\mathbf{B}$ and $\mathbf{F}$:

$$\begin{aligned}\text{Tr}(\mathbf{BF}) &= \frac{1}{2}B^A F^A \eta_{AB} = \frac{1}{2}B^a F^a - \frac{1}{2}B^2 F^2 \\ &= -\frac{1}{2}\phi_a(de^a + \epsilon^a{}_b \omega \wedge e^b) + \frac{1}{2}\phi(d\omega + e^0 \wedge e^1)\end{aligned}$$

, with $B^2 = -\phi$ raised by the Cartan-Killing metric η^{AB}. The RHS exactly coincides with the first order JT action Eq 2.17. It is thus possible to write the bulk JT action in terms of a BF (***B**ackground **F**ield*) gauge theory over a topologically trivial 2d manifold $\mathcal{M}$:

$$\boxed{\frac{1}{4}\int_{\mathcal{M}} d^2x\sqrt{g}\phi(R+2) = \int_{\mathcal{M}} \text{Tr}(\mathbf{BF})}. \tag{2.27}$$

Since the volume form of the BF action does not explicitly depend on the metric, the observables are topological in the bulk. As in Chern-Simons theory for example [80], these are topological knot observables, including Wilson loops and surfaces associated to both fields $\mathbf{A}$ and $\mathbf{B}$. Hence, the Hamiltonian of BF theory only has support on the boundary, where all interesting dynamics take place. Note that this is of course very similar to the second order JT formulation, which fixes the metric in the bulk, and reduces the dynamics to the boundary reparametrization modes.

An often underappreciated derivation to establish the on-shell equivalence between an $\mathfrak{sl}(2,\mathbb{R})$ BF theory and the bulk JT action, is to check that the equations of motion coincide with those obtained in section 1.4. In particular, variation with respect to the auxiliary $\mathbf{B}$-fields readily imposes the connections to be flat $\mathbf{F} = 0$. Unpacking the different components Eq 2.26, $F^a = 0$ imposes the no-torsion constraints, while F^2 forces the manifold to have negative curvature throughout $R = -2$, using Eqs 2.15, 2.16.
Variation with respect to the gauge connections on the other hand should reproduce the vacuum dilaton equations of motion Eq 1.62:

$$\begin{aligned}\int_{\mathcal{M}} \text{Tr}((\mathbf{B}\delta\mathbf{F}) &= \int_{\mathcal{M}} \text{Tr}(\mathbf{B}(d\delta\mathbf{A} + \mathbf{A}\wedge\delta\mathbf{A} + \delta\mathbf{A}\wedge\mathbf{A})) \\ &= -\int_{\mathcal{M}} \text{Tr}(d\mathbf{B}\wedge\delta\mathbf{A} - \mathbf{BA}\wedge\delta\mathbf{A} - \mathbf{B}\delta\mathbf{A}\wedge\mathbf{A}) + \int_{\partial\mathcal{M}} \text{Tr}(\mathbf{B}\delta\mathbf{A}|_\partial) \\ &= -\int_{\mathcal{M}} \text{Tr}(d\mathbf{B}\wedge\delta\mathbf{A} - \mathbf{BA}\wedge\delta\mathbf{A} - \delta\mathbf{A}\wedge\mathbf{AB})) \\ &= -\frac{1}{2}\int_{\mathcal{M}} \text{Tr}((\partial_\mu\mathbf{B}\delta\mathbf{A}_\nu - \partial_\nu\mathbf{B}\delta\mathbf{A}_\mu - \mathbf{BA}_\mu\delta\mathbf{A}_\nu + \mathbf{BA}_\nu\delta\mathbf{A}_\mu - \delta\mathbf{A}_\mu\mathbf{A}_\nu\mathbf{B} + \delta\mathbf{A}_\nu\mathbf{A}_\mu\mathbf{B})dx^\mu\wedge dx^\nu) \\ &= -\frac{1}{2}\int_{\mathcal{M}} \text{Tr}((\partial_\mu\mathbf{B} + \mathbf{A}_\mu\mathbf{B} - \mathbf{BA}_\mu)\delta\mathbf{A}_\nu dx^\mu\wedge dx^\nu + \dots).\end{aligned} \tag{2.28}$$

In the first line, I have used the commutativity of $\delta\mathbf{A}\wedge\mathbf{A} = \mathbf{A}\wedge\delta\mathbf{A}$, derived from $\delta\mathbf{A}\wedge\mathbf{A} \equiv \frac{1}{2}[\delta\mathbf{A}_\nu, \mathbf{A}_\mu]dx^\nu \wedge dx^\mu = -\frac{1}{2}[\delta\mathbf{A}_\nu, \mathbf{A}_\mu]dx^\mu\wedge dx^\nu = \frac{1}{2}[\mathbf{A}_\mu, \delta\mathbf{A}_\nu]dx^\mu\wedge dx^\nu = \mathbf{A}\wedge\delta\mathbf{A}$. In the second line, I have used partial integration, keeping in mind that the Leibnitz product rule of the exterior derivative between a wedge product of a p- and q-form is modified to $d(\omega^{(p)}\wedge\omega^{(q)}) = d\omega^{(p)}\wedge\omega^{(q)} + (-)^p\omega^{(p)}\wedge d\omega^{(q)}$. In this case, $\mathbf{B}$ is a zero-form, such that the standard Leibnitz rule applies. Partial integration yields a boundary term, which vanishes on compact surfaces $\partial\mathcal{M} = 0$. When placing the theory on a disk with $\partial\mathcal{M} \neq 0$, we can neglect it for now by imposing Dirichlet boundary conditions on the gauge fields $\delta\mathbf{A}_\tau = 0$. In the third line, I have used the cyclicity of the trace in the last term. In the fourth line, I have written out the wedge product explicitly, keeping only the term corresponding to $\delta\mathbf{A}_\nu$ in the last.

The equations of motion corresponding to variation of $\delta\mathbf{A}_\nu$ are simply

$$D_\mu\mathbf{B} = \partial_\mu\mathbf{B} + [\mathbf{A}_\mu, \mathbf{B}] = 0. \tag{2.29}$$

These can be unpacked by writing out the components explicitly $\mathbf{B} = B^I P_I$, $\mathbf{A} = A^I P_I$, where $B^I = B_J\eta^{IJ} = (-\phi^a, -\phi)$, and $A^I \equiv (e^a, \omega)$,

$$\partial_\mu(-\phi P_2 - \phi_0 P_0 - \phi_1 P_1) + [e^0 P_0 + e^1 P_1 + \omega P_2, -\phi P_2 - \phi^0 P_0 - \phi^1 P_1] = 0.$$

Using the $\mathfrak{sl}(2,\mathbb{R})$ algebra Eq 2.20, the equations corresponding to the P_2, P_0, P_1 components are respectively:

$$\partial_\mu\phi = e^1_\mu\phi^0 - e^0_\mu\phi^1 \tag{2.30}$$
$$\partial_\mu\phi^0 = e^1_\mu\phi - \omega_\mu\phi^1 \tag{2.31}$$
$$\partial_\mu\phi^1 = -e^0_\mu\phi + \omega_\mu\phi^0. \tag{2.32}$$

We can now take an additional ∂_ν derivative of the first equation

$$\partial_\nu\partial_\mu\phi = \partial_\nu e^1_\mu\phi^0 + e^1_\mu\partial_\nu\phi^0 - \partial_\nu e^0_\mu\phi^1 - e^0_\mu\partial_\nu\phi^1. \tag{2.33}$$

Subtracting $\Gamma^\alpha_{\nu\mu}\partial_\alpha\phi = \Gamma^\alpha_{\nu\mu}(e^1_\alpha\phi^0 - e^0_\alpha\phi^1)$, we can write the LHS in terms of a covariant derivative

$$\nabla_\nu\partial_\mu\phi = \phi^0(\partial_\nu e^1_\mu - \Gamma^\alpha_{\nu\mu}e^1_\alpha) - \phi^1(\partial_\nu e^0_\mu - \Gamma^\alpha_{\nu\mu}e^0_\alpha) + e^1_\mu\partial_\nu\phi^0 - e^0_\mu\partial_\nu\phi^1. \tag{2.34}$$

Using Eqs 2.31, 2.32, we rewrite the last two terms:

$$\nabla_\nu\partial_\mu\phi = \phi^0(\partial_\nu e^1_\mu - \Gamma^\alpha_{\nu\mu}e^1_\alpha) - \phi^1(\partial_\nu e^0_\mu - \Gamma^\alpha_{\nu\mu}e^0_\alpha) + e^1_\mu e^1_\nu\phi + e^0_\mu e^0_\nu\phi - e^1_\mu\omega_\nu\phi^1 - e^0_\mu\omega_\nu\phi^0. \tag{2.35}$$

From the definition of the metric tensor, we write $g_{\mu\nu} = e^a_\mu e^b_\nu\delta_{ab} = e^0_\mu e^0_\nu + e^1_\mu e^1_\nu$. The first two terms contain respectively $\partial_\nu e^1_\mu - \Gamma^\alpha_{\nu\mu}e^1_\alpha - \omega_\nu e^0_\mu$, and $\partial_\nu e^0_\mu - \Gamma^\alpha_{\nu\mu}e^0_\alpha + \omega_\nu e^1_\mu$. Using the antisymmetry property in 2d ${\omega^a}_b = \omega{\epsilon^a}_b$, these two terms vanish from the frame field postulate Eq 2.8: $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^a_{\mu\ b}e^b_\nu - \Gamma^\sigma_{\mu\nu}e^a_\sigma = 0$.

Therefore, the equations of motion obtained from varying with respect to the gauge connections yield:

$$\boxed{\nabla_\nu\nabla_\mu\phi = g_{\mu\nu}\phi} \tag{2.36}$$

, where I have used that any covariant derivative on a scalar field reduces to the ordinary partial derivative $\nabla_\nu\nabla_\mu\phi = \nabla_\nu\partial_\mu\phi$.

We can massage the equation of motion of the dilaton Eq 1.62, obtained from varying the second order JT action with respect to the metric, to this form by contracting with the inverse metric $g^{\mu\nu}$ (and $g_{\mu\nu}g^{\mu\nu}=2$):

$$\begin{aligned} g^{\mu\nu}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi + \phi g_{\mu\nu}) &= 0 \\ \leftrightarrow \quad -\nabla^2\phi + 2\phi &= 0. \end{aligned}$$

Plugging this form of the Laplace-Beltrami operator back in the original equation of motion, reduces the latter to

$$\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\phi = 0 \tag{2.37}$$

, which exactly coincides with the equation of motion obtained in the BF theory above.
These basic facts establish a complete on-shell equivalence between the bulk JT action in the second order metric formulation, and a topological $\mathfrak{sl}(2,\mathbb{R})$ BF theory in the first order form. Note that many different authors can use different isomorphisms of the $\mathfrak{sl}(2,\mathbb{R})$ algebra and the explicit form of the triplet fields. In particular, we have used some slight modifications of the conventions compared to [40].
Infinitesimal gauge transformations parameterized by a parameter $\Theta = \Theta^I J_I$ act on the corresponding gauge connections as transformations in the fibre bundle of $\mathfrak{sl}(2,\mathbb{R})$: $\mathbf{A}\to\mathbf{A}+d\Theta+[\mathbf{A},\Theta]=\mathbf{A}+D\Theta$, while they act on $\mathbf{B}$ as an element of the adjoint representation $\mathbf{B}\to\mathbf{B}+[\mathbf{B},\Theta]$. When the gauge connection is flat with $\mathbf{F}=0$, infinitesimal gauge transformation are related to diffeomorphisms generated by a vector fields ξ^μ, that act on the metric as $\delta g_{\mu\nu}=\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu$ as $\Theta^a\sim e^a_\mu(x)\xi^\mu(x),\ \Theta^0\sim\omega_\mu(x)\xi^\mu(x)$ [23].
Note that our conventions fix the form of the covariant derivative, introduced in Eq 2.29, symbolically as:

$$D = d + \mathbf{A}. \tag{2.38}$$

In local coordinates, it acts on adjoint vectors as $D_\mu\mathbf{B}=\partial_\mu\mathbf{B}+[\mathbf{A}_\mu,\mathbf{B}]$. This ensures that e.g. the combination $D_\mu\mathbf{B}$ transforms in the adjoint representation; $\delta(D_\mu\mathbf{B})=[D_\mu\mathbf{B},\Theta]$. Indeed,

$$\begin{aligned} \delta(D_\mu\mathbf{B}) \equiv \delta(\partial_\mu\mathbf{B}+[\mathbf{A}_\mu,\mathbf{B}]) &= \partial_\mu\delta\mathbf{B}+[\delta\mathbf{A}_\mu,\mathbf{B}]+[\mathbf{A}_\mu,\delta\mathbf{B}] \\ &= [\partial_\mu\mathbf{B},\Theta]+[\mathbf{B},\partial_\mu\Theta]+[\partial_\mu\Theta,\mathbf{B}]+[[\mathbf{A}_\mu,\Theta],\mathbf{B}]+[\mathbf{A}_\mu,\delta\mathbf{B}] \\ &= [\partial_\mu\mathbf{B},\Theta]+[\mathbf{A}_\mu,[\Theta,\mathbf{B}]]-[\Theta,[\mathbf{A}_\mu,\mathbf{B}]]+[\mathbf{A}_\mu,\delta\mathbf{B}]. \end{aligned}$$

To obtain the last line, I have used the Jacobi identity, written conveniently as $[[A_\mu,\Theta],\mathbf{B}]=[A_\mu,[\Theta,\mathbf{B}]]-[\Theta,[A_\mu,\mathbf{B}]]$. Using again the form of the transformation $\delta\mathbf{B}=[\mathbf{B},\Theta]$ yields

$$\delta(D_\mu\mathbf{B}) = [\partial_\mu\mathbf{B},\Theta]+[[\mathbf{A}_\mu,\mathbf{B}],\Theta]-[\mathbf{A}_\mu,\delta\mathbf{B}]+[\mathbf{A}_\mu,\delta\mathbf{B}] = [D_\mu\mathbf{B},\Theta].$$

This allows to express the field strength as the covariant derivative acting on the gauge field $\mathbf{A}$;

$$\mathbf{F} = D\mathbf{A} = d\mathbf{A}+\mathbf{A}\wedge\mathbf{A} \tag{2.39}$$

, or in local coordinates $\mathbf{F}=\frac{1}{2}\mathbf{F}_{\mu\nu}dx^\mu\wedge dx^\nu$ as the commutator of two covariant derivatives acting on a field ψ in the fundamental representation (for which the covariant derivative acts on the field as $D_\mu\psi=\partial_\mu\psi+A_\mu\psi$);

$$\mathbf{F}_{\mu\nu}\psi = [D_\mu,D_\nu]\psi = (\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+[\mathbf{A}_\mu,\mathbf{A}_\nu])\psi. \tag{2.40}$$

2.2.1 General dilaton-gravity models

It is important to realize that the BF formalism of JT gravity is only one datapoint of a more general equivalence between general 2d dilaton-gravity models, and topological gauge theories. The latter are the Poisson-sigma models, studied e.g. in [81] [82]. The Poisson sigma models, like the non-linear 2d sigma models, are defined on an ambient two-dimensional manifold M, whose target space N is endowed with a set of bosonic coordinates B_I. In this case, the target space is a Poisson manifold, equipped with a Poisson structure $P = P^{IJ}\frac{d}{dB^I}\wedge\frac{d}{dB^J}$, where P^{IJ} are anticommuting bilinear Poisson structure constants that satisfy the Jacobi identity and generate a Poisson bracket on N: $\{f,g\} = P^{IJ}\frac{\partial f}{\partial B^I}\frac{\partial g}{\partial B^J}$ [83]. We have seen that the most general model of 2d dilaton-gravity Eq 1.46 is parameterized by a single dilaton potential $U(\phi)$. In Euclidean signature,

$$I = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{g}(\phi R - U(\phi)). \tag{2.41}$$

Again using the basic relations Eqs 2.15, 2.16, and introducing Lagrange multipliers ϕ^a enforcing the no-torsion constraint, we can repeat the steps leading up to Eq 2.17, to obtain the dilaton-gravity action in its first order formulation:

$$\frac{1}{4}\int_{\mathcal{M}} d^2x\sqrt{g}(\phi R - U(\phi)) \;\simeq\; \frac{1}{2}\int_{\mathcal{M}}[\left(\phi d\omega - \frac{U(\phi)}{2}e^0\wedge e^1\right) - \phi_a(de^a + \epsilon^a{}_b\omega\wedge e^b)]. \tag{2.42}$$

We again introduce a triplet of gauge connections $A^I = (e^a,\omega)$ and a triplet of auxiliary fields $B_I = (-\phi^a,\phi)$ ($I = 0,1,2$ and $a = 0,1$), to rewrite the action in terms of the antisymmetric Poisson brackets $\{B_I, B_J\}_P \equiv P_{IJ}(B)$. The form of these triplets coincides with the choice made for pure JT gravity above. Since the Poisson brackets satisfy the Jacobi identity $\partial_L P^{[IJ|}P^{L|K]} = 0$, they span an algebra called the Poisson sigma algebra. We can rewrite the first order action as [25]:

$$\boxed{\frac{1}{4}\int_{\mathcal{M}} d^2x\sqrt{g}\phi(R - U(\phi)) \;\simeq\; \frac{1}{2}\int_{\mathcal{M}}\left(A^I\wedge dB_I + \frac{1}{2}P_{IJ}(B)\,A^I\wedge A^J\right)} \tag{2.43}$$

, where the Poisson brackets are defined as

$$P_{01} = \{B_0, B_1\}_P = -\frac{U(B_2)}{2}, \qquad P_{a2} = \{B_a, B_2\}_P = \epsilon_{ab}B_b. \tag{2.44}$$

This can readily be checked by inserting the proper values of B_I, A^I, and the explicit antisymmetric Poisson bracket relations:

$$\begin{aligned}
&\int_{\mathcal{M}}\left(A^2\wedge dB_2 + A^a\wedge dB_a + \frac{1}{2}P_{01}A^0\wedge A^1 + \frac{1}{2}P_{10}A^1\wedge A^0 + \frac{1}{2}P_{a2}A^a\wedge A^2 + \frac{1}{2}P_{2a}A^2\wedge A^a\right)\\
&= \int_{\mathcal{M}}\left(dA^2B_2 + dA^aB_a + P_{01}A^0\wedge A^1 + P_{a2}A^a\wedge A^2\right)\\
&= \int_{\mathcal{M}}\left(d\omega\phi - de^a\phi^a - \frac{U(\phi)}{2}e^0\wedge e^1 - \epsilon_{ab}\phi^b e^a\wedge\omega\right)\\
&= \int_{\mathcal{M}}\left(d\omega\phi - de^a\phi^a - \frac{U(\phi)}{2}e^0\wedge e^1 - \epsilon_{ab}\phi^a\omega\wedge e^b\right).
\end{aligned}$$

This form coincides with Eq 2.42. In the second line, I have used the antisymmetry of both the Poisson brackets and wedge product, and have implemented a partial integration, taking into account that the A^I gauge connections are one-forms in the general Leibnitz rule $d(\omega^{(p)} \wedge \omega^{(q)}) = d\omega^{(p)} \wedge \omega^{(q)} + (-)^p \omega^{(p)} \wedge d\omega^{(q)}$.
We note that in the case of JT gravity, the form of the dilaton potential $U(\phi) = -2\phi$ reduces the non-linear Poisson sigma algebra Eq 2.44 exactly to the $\mathfrak{sl}(2,\mathbb{R})$ algebra of Eq 2.20. In this case, we can expend the triplets into generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra $\mathbf{B} = B_I J^I$, $\mathbf{A} = A^I J_I$, normalized in the same way $\text{Tr}(J_I J^I) = \delta_I{}^J/2$. Since the algebra is in this case just linear, we define a field strength and expand it into generators $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = d\mathbf{A} + \frac{1}{2}\epsilon_{IJ}{}^K P_K A^I \wedge A^J$. Partially integrating the first term in the Poisson sigma action Eq 2.43 immediately recovers the BF action:

$$I \sim \frac{1}{2}\int_{\mathcal{M}} B_I F^I = \int_{\mathcal{M}} \text{Tr}(\mathbf{BF}). \tag{2.45}$$

2.2.2 Quantum BF theory

Although the similarity between topological gauge theories and gravity allows to gain deeper structural understanding on the quantization of boundary correlators, there exist profound structural differences between the full non-perturbative solutions of gravity and gauge theory. These subtleties were already noted in the context of 3d gravity [84]. Therefore, before I go over to the full quantum solutions, I list some of these subtleties here already.
First of all, $\mathbf{B}$ acts as a Lagrange multiplier in the BF path integral, enforcing the constraint $\mathbf{F} = 0$ also off-shell. The resulting path integral consequently calculates the volume of the moduli space of flat connections. Within the moduli space, regular gauge connections can correspond to singular geometries in the gravity theory. For example, $\mathbf{A} = 0$ corresponds to a non-invertible metric in 3d Chern-Simons theory. One should therefore restrict the integration space to smooth geometries only. In 2d $\text{SL}(2,\mathbb{R})$ gravity, it turns out that the subset of smooth geometries corresponds to the Teichmüller subspace $\mathcal{T}(\Sigma)$ of connected hyperbolic components in the moduli space of flat gauge connections on any given Riemann surface Σ. We will see that this effectively restricts the description of the full $\text{SL}(2,\mathbb{R})$ gauge group to the subsemigroup $\text{SL}^+(2,\mathbb{R})$.
Furthermore, gravity contains large diffeomorphisms that are invisible within the gauge transformations of the BF description. These are the diffeomorphisms of the modular group $\text{SL}(2,\mathbb{Z})$ of the torus [52]. In general, we must quotient the Teichmüller space by the mapping class group $MCG(\Sigma)$ of discrete transformations corresponding to large diffeomorphisms.
It is also not a priori clear that the natural measure corresponding to BF theory translates to the measure that we have found earlier to describe the Schwarzian path integral. However, we will show along the lines of [30] that the natural symplectic measure of BF theory reproduces the measure at the asymptotic boundary encoding the Schwarzian dynamics. This general definition furthermore reproduces the Weil-Peterson measure on hyperbolic surfaces.
Finally, one should realize that the gravitational path integral naturally contains a sum over spacetime topologies, consistent with the prescribed boundary conditions. For a predefined number of boundaries, this is a genus expansion of non-trivial Euclidean wormholes. In contrast, gauge theory is defined on a fixed manifold with a predefined genus. For example on a 2d Riemann surface, the constrained path integral of pure Einstein gravity is restricted to sum over metric tensors with fixed Euler characteristic determined by the Gauss-Bonnet

theorem.
The higher topological contributions are non-perturbative corrections to the disk correlation functions obtained in the gauge theory, and should be added by hand. These important effects give rise to the notion of ensemble averaging [30], and a unitary Page curve in the black hole evaporation process [35] [34].

2.3 Recovering the Schwarzian boundary action

2.3.1 Boundary conditions of the BF action

One of the attractive features of the gauge BF formulation of JT gravity, is the ability to derive the symplectic form of the boundary Schwarzian reparametrization modes directly from the bulk theory. The symplectic form was derived along the lines of [20] in Eq C.15 as the natural symplectic form on the space of coadjoint orbits of the Virasoro algebra. Here, I will review the argument of [30], including the proper boundary conditions to obtain the Schwarzian action in holography.
Since the bulk BF action is topological, we need to include a boundary term that correctly reproduces the Schwarzian boundary action of Eq 1.93. This action is readily obtained in the second order form from the total JT action upon including the natural GHY boundary term. The required boundary conditions are isometric boundary lengths for the induced boundary metric, and a diverging dilaton backreaction profile:

$$g_{\tau\tau} = \frac{1}{\epsilon^2}, \qquad \phi = \frac{a}{2\epsilon}, \qquad \text{for } \epsilon \to 0. \tag{2.46}$$

To study JT gravity on a disk in the first order form, we need to add an appropriate boundary action. Restoring the prefactor in Eq 2.27, one usually studies:

$$I_{JT} = -\frac{1}{4\pi G}\int_{\mathcal{M}} \mathrm{Tr}(\mathbf{BF}) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{BA}). \tag{2.47}$$

One can also motivate the presence of this boundary term by realizing that we get precisely this term when dimensionally reducing 3d Chern-Simons theory to the 2d BF model [25]. The analogous boundary condition on the auxiliary field to Eq 2.46 would be to fix ($\gamma = \frac{a}{2}$)

$$\mathbf{B}|_\partial = \gamma \mathbf{A}|_\partial \tag{2.48}$$

, since the metric along the boundary $g_{\tau\tau}$ is second order in the frame fields comprising $\mathbf{A} \sim 1/\epsilon$. In most cases, I will indicate the boundary value of the gauge triplet as the one-form $\mathbf{A}|_\partial = \mathbf{A}_\tau d\tau$.
Imposing this boundary condition makes for a well-defined variational problem, since the form of the boundary term compensates the boundary term present in the variation of the bulk action (Eq 2.28). In that case, the boundary term turned up due to partial integration in the variations of the bulk gauge fields. With the addition of the boundary term in Eq 2.47, varying both $\mathbf{A}$ and $\mathbf{B}$ leads to:

$$\delta I \propto -\int_{\mathcal{M}} (\text{bulk e.o.m.}) - \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{B}\delta\mathbf{A}|_\partial - \mathbf{A}\delta\mathbf{B}|_\partial). \tag{2.49}$$

Imposing the boundary condition Eq 2.48 on the physical boundary fields indeed leads to a vanishing boundary term. In this sense, the boundary term can also be understood as the natural GHY boundary term in the total BF action on non-compact surfaces ($\partial\mathcal{M} \neq 0$) that changes the boundary condition from Dirichlet to Eq 2.48. We might rescale $\mathbf{B} \to 4\pi G\mathbf{B}$ to obtain a more natural BF action without loss of generality;

$$\boxed{I_{BF} \simeq -\int_{\mathcal{M}} \mathrm{Tr}(\mathbf{BF}) + \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{BA})}. \tag{2.50}$$

The rescaled boundary condition Eq 2.48 then becomes:

$$\boxed{\mathbf{B}|_\partial = \frac{\gamma}{4\pi G}\mathbf{A}|_\partial = 2C\mathbf{A}|_\partial \equiv \mathbf{A}|_\partial} \tag{2.51}$$

, where $C = \frac{a}{16\pi G_N}$ is defined as in chapter 1 (c.f. Eq 1.93). In the last identity, I have defined $C \equiv 1/2$, using conventional notation of e.g. [21]. This is a choice which will simplify a lot of the final expressions. In this section, I leave it explicit though, to establish full equivalence with earlier notations of the Schwarzian boundary action. On shell, the variation with respect to $\mathbf{B}$ forces $\mathbf{F} = 0$, and the bulk term vanishes. Inserting the boundary conditions above sets

$$I_{JT} \simeq C\int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{A}|_\partial^2). \tag{2.52}$$

We now aim to reproduce the Schwarzian boundary action, which is most convenient in Rindler coordinates Eq 1.61. Properly rescaling the periodic time coordinate, and switching to Euclidean signature $\theta = it$, the thermal patch is given by

$$ds^2 = dr^2 + \sinh^2 r\, d\theta^2. \tag{2.53}$$

Here, we interpret $\theta(\tau)$ as the temporal black hole coordinate labeling the reparametrization modes in terms of the proper time τ. The spatial limit $\epsilon \to 0$ is equivalent to r going to infinity. In this limit, we approximate $\sinh^2 r \to \frac{1}{4}e^{2r} + \dots$. Denoting the first order correction in τ as $S(\tau)$, we approximate the metric along the boundary as:

$$ds^2 = dr^2 + \left(\frac{1}{4}e^{2r} - S(\tau) + \dots\right) d\tau^2. \tag{2.54}$$

To first order in large $r \to \infty$, the Ricci scalar is indeed $R = -2$. Calculating the extrinsic trace of a constant r hypersurface yields

$$K = \frac{e^{2r}}{e^{2r} - 4S(\tau)} \to 1 + 4e^{-2r}S(\tau) + \dots \tag{2.55}$$

$S(\tau)$ can therefore be interpreted as a function that captures the leading order correction away from AdS_2 in the GHY boundary term of Eq 1.48. However, we have not yet shown that it actually coincides with the Schwarzian derivative until we know its transformation behaviour under arbitrary reparametrizations.

Now, we would like to translate this dynamics to the BF theory. This requires an additional constraint on the asymptotic gauge fields $\mathbf{A}|_\partial$ in order to recover the asymptotic behavior of the metric Eq 2.54 in its second-order form. First of all, the frame fields of the diagonal asymptotic metric Eq 2.54 can be readily read off:

$$e^0 = dr, \qquad e^1 = \left(\frac{1}{2}e^r - S(\tau)e^{-r} + \mathcal{O}(e^{-2r})\right) d\tau. \tag{2.56}$$

The corresponding spin connections are found from the no-torsion constraint $de^a + \omega^a{}_b \wedge e^b = 0$, which in local coordinates reads $\partial_\mu e^a_\nu \, dx^\mu \wedge dx^\nu + \omega^a_\mu{}_b e^b_\nu \, dx^\mu \wedge dx^\nu = 0$. We check that

$$\omega = -\left(\frac{1}{2}e^r + S(\tau)e^{-r}\right) d\tau \tag{2.57}$$

satisfies these equations. $a = 0$ is trivially satisfied for $\epsilon^0{}_1 = 1$ with $e^1_r = 0$ and $\omega_r = 0$. For $a = 1$, we verify ($\epsilon^1{}_0 = -1$)

$$\partial_r \left(\frac{1}{2}e^r - S(\tau)e^{-r}\right) dr \wedge d\tau + \left(\frac{1}{2}e^r + S(\tau)e^{-r}\right) d\tau \wedge dr = 0.$$

These one-forms form an $\mathfrak{sl}(2,\mathbb{R})$ triplet $\mathbf{A} = A^I P_I$, where P_I ($I = 0, 1, 2$) are the generators satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra Eq 2.20 and $A^I = (e^0, e^1, \omega)$ as before. Using the fundamental representation matrices Eq 2.23, the asymptotic behaviour of the gauge connection is given by

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} -e^0 & e^1 - \omega \\ e^1 + \omega & e^0 \end{pmatrix} = \frac{dr}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{d\tau}{2}\begin{pmatrix} 0 & e^r \\ -2S(\tau)e^{-r} & 0 \end{pmatrix}. \tag{2.58}$$

The last term defines the leading order boundary action in the asymptotic region $r \to \infty$. Fixing the asymptotic form of the gauge fields will lead to the *coset boundary conditions* later. Plugging in this explicit form in the BF boundary action Eq 2.52, we obtain:

$$I = C \int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{A}^2) = -C \int d\tau S(\tau). \tag{2.59}$$

Therefore, identifying the leading asymptotic $S(\tau)$ correction with the Schwarzian derivative immediately yields the identification of the BF boundary term with the Schwarzian theory. This identification follows from studying its behaviour under reparameterizations.
The general diffeomporhisms of the metric tensor are equivalent to gauge transformations on the frame fields of the $\mathfrak{sl}(2,\mathbb{R})$-triplet $\mathbf{A}$. Parameterized by an infinitesimal parameter $\Theta(\tau)$, these transform in the fibre bundle under local transformations as:

$$\mathbf{A} \to \mathbf{A} + d\Theta + [\mathbf{A}, \Theta]. \tag{2.60}$$

The exterior derivative acts only on the proper boundary τ-coordinate. Since the gauge transformations reach the boundary, they corresponds to real physical symmetries of the action. The constrained gauge transformations should still obey the asymptotic boundary condition on $\mathbf{A}$ (Eq 2.58) for large $r \to \infty$. It can readily be checked using e.g. Mathematica, that the most general gauge transformations which leave the coset boundary condition invariant, are parameterized by [30]:

$$\Theta(\tau, r) = \begin{pmatrix} \frac{1}{2}\epsilon'(\tau) & \frac{1}{2}e^r\epsilon(\tau) \\ -e^{-r}(S(\tau)\epsilon(\tau) + \epsilon''(\tau)) & -\frac{1}{2}\epsilon'(\tau) \end{pmatrix} \tag{2.61}$$

, in terms of an infinitesimal degree of freedom $\epsilon(\tau)$. This transformation preserves the asymptotic form Eq 2.54, but induces a transformation of the first order correction:

$$S(\tau) \to S(\tau) + \epsilon'''(\tau) + \epsilon(\tau)S'(\tau) + 2\epsilon'(\tau)S(\tau). \tag{2.62}$$

This is the same transformation law of the Schwarzian derivative $\left\{\tan\frac{f(\tau)}{2},\tau\right\}$ under $f(\tau)\to f(\tau+\epsilon(\tau))$, thereby confirming the identification:

$$S(\tau)\equiv\left\{\tan\frac{f(\tau)}{2},\tau\right\}. \tag{2.63}$$

2.3.2 Deriving the symplectic form in the BF perspective

It is interesting that the construction above allows us to derive the symplectic form of the coadjoint orbits of the Virasoro algebra Eq C.15 directly from the bulk BF perspective [30].

Path integrating out $\mathbf{B}$ constrains the integration space to flat $\mathbf{F}=0$ connections only, which admit the natural antisymmetric symplectic form [30] [85]:

$$\boxed{\omega(\delta_1\mathbf{A},\delta_2\mathbf{A})=2\int_{\mathcal{M}}\mathrm{Tr}(\delta_1\mathbf{A}\wedge\delta_2\mathbf{A})} \tag{2.64}$$

, where $\delta_1\mathbf{A},\delta_2\mathbf{A}$ are one-forms on the tangent space of gauge fields that parameterize infinitesimal variations in $\mathbf{A}$. The tangent space is defined by infinitesimal gauge fields $\delta\mathbf{A}$, such that $\mathbf{A}+\delta\mathbf{A}$ is still flat to linear order;

$$d(\delta\mathbf{A})+\mathbf{A}\wedge\delta\mathbf{A}+\delta\mathbf{A}\wedge\mathbf{A}=0. \tag{2.65}$$

Up to possible boundary terms, the symplectic measure is naturally invariant under infinitesimal gauge transformations labeled by Θ. These act on the space of one-forms as $\delta_2\mathbf{A}\to\delta_2\mathbf{A}+d\Theta+[\mathbf{A},\Theta]$. Up to possible boundary terms, we have indeed;

$$\begin{aligned}2\int\mathrm{Tr}(\delta_1\mathbf{A}\wedge(d\Theta+[\mathbf{A},\Theta]))&=2\int\mathrm{Tr}(\delta_1(d\mathbf{A})\Theta+\delta_1\mathbf{A}\wedge\mathbf{A}\Theta-\delta_1\mathbf{A}\wedge\Theta\mathbf{A})\\&=2\int\mathrm{Tr}(\delta_1(d\mathbf{A})\Theta+\delta_1\mathbf{A}\wedge\mathbf{A}\Theta+\mathbf{A}\wedge\delta_1\mathbf{A}\Theta)=0.\end{aligned}$$

This vanishes from the definition of the tangent space of flat gauge connections. In the last term, I have used cyclicity of the trace: $\mathrm{Tr}(\delta_1\mathbf{A}\wedge\Theta\mathbf{A})=\mathrm{Tr}(\delta_1\mathbf{A}_\mu\Theta\mathbf{A}_\nu)\;dx^\mu\wedge dx^\nu=\mathrm{Tr}(\mathbf{A}_\nu\delta_1\mathbf{A}_\mu\Theta)\;dx^\mu\wedge dx^\nu=-\mathrm{Tr}(\mathbf{A}_\nu\delta_1\mathbf{A}_\mu\Theta)\;dx^\nu\wedge dx^\mu=-\mathrm{Tr}(\mathbf{A}\wedge\delta_1\mathbf{A}\Theta)$.
It is also closed on the tangent space of flat gauge connections, where using the general Leibnitz rule yields:

$$d\omega=2\int\mathrm{Tr}(\delta_1 d\mathbf{A}\wedge\delta_2\mathbf{A}-\delta_1\mathbf{A}\wedge\delta_2 d\mathbf{A})=2\int\mathrm{Tr}(\delta_1 d\mathbf{A}\wedge\delta_2\mathbf{A}-\delta_1 d\mathbf{A}\wedge\delta_2\mathbf{A})=0.$$

In the second line, I have performed a partial integration, again using the generalized Leibnitz rule on forms.

For flat gauge connections, we can start from $\mathbf{A}=0$, and take the most general tangent element to be a pure gauge transformation that has the limiting form Eq 2.61. The tangent space therefore consists of pure gauge fields, labeled by

$\delta_i \mathbf{A} = d\Theta_i + [\mathbf{A}, \Theta_i]$. We can work out the symplectic form in this case:

$$
\begin{aligned}
\int_{\mathcal{M}} \mathrm{Tr}(\delta_1 \mathbf{A} \wedge \delta_2 \mathbf{A}) &= \int_{\mathcal{M}} \mathrm{Tr}((d\Theta_1 + [\mathbf{A}, \Theta_1]) \wedge (d\Theta_2 + [\mathbf{A}, \Theta_2])) \\
&= \int_{\mathcal{M}} \mathrm{Tr}(d\Theta_1 \wedge (d\Theta_2 + [\mathbf{A}, \Theta_2]) + [\mathbf{A}, \Theta_1] \wedge (d\Theta_2 + [\mathbf{A}, \Theta_2])) \\
&= \int_{\partial\mathcal{M}} \mathrm{Tr}(\Theta_1 \wedge (d\Theta_2 + [\mathbf{A}, \Theta_2])) \\
&\quad + \int_{\mathcal{M}} \mathrm{Tr}(-\Theta_1 d[\mathbf{A}, \Theta_2] + [\mathbf{A}, \Theta_1] \wedge d\Theta_2 + [\mathbf{A}, \Theta_1] \wedge [\mathbf{A}, \Theta_2]).
\end{aligned}
$$

Working out the last term explicitly:

$$
\begin{aligned}
\mathrm{Tr}([\mathbf{A}, \Theta_1] \wedge [\mathbf{A}, \Theta_2]) &= \mathrm{Tr}(\cancel{\mathbf{A}\Theta_1 \wedge \mathbf{A}\Theta_2} - \Theta_1 \mathbf{A} \wedge \mathbf{A}\Theta_2 - \mathbf{A}\Theta_1 \wedge \Theta_2 \mathbf{A} + \cancel{\Theta_1 \mathbf{A} \wedge \Theta_2 \mathbf{A}}) \\
&= -\mathrm{Tr}(\Theta_1 \mathbf{A} \wedge \mathbf{A}\Theta_2 + \mathbf{A}\Theta_1 \wedge \Theta_2 \mathbf{A})
\end{aligned} \tag{2.66}
$$

, where $\mathrm{Tr}(\mathbf{A}\Theta_1 \wedge \mathbf{A}\Theta_2 + \Theta_1 \mathbf{A} \wedge \Theta_2 \mathbf{A}) = 0$, since the last term can be rewritten as $\mathrm{Tr}(\Theta_1 \mathbf{A} \wedge \Theta_2 \mathbf{A}) = \mathrm{Tr}(\Theta_1 \mathbf{A}_\mu \Theta_2 \mathbf{A}_\nu) dx^\mu \wedge dx^\nu = \mathrm{Tr}(\mathbf{A}_\nu \Theta_1 \mathbf{A}_\mu \Theta_2) dx^\mu \wedge dx^\nu = -\mathrm{Tr}(\mathbf{A}\Theta_1 \wedge \mathbf{A}\Theta_2)$.
Furthermore $d[\mathbf{A}, \Theta_2] = [d\mathbf{A}, \Theta_2] - [\mathbf{A}\wedge, d\Theta_2]$, with $-\mathrm{Tr}(\Theta_1 [d\mathbf{A}, \Theta_2]) = -\mathrm{Tr}(\Theta_1 d\mathbf{A}\Theta_2 - \Theta_1 \Theta_2 d\mathbf{A}) = -\mathrm{Tr}(\Theta_2 \Theta_1 d\mathbf{A} - \Theta_1 \Theta_2 d\mathbf{A})$. Using $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0$,

$$
-\mathrm{Tr}(\Theta_1 [d\mathbf{A}, \Theta_2]) = \mathrm{Tr}(\Theta_2 \Theta_1 \mathbf{A} \wedge \mathbf{A} - \Theta_1 \Theta_2 \mathbf{A} \wedge \mathbf{A}).
$$

This term cancels with Eq 2.66

$$
-\mathrm{Tr}(\Theta_1 [d\mathbf{A}, \Theta_2]) + \mathrm{Tr}([\mathbf{A}, \Theta_1] \wedge [\mathbf{A}, \Theta_2]) = \mathrm{Tr}(\cancel{\Theta_2 \Theta_1 \mathbf{A} \wedge \mathbf{A}} - \cancel{\Theta_1 \Theta_2 \mathbf{A} \wedge \mathbf{A}}) - \mathrm{Tr}(\cancel{\Theta_1 \mathbf{A} \wedge \mathbf{A}\Theta_2} + \cancel{\mathbf{A}\Theta_1 \wedge \Theta_2 \mathbf{A}}).
$$

Using the same techniques, we work out

$$
\begin{aligned}
\mathrm{Tr}(\Theta_1 [\mathbf{A}\wedge, d\Theta_2]) &= \mathrm{Tr}(\cancel{\Theta_1 \mathbf{A} \wedge d\Theta_2} + \cancel{\Theta_1 d\Theta_2 \wedge \mathbf{A}}) \\
\mathrm{Tr}([\mathbf{A}, \Theta_1] \wedge d\Theta_2) &= \mathrm{Tr}(\cancel{\mathbf{A}\Theta_1 \wedge d\Theta_2} - \cancel{\Theta_1 \mathbf{A} \wedge d\Theta_2}).
\end{aligned} \tag{2.67}
$$

The indicated term vanishes upon adding both terms in the total symplectic form. Therefore, we see that all bulk terms mutually cancel, and only the boundary term due to partial integration survives,

$$
\omega(\delta_1 \mathbf{A}, \delta_2 \mathbf{A}) = 2 \int_{\mathcal{M}} \mathrm{Tr}(\delta_1 \mathbf{A} \wedge \delta_2 \mathbf{A}) = 2 \int_{\partial\mathcal{M}} \mathrm{Tr}(\Theta_1 (d\Theta_2 + [\mathbf{A}, \Theta_2])). \tag{2.68}
$$

Inserting the asymptotic coset behaviour of $\mathbf{A}_\tau$ Eq 2.58 and the restricted gauge parameter Eq 2.61, it is readily checked with Mathematica that the symplectic measure boils down to:

$$
\begin{aligned}
\omega(\delta_1 \mathbf{A}, \delta_2 \mathbf{A}) &= -\int_0^\beta d\tau \left[2\epsilon_1(\tau)\epsilon_2'(\tau) S(\tau) + \epsilon_1(\tau)\epsilon_2(\tau) S'(\tau) + \epsilon_1(\tau)\epsilon_2'''(\tau)\right] \\
&= \int_0^\beta d\tau \left[\epsilon_1'(\tau)\epsilon_2''(\tau) - S(\tau)\left(\epsilon_1(\tau)\epsilon_2'(\tau) - \epsilon_1'(\tau)\epsilon_2(\tau)\right)\right]
\end{aligned} \tag{2.69}
$$

, assuming that all functions are periodic on $[0, \beta]$. Introducing an abstract exterior derivative that works only the $\epsilon(\tau)$ fields and that formally commutes with ∂_τ, this is written more suggestively as:

$$\omega(\delta_1\mathbf{A},\delta_2\mathbf{A}) = \frac{1}{2}\int_0^\beta d\tau\left(d\epsilon'(\tau)\wedge d\epsilon''(\tau) - 2S(\tau)d\epsilon(\tau)\wedge d\epsilon'(\tau)\right). \tag{2.70}$$

This is the same symplectic measure that was derived in Eq C.15 in the context of the coadjoint orbits of the Virasoro algebra.

2.4 Exact solutions of 2d Yang-Mills theory

It is well known that generic $SU(N)$ Yang-Mills (YM) theories might be dual to string theories in the limit of large N. In fact, the very genesis of string theory was to formulate a theory of the strong interaction. Unfortunately, it is in general difficult to write down the correct string analogue of YM in anything other than a phenomenological approach. The motivations on how to proceed are that the weak coupling, large N diagrammatic expansion of 't Hooft planar diagrams connects with large N matrix theories of string theory. Furthermore, the natural variables in YM are Wilson loops of holonomy variables. These Wilson loops create rings of glue that make up the Hilbert space of string states. It is in this discussion that 2d Yang-Mills materialized, where the loop variables can be defined precisely and exact expressions of the amplitudes exist.
In the context of JT gravity, we will take inspiration of 2d YM to calculate the exact quantum amplitudes, and identify JT gravity as a constrained non-compact generalization of the well known story for 2d YM.

2.4.1 YM$_2$ action and sDiff invariance

For now, let us consider a compact simple Lie group G (these may be the classical Lie groups $(S)U(N)$, $(S)O(N)$ or $USp(2N)$), and its associated Lie algebra $\mathfrak{g}$. For simple Lie algebras, the generators can be chosen to be trace orthogonal with respect to the Cartan-Killing metric $\kappa(T_a,T_b)/2 = \kappa_{ab}/2 = \text{Tr}(T_aT_b) = N\delta_{ab}$. N denotes an a priori fixed normalization constant. The classical Yang-Mills action on a two-dimensional manifold $\mathcal{M}$ equipped with metric $g_{\mu\nu}$ reads, in local coordinates:

$$I[A] = -\frac{1}{4e^2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\text{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{2.71}$$

For the compact groups, the invariant quadratic form Tr can simply be taken as the trace in the fundamental representation of G. F is the curvature two-form associated with the gauge field A. These fields reside in the algebra. Introducing a Hodge star operation acting on the frame fields as $(p+q=D)$:

$$*e^{a_1}\wedge\ldots e^{a_p} \equiv \frac{1}{q!}e^{b_1}\wedge\cdots\wedge e^{b_q}\epsilon_{b_1\ldots b_q}{}^{a_1\ldots a_p} \tag{2.72}$$

induces a duality transformation between a p-form $\omega^{(p)} = \frac{1}{p!}\omega_{a_1\dots a_p}e^{a_1}\wedge\dots\wedge e^{a_p}$ and a q-form $\Omega^{(q)} = \frac{1}{q!}\Omega_{b_1\dots b_q}e^{b_1}\wedge\dots e^{b_q}$ as $\Omega^{(q)} = {}^*\omega^{(p)}$, where the frame field components transform as:

$$\Omega_{b_1\dots b_q} = ({}^*\omega)_{b_1\dots b_q} = \frac{1}{p!}\epsilon_{b_1\dots b_q}{}^{a_1\dots a_p}\omega_{a_1\dots a_p}. \tag{2.73}$$

For Euclidean manifolds, this operation satisfies the important involutive property[2] ${}^*({}^*\omega^{(p)}) = (-)^{pq}\omega^{(p)}$. This means that for $D = 2$, the Hodge star transforms a 2-form into a 0-form, whose action is involutive ($** = \mathbf{1}$). The definition implies that the star operation on $\mathbf{1}$ is:

$$*\mathbf{1} = \frac{1}{D!}\epsilon_{a_1\dots a_D}e^{a_1}\wedge\dots\wedge e^{a_D} = e^0\wedge\dots\wedge e^D = \sqrt{g}d^Dx. \tag{2.74}$$

Using the identities Eq 2.3, one can readily translate the star duality transformation to the coordinate basis ${}^*(dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}) = \frac{1}{q!}eg^{\mu_1\rho_1}\dots g^{\mu_p\rho_p}dx^{\nu_1}\wedge\dots\wedge dx^{\nu_q}\epsilon_{\nu_1\dots\nu_q\rho_1\dots\rho_p}$, acting on the components as:

$$({}^*\omega)_{\mu_1\dots\mu_q} = \frac{1}{p!}e\epsilon_{\mu_1\dots\mu_q\rho_1\dots\rho_p}g^{\nu_1\rho_1}\dots g^{\nu_p\rho_p}\omega_{\nu_1\dots\nu_p}. \tag{2.75}$$

Furthermore, using $e^{b_1}\wedge\dots\wedge e^{b_q}\wedge e^{c_1}\wedge\dots\wedge e^{c_p} = \epsilon^{b_1\dots b_q c_1\dots c_p}\sqrt{g}d^Dx$, we have (see footnote 2):

$$\int {}^*\omega^{(p)}\wedge\omega^{(p)} = \int\frac{d^Dx\sqrt{g}}{(p!)^2q!}\epsilon_{b_1\dots b_q}{}^{a_1\dots a_p}\epsilon^{b_1\dots b_q}{}_{c_1\dots c_p}\omega_{a_1\dots a_p}\omega^{c_1\dots c_p} = \frac{1}{p!}\int d^Dx\sqrt{g}\,\omega_{\mu_1\dots\mu_p}\omega^{\mu_1\dots\mu_p}. \tag{2.76}$$

Therefore, the 2d Yang-Mills action (YM$_2$) may be written more conveniently as:

$$\boxed{I[A] = -\frac{1}{2e^2}\int_{\mathcal{M}}\mathrm{Tr}(*F\wedge F)}. \tag{2.77}$$

Using $*1 = \sqrt{g}d^2x$, we write the top-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$ in $D = 2$ in terms of a scalar zero-form f

$$F_{\mu\nu} \equiv ({}^*f)_{\mu\nu} = \sqrt{g}\epsilon_{\mu\nu}f \tag{2.78}$$

, allowing us to write $F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$ as $F = \mu f$, where $\mu \equiv \sqrt{g}d^2x$ is the canonical volume form. Using the involutive property of the Hodge star on two-forms, the latter can be inverted to:

$$F = \mu f \quad \leftrightarrow \quad f = {}^*F. \tag{2.79}$$

The action Eq 2.77 may therefore be rewritten as [45]:

$$I[A] = -\frac{1}{2e^2}\int_{\mathcal{M}}\mu\,\mathrm{Tr}(f^2). \tag{2.80}$$

We see that the action only depends on the metric through the volume element μ. The gauge symmetry is augmented to the group of area-preserving diffeomorphisms sDiff($\mathcal{M}$). This makes it almost generally

[2] Using the generalized Levi-Civita identity in Euclidean signature $\epsilon_{a_1\dots a_p b_1\dots b_q}\epsilon^{a_1\dots a_p c_1\dots c_q} = p!q!\delta^{c_1\dots c_q}_{b_1\dots b_q}$, with $\delta^{c_1\dots c_q}_{b_1\dots b_q}$ the antisymmetric q-index Kronecker Kronecker delta, defined as $\delta^{c_1\dots c_q}_{b_1\dots b_q} \equiv \delta^{[c_1}_{b_1}\dots\delta^{c_q]}_{b_q}$, where antisymmetrization with weight one is understood.

covariant, and makes the theory almost topological. This will be exact in the limit of vanishing coupling constant e. To see this, we write YM_2 in terms of a topological BF gauge theory, in the presence of a defect:

$$I[A,B] = -\int_{\mathcal{M}} \mathrm{Tr}(BF) + \frac{e^2}{2}\int_{\mathcal{M}} \mu \mathrm{Tr}(B^2) \tag{2.81}$$

, where B is a field in algebra $\mathfrak{g}$. Using the on-shell equations of motion of the background field B ($F = e^2\mu B$) in the action, yields again the YM_2 action,

$$I[A,B] = -\frac{1}{2}\int \mathrm{Tr}\left(\frac{F^2}{e^2\mu}\right) = -\frac{1}{2e^2}\int \mu \mathrm{Tr}(f^2).$$

This relation is exact also off-shell since the action is of Gaussian type. Since the BF action is purely topological, we see that this is broken by an amount proportional to e^2. From the action Eq 2.81, we see that the coupling constant e^2 and the area $A = \int \mu$ appear together, such that a vanishing area ($A \to 0$) limit is functionally equivalent to a weak coupling limit. In this limit, the action becomes fully topological. Hence, the small area limit must reproduce the results of a topological field theory.

2.4.2 Hilbert space on S^1

We first consider the Hilbert space on a spatial circle S^1, acting as one of the boundaries of the cylinder. The space is equipped with the flat space metric $ds^2 = dt^2 + dx^2$, where x is periodic in L. Since there is only one spatial direction, the field strength has no magnetic components. Further fixing to the temporal gauge $A_t \equiv 0$, the only non-vanishing component of the field strength is associated to $F_{tx} \equiv E$, where $E \equiv \dot{A}_x$, and the dot represents the derivative with respect to the temporal coordinate. The Gauss-law constraint is obtained by varying the action 2.71 with respect to A_0, yielding $D_1F_{10} = 0$, with D the covariant derivative in the adjoint representation of G. In the flat space metric, the YM_2 Lagrangian in temporal gauge is simply

$$\mathscr{L} = -\frac{1}{2e^2}\mathrm{Tr}(E^2) = -\frac{N}{2e^2}(E^aE^b)\delta_{ab}, \qquad a \in (1,\ldots,\dim(\mathfrak{g})) \tag{2.82}$$

, with the Gauss law constraint $D_1F_{10} = 0$ imposed a posteriori. The conjugate momenta associated to the gauge fields A^a for the Lagrangian above are simply

$$\Pi_a = \frac{\partial \mathscr{L}}{\partial \dot{A}^a} = -\frac{N}{e^2}E^a = -\frac{N}{e^2}\dot{A}^a. \tag{2.83}$$

In the canonical quantization of YM_2 in the temporal gauge on the circle, one promotes these fields to operators ($A^a(x) \to \hat{A}^a(x)$, $\Pi_a(x) \to \hat{\Pi}_a$), satisfying the canonical commutation relations $[\hat{A}^a, \hat{\Pi}_b(y)] = \delta^a_b\delta(x-y)$. The commutation relations are realized as $\hat{\Pi}_a(x) = -\frac{\delta}{\delta A^a(x)}$. The eigenstates are $|A(x)\rangle$, whose overlap with arbitrary $|\Psi\rangle$ determine the wave functionals $\Psi[A] = \langle A(x) \mid \Psi\rangle$. The Gauss law constraint is imposed on the physical states by demanding $\hat{D_x E}\,|\mathrm{physical}\rangle = 0$. The Hamiltonian $\mathscr{H} \equiv \Pi_a\dot{A}^a - \mathscr{L}$ associated to Eq 2.82 is:

$$\mathscr{H} = -\frac{N}{2e^2}\hat{E}^a\hat{E}^a = -\frac{e^2}{2N}\frac{\delta}{\delta A^a(x)}\frac{\delta}{\delta A^a(x)}. \tag{2.84}$$

The Gauss law constraint becomes an operator identity acting on the wave functionals $\Psi[A(x)]$:

$$\left(\hat{D}_x \frac{\delta}{\delta A(x)}\right)_a \Psi[A(x)] = 0. \tag{2.85}$$

This constraint may be solved by realizing that the operator $(D_x E)_a$ generates gauge transformations under $\mathbf{A} \to \mathbf{A} + D_x\Theta$ [86], where the associated Noether charge is:

$$Q \equiv \int_0^L dx \left(\frac{\partial \mathscr{L}}{\partial \dot{A}_a}\delta A_a\right) = -\frac{N}{e^2}\int_0^L dx \,\mathrm{Tr}(E(D_x\Theta)) \simeq \frac{N}{e^2}\int_0^L dx \,\mathrm{Tr}((D_x E)\Theta).$$

Note that we could perform a partial integration in the second line since the covariant derivative is taken in the adjoint representation. Therefore, after quantization, the Gauss law constraint on physical wave functionals demands that they are gauge invariant. Demanding invariance under x-independent gauge transformations shows that they can only depend on the conjugacy class of $U = \mathcal{P}\exp(-\oint A) = \mathcal{P}\exp(-\int_0^L dx A_x)$, where $\mathcal{P}$ denotes the path-ordening operation;

$$\Psi[A(x)] \equiv \Psi\left[\mathcal{P}\exp\left(-\int_0^L dx A_x\right)\right]. \tag{2.86}$$

The physical Hilbert space of states therefore consists of square-integrable L^2 class functions of G [45], denoted as $\mathcal{H} = L^2(\mathcal{A}/G)$. These satisfy the property $\Psi(U) = \Psi(g(x)Ug^{-1}(x))$, for every $g \in G$ and some boundary point x on the circle. E.g., these are functions defined up to conjugacy class equivalence. Since this is by definition a finite gauge transformation on the gauge group G, the class functions Ψ are indeed gauge invariant.

U describes the holonomy around some closed loop embedded in $\mathcal{M}$. The Hilbert space of class functions around the holonomy U of a compact group is spanned by an orthonormal basis of characters $\{\chi_R(U)\}$ of U around S^1, evaluated as the trace in the unitary irreducible representations R of G.
We can associate them with wavefunctions corresponding to the overlap of the representation basis $\{|R\rangle\}$ with the holonomy elements $\{|U\rangle\}$:

$$\langle U \mid R\rangle = \chi_R(U) \equiv \mathrm{Tr}(R(U)). \tag{2.87}$$

The Haar measure on the space of conjugacy class group elements of a compact group G constitutes a natural positive definite inner product on the space of square integrable class functions $\mathcal{H}$

$$\int dU\, f(U^{-1})h(U) = \langle f \mid h\rangle. \tag{2.88}$$

This also defines a completeness and normalization relation of the holonomy elements:

$$\int dU\, |U\rangle\langle U| = \mathbf{1}, \qquad \langle U \mid U'\rangle = \delta(U - U') \tag{2.89}$$

, where the square integrable class functions determine the overlap with the state vectors $|f\rangle \in \mathcal{H}$:

$$\langle U \mid f\rangle = f(U), \qquad \langle f \mid U\rangle = f(U^{-1}). \tag{2.90}$$

The wavefunctions $\{\chi_R(U)\}$ indeed form a complete basis on $\mathcal{H}$ by virtue of the grand-orthogonality theorem and the Peter-Weyl theorem [23]:

$$\boxed{\int dU\, \chi_{R'}(U^{-1})\chi_R(U) = \delta_{R'R}, \qquad \sum_R \chi_R(U)\chi_R(U'^{-1}) = \delta(U-U')}. \tag{2.91}$$

$\delta(U-U')$ denotes the delta function imposing $U \equiv U'$ on the Lie group with respect to the Haar measure dU, normalized to 1. These relations respectively constitute an orthogonality, and completeness relation on the representation basis, in the sense that:

$$\langle R' \mid R\rangle = \delta_{R'R}, \qquad \sum_R |R\rangle\langle R| = \mathbf{1}. \tag{2.92}$$

Note that the Hilbert space requirement $\langle U \mid R\rangle = \langle R \mid U\rangle^*$ restrict the traces to the subset of unitary representations: $\chi_R(U)^* = \chi_R(U^{-1})$.

The Hamiltonian $H = \int_0^L dx \mathscr{H}$ associated to the YM_2 Lagrangian (in temporal gauge) Eq 2.84, is diagonalized by acting on the representation basis, since by definition of the holonomy $U = \mathcal{P}\exp(-\oint A)$ around the circle

$$\frac{\delta}{\delta A^a}\chi_R(U) = -\chi_R(T_a U), \quad \rightarrow \quad H\chi_R(U) = e^2 L \mathcal{C}_2(R)\chi_R(U)$$

, where $\mathcal{C}_2(R) = -T_a T_a/(2N)$ is our definition of the quadratic Casimir defined in general as:

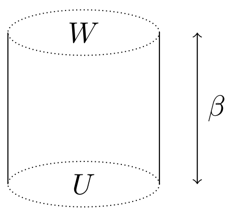
$$\boxed{\mathcal{C}_2 = -\kappa^{ab}T_a T_b}. \tag{2.93}$$

κ^{ab} is the inverse of the Cartan-Killing metric defined earlier as $\text{Tr}(T_a T_b) = \kappa_{ab}/2$. In the case at hand, the generators are normalized according to $\text{Tr}(T_a T_b) = N\delta_{ab}$, leading to the Cartan-Killing metric $\kappa_{ab} = 2N\delta_{ab}$, and its inverse $\kappa^{ab} = \frac{\delta^{ab}}{2N}$. I will continue to use this convention of the quadratic Casimir throughout this thesis.

2.4.3 Exact amplitudes of YM_2

We first consider the YM_2 Euclidean path integral on an extended cylinder that computes the propagation amplitude from boundary holonomy U to W. In the Hamiltonian formulation, we can interpret the path integral as the evolution operator between these two states

$$\langle W| e^{-\beta H} |U\rangle = \int_U^W \mathcal{D}B\mathcal{D}A\, e^{-I_{\text{YM}_2}[B,A]} \quad = \tag{2.94}$$



We may diagonalize the Hamiltonian on slices of S_1 in the representation basis using the completeness relation Eq 2.92 and the definition of the character wavefunctions:

$$\boxed{Z_{\text{cyl}}(\beta, W, U) = \langle W| e^{-\beta H} |U\rangle = \sum_R \chi_R(W)\chi_R(U^{-1})e^{-e^2 L\beta C_2(R)}}. \tag{2.95}$$

The exponent is the heat kernel of the group G [85], weighted by the combination of the coupling strength and the total propagated area[3]: $e^2 a = e^2\beta L$. Although one has to choose a preferred temporal coordinate in the canonical quantization to define the Hamiltonian, the partition function only depends on the surface area as a whole. This is what we expect from the sDiff invariance. Due to this property, we can take the general ansatz for the heat kernel to be

$$e^{-e^2 a C_2(R)} \tag{2.96}$$

, independent of the topology of $\mathcal{M}$. Note that in the limit of vanishing $e^2 a \to 0$, the heat kernel goes to the identity and the theory Eq 2.81 becomes completely topological. In this limit, the completeness property Eq 2.91 demands $U \equiv W$ as expected for a theory with only flat $F = 0$ connections that cannot change the boundary holonomy by an infinitesimal amount.

A novel property of YM_2 is the ability to glue partition functions together along a common holonomy element, as a consequence of the orthogonality of the characters [45]:

$$\begin{aligned}\int dg\ Z_{\text{cyl}}(\beta_1, U, g) Z_{\text{cyl}}(\beta_2, g, W) &= \sum_{RR'} \chi_R(U)\chi_{R'}(W^{-1})e^{-e^2 L\beta_1 C_2(R)}e^{-e^2 L\beta_2 C_2(R')}\int dg\ \chi_R(g^{-1})\chi_{R'}(g)\\ &= Z_{\text{cyl}}(\beta_1+\beta_2, U, W).\end{aligned}$$

The cutting and gluing technique allows to solve more general YM_2 amplitudes using a surgery approach. In this way, we can obtain the disk amplitude with area a by gluing a cap wavefunction $\Gamma(0, U)$ on one side of the cylinder,

$$Z_{\text{disk}}(a, U) = \int dg\ Z_{\text{cyl}}(a, U, g)\Gamma(0, g).$$

This cap amplitude can be considered in the limit of vanishing area. Since this limit is equivalent to the topological limit, the gauge connection is again restricted to $A \to 0$. This in turn restricts the holonomy on one side of the cylinder to $W = \exp(-\oint A) = \mathbf{1}$. Since the character of the identity element by definition yields the dimension of the representation $\chi_R(\mathbf{1}) \equiv \dim(R)$, the cap amplitude for vanishing surface area $a \to 0$ reads:

$$\Gamma(0, g) = \sum_R \dim(R)\chi_R(g) = \delta(g - \mathbf{1}). \tag{2.97}$$

The last identity is a consequence of the Peter-Weyl theorem Eq 2.91. Gluing this cap along one outer boundary

[3] We denote the area with small letter a to avoid confusion in notation with the gauge connection A.

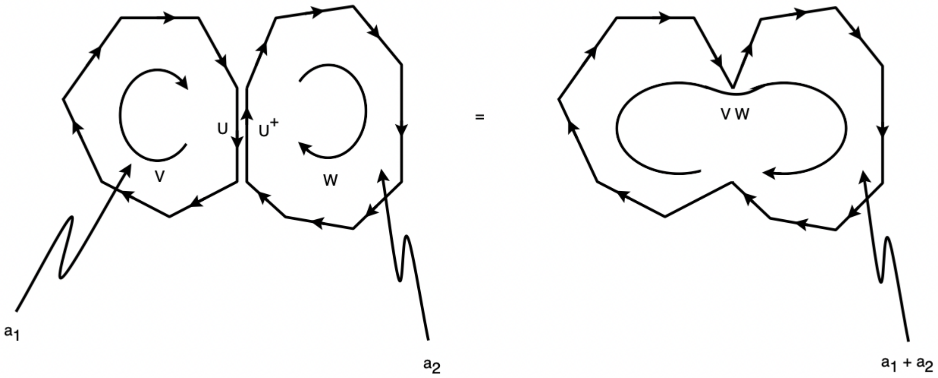


Figure 2.1: Gluing two different plaquettes together along the boundary group element U_l. Taken from [45].

holonomy of the cylinder yields the partition function at finite area a:

$$\boxed{Z_{\text{disk}}(a, U) = \sum_R \dim(R)\chi_R(U)e^{-e^2 a\mathcal{C}_2(R)}}. \tag{2.98}$$

Using the area-preserving sDiff invariance, the amplitude above is equivalent to the disk partition function by flattening the cone area to a disk shape.

Cutting and gluing axioms

We may now consider the disk amplitude as the fundamental plaquette amplitude $Z_{\text{disk}}(a, U) \equiv \Gamma(a, U)$ that constitutes more general higher-genus manifolds $\mathcal{M}$. These can be constructed by gluing together the plaquettes along the boundary edges. We consider the plaquette amplitude with edges (denoted by L) and vertices (denoted by V) [86]. We can decompose the total boundary holonomy U into different segments $U_l \in G$ along $l \in L$. These are constrained by group multiplication to yield the total holonomy $U = \prod_l U_l$ along the plaquette. We may use the segment U_l to parallel transport the vertex $x \in V$ to $y \in V$ along l: $y = U_l x = \mathcal{P}\exp(-\int_l A)x$. A finite gauge transformation acts on U_l as $U_l \to g(y)U_l g(x)^{-1}$. Parallel transport in the opposite direction is defined as U_l^{-1}.
Decomposing the area of the manifold $\mathcal{M}$ into a cell decomposition of plaquettes $\Gamma(a_p, U_p)$ with $a = \sum_p a_p$, the total partition function along $\mathcal{M}$ is defined by gluing along the coincident boundary holonomy segments $U_L = \{U_l\}$;

$$Z = \int \prod_{U_l \in U_L} \prod_{p \in P} \Gamma(a_p, U_p). \tag{2.99}$$

As an example in figure 2.1, we glue two plaquettes of area a_1 and a_2 along the boundary group element U.

The arrows indicate the direction of the group element U. The resulting partition function is given by:

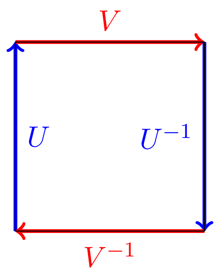
$$\begin{aligned}\int dU\Gamma(a_1, VU)\Gamma(a_2, U^{-1}W) &= \sum_{RR'} \text{dim}(R)\text{dim}(R')e^{-e^2 a_1 \mathcal{C}_2(R)} e^{-e^2 a_2 \mathcal{C}_2(R')} \int dU \chi_R(VU)\chi_{R'}(U^{-1}W) \\ &= \sum_R \text{dim}(R)\chi_R(VW)e^{-e^2 a \mathcal{C}_2(R)} = Z(a, VW)\end{aligned}$$

, with $a = a_1 + a_2$ the total area. In the last identity, I have used the generalized character orthogonality relation

$$\int dU \chi_{R'}(VU)\chi_R(U^{-1}W) = \delta_{RR'} \frac{\chi_R(VW)}{\text{dim}(R)}. \tag{2.100}$$

We may also consider more general partition functions on a topologically non-trivial manifold $\mathcal{M}$. Take e.g. $\mathcal{M} = T^2 = S^1 \times S^1$ a genus one torus. We may construct its amplitude from the plaquette amplitude by decomposing the boundary holonomy as $UVU^{-1}V^{-1}$, and gluing along V and U as demonstrated below

$$Z(a) = \int dU dV$$



$$= \int dU dV \ \Gamma(a, UVU^{-1}V^{-1}).$$

To proceed, we use the identity

$$\int dU \ \chi_R(UVU^{-1}W) = \frac{\chi_R(V)\chi_R(W)}{\text{dim}(R)} \tag{2.101}$$

, to write

$$Z(a) = \sum_R e^{-e^2 a \mathcal{C}_2(R)} \int dV \ \chi_R(V)\chi_R(V^{-1}) = \sum_R e^{-e^2 a \mathcal{C}_2(R)}$$

where we again made use of the grand orthogonality theorem Eq 2.91.

It is a general result in topology that any compact orientable genus g manifold $\mathcal{M}$ can be decomposed in a similar fashion by considering a $4g$ polygon, and writing the boundary holonomy as $U_1 W_1 U_1^{-1} W_1^{-1} U_2 \dots W_g^{-1}$ [86]. This leads to $2g$ pairs of edges that should be identified. Using Eq 2.101 to perform the first $2g-1$ integrals generates a factor $\text{dim}^{-(2g-1)}(R)$. For the last integral, we should use the grand orthogonality theorem instead. Together with the factor $\text{dim}(R)$ in the plaquette partition function 2.98 leads to the general expression of the partition function on an arbitrary orientable genus g surface

$$\boxed{Z_g(a) = \sum_R (\text{dim}(R))^\chi \ e^{-e^2 a \mathcal{C}_2(R)}} \tag{2.102}$$

, where $\chi = 2 - 2g$ is the Euler characteristic on a compact genus g surface. Note that this amplitude is also valid for a spherical ($\chi = 2$) topology, although this requires different techniques from the ones explained above. In particular, one obtains the spherical topology by simply gluing two disk partition functions of area a_1 and a_2 together, while smoothly deforming the total surface area to a spherical geometry with a total surface area $a = a_1 + a_2$.

2.5 BF- and Particle-on-a-group theory

We may exploit the machinery of YM_2 to derive the exact quantum amplitudes of a BF theory Eq 2.50 over a non-compact manifold $\mathcal{M}$ and compact gauge group G with the specific boundary condition Eq 2.51, which I repeat here for convenience;

$$I_{BF} = -\int_{\mathcal{M}} \text{Tr}(\mathbf{BF}) + \frac{1}{2}\int_{\partial\mathcal{M}} d\tau \text{Tr}(\mathbf{BA}), \qquad \mathbf{B}|_\partial = 2C\mathbf{A}|_\partial \equiv \mathbf{A}|_\partial. \tag{2.103}$$

I again set $C = 1/2$ for convenience in notation. The boundary term was added to obtain a proper variational problem, or as argued in [22], is the natural result when dimensionally reducing 3d Chern-Simons with a boundary to 2d. In [23], the boundary term was interpreted as a defect that changes the natural Dirichlet boundary condition of BF to the one needed to reproduce the Schwarzian. The defect takes the form above along the boundary loop $\partial\mathcal{M}$ parameterized by $F(\tau)$ in terms of the proper time τ. The only restriction on the loop is for it to have a total renormalized length β. This requires the reintroduction of a metric along the boundary.
Since the metric is fixed inside the bulk for flat connections, the bulk is completely topological, and all dynamics take place on the boundary. An alternative perspective why all dynamics take place at the boundary proceeds by introducing a 1d metric $\gamma_{\tau\tau}$ along the boundary curve [22], which changes the natural boundary condition to $\mathbf{B}|_\partial = \gamma_{\tau\tau}\mathbf{A}|_\partial$. The energy-momentum tensor is calculated directly by varying the action with respect to the metric using the standard dictionary [22]:

$$T_{\tau\tau} = \frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\tau\tau}} = \text{Tr}(\mathbf{B}^2) = \text{Tr}(\mathbf{A}^2) \tag{2.104}$$

, setting $\gamma_{\tau\tau} = 1$ back at the end. For $\mathbf{A}|_\partial = g\partial_\tau g^{-1}$ pure gauge, this Hamiltonian is indeed proportional to the quadratic Casimir after canonical quantization on the particle-on-a-group manifold. Therefore, the dynamical observables split into topological Wilson loops in the bulk, and quasi-topological Wilson lines anchored to the boundary. Although YM_2 is only quasi-topological (the topological defect proportional to the coupling parameter extends along the entire bulk, c.f. Eq 2.81), while pure BF is fully topological in the bulk, they share the same class of observables. This is due to the lack of propagating degrees of freedom in YM_2. In particular, the number of on-shell degrees of freedom of massless gauge bosons is $D - 2$ [59]. Therefore, the lack of transversal degrees of freedom for $D = 2$ renders the bulk of YM_2 free of local dynamics. The interesting gauge-invariant observables are therefore only defined globally, and are e.g. the Wilson loops along some closed path I in $\mathcal{M}$;

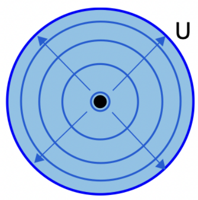
$$W_R^I(\mathbf{A}) = \chi_R\left(\mathcal{P}(\exp - \oint_I \mathbf{A})\right). \tag{2.105}$$

In a topologically trivial theory, all Wilson loops that do not intersect or encircle a defect are contractible and do not exhibit interesting dynamics. In the present context, we will mainly be interested in boundary-anchored Wilson lines instead, where both endpoints are located at the boundary. In the context of holography, these are in one-to-one relation with bilocal operators in the Schwarzian particle-on-a-group theory.
Restoring the factor $2C$, we may insert the boundary conditions back into the boundary action and obtain a theory that is structurally very similar to YM_2 in Eq 2.81,

$$I_{BF} = -\int_{\mathcal{M}} \mathrm{Tr}(\mathbf{BF}) + \frac{1}{4C}\int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{B}^2). \tag{2.106}$$

In particular, both theories share an identical Hilbert space structure on the thermal circle, determined by the Peter-Weyl theorem.
The Hilbert space consists of square-integrable class functions, spanned by a complete set of characters $\{\chi_R(U)\}$ determined by the overlap of the representation basis on the holonomy basis: $\chi_R(U) \equiv \langle U \mid R\rangle = \mathrm{Tr}(R(U))$. The only difference is the spatial extension of the topological defect from the entire bulk in the case of YM_2 to the boundary circle $\partial\mathcal{M}$ for BF theory. Nonetheless, as argued in [23], we can consider the BF theory with the boundary defect as a continuous gluing of $e^2 = 0$ radial concentric YM_2 slices inside the boundary changing defect, and a YM_2 theory of charge $e^2 = 1/2C$ along the defect loop $\partial\mathcal{M}$. This effectively yields the same general amplitudes of YM_2, where the dependence on area a is replaced with a dependence on proper length β of the boundary circle.
The general BF disk amplitude with a boundary holonomy U is readily obtained from the general result Eq 2.98 by substituting $e^2 a \to \beta/2C$ (compare Eq 2.81 and Eq 2.106), leading to:

$$Z_{\text{disk}}(\beta, U) = \sum_R \dim(R)\chi_R(U)\; e^{-\frac{\beta C_2(R)}{2C}} = \tag{2.107}$$



2.5.1 Open channel approach

To compute correlators with the insertion of a boundary-anchored Wilson line, it will be more convenient to employ an open channel slicing, developed in [22]. In particular, in the closed channel approach of the previous section, we have covered the disk amplitude using radially outgoing concentric circles. These circles eventually end on the boundary holonomy U. This slicing is rather inconvenient when quantizing the theory on an interval for Wilson lines anchored at the boundary[4]. The following section will largely reformulate the arguments of sections 2 and 3 in the aforementioned paper [22], where working out the derivations in detail required making some modifications for consistency.
In an open slicing approach, one splits the boundary holonomy into two intervals, characterized by two group states $|g\rangle$, $|h\rangle$, such that the holonomy upon encircling the boundary is given by $|U\rangle = |g \cdot h^{-1}\rangle$. Instead of

[4]See [23] for a quantization with boundary-anchored Wilson lines in a circular slicing approach. In this context, one considers the Wilson lines as Wilson loops extending to the boundary, where they set the gauge fields to zero at the boundary $\mathbf{A}|_\partial = 0$.

considering a basis of holonomy elements that are invariant under conjugation, we define a basis of genuine group elements $|g\rangle$. This serves as the configurational basis on the Hilbert space of square integrable functions $|f\rangle \in L^2$, whose overlap with the group elements $|g\rangle$ defines the wavefunction $f(g)$:

$$\langle g \mid f\rangle = f(g), \qquad \langle f \mid g\rangle = f(g^{-1}). \tag{2.108}$$

The inner product on square integrable functions defines a completeness relation of the configuration basis of group elements $|g\rangle$:

$$\langle f \mid h\rangle = \int dg f(g^{-1})h(g), \quad \rightarrow \quad \int dg\, |g\rangle\langle g| = \mathbf{1}. \tag{2.109}$$

dg is the Haar measure on the group manifold, normalized by the group volume. The natural basis conjugate to the basis of group elements are the representation matrices $|R, ab\rangle$, whose normalized overlaps are:

$$\psi^R_{ab}(g) = \langle g \mid R, ab\rangle \equiv \sqrt{\text{dim(R)}}R_{ab}(g). \tag{2.110}$$

$R_{ab}(g)\delta_{RR'} = \langle R, a \mid g \mid R', b\rangle$ is the matrix element of g evaluated in the representation R.
The representation matrices span a complete basis by virtue of the Peter-Weyl theorem, satisfying an orthogonality and completeness relation:

$$\boxed{\int dg\; R_{ab}(g)R'_{cd}(g^{-1}) = \frac{\delta_{RR'}}{\text{dim}(R)}\delta_{ad}\delta_{bc}, \qquad \sum_{R,m,n} \text{dim}(R)\; R_{mn}(g_1)R_{nm}(g_2^{-1}) = \delta(g_1 - g_2)}. \tag{2.111}$$

Note that this orthogonality theorem is compatible with the corresponding orthogonality theorem for characters Eq 2.91, by setting $a = b$, $c = d$, and taking the sum over a, c. The completeness relation on representation matrices is a specific rewriting of the completeness relation of the characters, by noting that $\chi_R(\mathbf{1}) = \text{dim}(R)$, and $\sum_{mn} R_{mn}(g_1)R_{nm}(g_2^{-1}) = \chi(g_1 g_2^{-1})$.

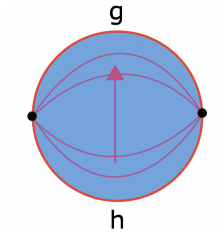
Writing the normalized overlap as $\langle g \mid R, ab\rangle = \sqrt{\text{dim(R)}}R_{ab}(g)$ and $\langle R, ab \mid g\rangle = \sqrt{\text{dim(R)}}R_{ba}(g^{-1})$, these relations are written more suggestively as:

$$\langle R, ab \mid R', mn\rangle = \delta_{RR'}\delta_{am}\delta_{bn}, \qquad \sum_{R,mn} |R, mn\rangle\langle R, mn| = \mathbf{1}. \tag{2.112}$$

Note that defining a Hilbert space structure is again only valid if $\langle g \mid R, mn\rangle = \langle R, mn \mid g\rangle^*$. This in turn implies $R_{nm}(g^{-1}) = R_{mn}(g)^*$, which restricts the representation matrices to be unitary.
The class wavefunctions of the closed channel approach $\langle U \mid R\rangle = \chi_R(U)$ can be decomposed into matrix elements of the open channel approach according to $\chi_R(U) = \sum_i R_{ii}(U)$. Similarly, the representation states $|R\rangle$ of the closed channel can be decomposed into states $|R, ii\rangle$ of the open channel. The holonomy elements $|U\rangle \sim |g \cdot U \cdot g^{-1}\rangle$ of the closed channel decompose into all group elements in its conjugacy class in the open channel.
We can now compute the disk partition function by vertically propagating $|h\rangle$ along a boundary of length β, and computing its overlap with $|g\rangle$. The Hamiltonian governing the evolution is again proportional to the quadratic Casimir, diagonalized by the representation matrices: $H(R) = \mathcal{C}_2(R)$. Inserting a completeness relation of the

representation basis, we diagonalize the Hamiltonian:

$$Z_{\text{disk}}(g,h) = \langle g \mid e^{-\beta H} \mid h \rangle \quad =$$



$$= \sum_{R,ab} \langle g \mid R, a, b \rangle \langle R, a, b \mid h \rangle \, e^{-\beta \mathcal{C}_2(R)} = \sum_{R,ab} \text{dim}(R)\ R_{ab}(g) R_{ba}(h^{-1}) e^{-\beta \mathcal{C}_2(R)}. \tag{2.113}$$

Performing the sum over ab, and recognizing that $\sum_{R,ab} R_{ab}(g) R_{ba}(h^{-1}) = \text{Tr}(R(U))$ for $U = gh^{-1}$, we arrive at the original closed slicing amplitude Eq 2.107. Additionally showing that the amplitude of the cylinder agrees with that of the radial slicing proves full equivalence between the two perspectives by virtue of the cutting and gluing axioms.

We arrive at the cylinder partition function by splitting g into multiple boundary segments $g = g_A \cdot g_B \cdot g_C$, and using the defining property of the representation matrices $R_{ab}(g) = \sum_{x,y} R_{ax}(g_A) R_{xy}(g_B) R_{yb}(g_C)$. This yields an intermediate rectangular amplitude:

$$Z_{\text{rect}}(g_A, g_B, g_C, h) = \sum_{R,a,b,x,y} \text{dim}(R)\ R_{ax}(g_A) R_{xy}(g_B) R_{yb}(g_C) R_{ba}(h^{-1}) e^{-\beta \mathcal{C}_2(R)}.$$

Gluing the opposite ends $g_B = h$ together using the orthogonality theorem yields the amplitude of the cylinder with boundary holonomy elements g_A, g_C of Eq 2.95:

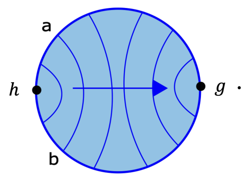
$$Z_{\text{cyl}}(g_A, g_C) = \sum_{R,ab} \chi_R(g_A) \chi_R(g_C) e^{-\beta \mathcal{C}_2(R)}. \tag{2.114}$$

Physical boundary amplitudes are characterized by trivial holonomies along the boundary $g, h \to \mathbf{1}$. Since the representation matrices are normalized according to $R_{ab}(\mathbf{1}) \equiv \delta_{ab}$, the thermal disk partition function Eq 2.113 simply becomes:

$$\boxed{Z_{\text{disk}} = \sum_{R} (\text{dim}(R))^2 e^{-\beta \mathcal{C}_2(R)}}. \tag{2.115}$$

An alternative, perhaps more insightful Hilbert space slicing to calculate the disk amplitude, is the point-defect channel developed in [64] [22]. In this case, the boundary states are associated with point-like defects instead of being smeared along the entire boundary. The calculation of the disk amplitude is precisely the same as before [24]:

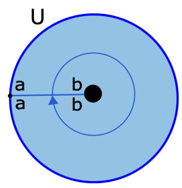
$$Z_{\text{disk}}(g,h) = \langle g \mid e^{-\beta H} \mid h \rangle \quad = \tag{2.116}$$



Intuitively, the timeflow of the Cauchy slices in the bulk corresponds to a physical propagation along the boundary with length β. The calculation is identical and proceeds by inserting a complete set of eigenstates in the representation basis that end on either side of the physical boundary.

The so-called angular slicing [24] defines yet another Hilbert-space slicing to cover the disk partition function. Here, one covers the disk into Cauchy slices that extend from the asymptotic boundary to a single point in the bulk. The representation matrices $R_{ab}(g)$ where a, b label the points at the boundary and in the bulk respectively, constitute a complete basis on this Hilbert space slicing. Consequently, one covers the entire disk by propagating these slices clockwise over a distance β. These slices start from the unit group element $\mathbf{1}$, and eventually pick up a total holonomy U along the boundary. Concretely, the Hamiltonian is diagonalized by inserting a completeness relation,

$$\langle U \mid e^{-\beta H} \mid \mathbf{1}\rangle = \sum_{R,a,b} \dim(R) R_{ab}(U) R_{ba}(\mathbf{1}) e^{-\beta \mathcal{C}_2(R)} = \tag{2.117}$$



The radial- Eq 2.107, as well as the open- Eq 2.113, point-defect- Eq 2.116, and angular slicings Eq 2.117 all have matching results.

2.5.2 Boundary-anchored Wilson lines

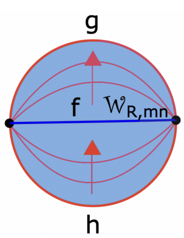
Single Wilson line insertion

We consider the matrix element of a single Wilson line from boundary point τ_i to τ_f, evaluated in representation R:

$$\mathcal{W}_{R,nm}(\tau_1,\tau_2) = \mathcal{P}\exp\left(-\int_{\tau_1}^{\tau_2} d\tau\; R(\mathbf{A})\right)_{nm}. \tag{2.118}$$

Note that reading the operator identity from right to left, m is associated with the initial time τ_1, while n is associated with the final time τ_2. This is a distinction that will become important when considering correlation functions of multiple Wilson line insertions. We may construct the amplitude of a single Wilson line insertion in the Euclidean Path integral between boundary group elements h and g by an open-channel Hamiltonian evolution [22]:

$$\begin{aligned}\langle \mathcal{W}_{R,nm}(\tau_i,\tau_f)\rangle_{h\to g} &= \int_h^g \mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A}\; \mathcal{W}_{R,nm}(\tau_1,\tau_2)\; e^{-I_{BF}[\mathbf{B},\mathbf{A}]} \\ &= \langle g \mid e^{-\beta H}\mathcal{W}_{R,nm}(\tau_1,\tau_2) \mid h\rangle \quad = \quad \int df\end{aligned} \tag{2.119}$$



The amplitude above may be solved by inserting a completeness integral over group elements f along the Wilson line, which diagonalizes the Wilson line operator into its representation matrix elements;

$$\mathcal{W}_{R,nm}(\tau_1,\tau_2) = \int df \; R_{nm}(f) \; |f\rangle\langle f| \,. \tag{2.120}$$

We can interpret this in terms of the cutting and gluing techniques of YM$_2$. In particular, gluing two half disks with boundary lengths $\beta_1 = |\tau_2-\tau_1|$ and $\beta_2 = \beta - |\tau_2 - \tau_1|$ respectively along the joint group element f yields, after diagonalizing in the representation basis:

$$\begin{aligned}\langle g \mid e^{-\beta H}\mathcal{W}_{R,nm} \mid h\rangle &= \int df \; \langle g \mid e^{-\beta_1 H} \mid f\rangle R_{nm}(f) \langle f \mid e^{-\beta_2 H} \mid h\rangle \qquad (2.121)\\ &= \sum_{R_i,n_i,m_i} \text{dim}(R_1)\text{dim}(R_2) \; R_{1,n_1m_1}(g)R_{2,m_2n_2}(h^{-1})e^{-\beta_1 \mathcal{C}_2(R_1)}e^{-\beta_2\mathcal{C}_2(R_2)}\\ &\quad\times \int df \; R_{1,m_1,n_1}(f^{-1})R_{nm}(f)R_{2,n_2m_2}(f). \qquad (2.122)\end{aligned}$$

We can compute the integral by using the Clebsch-Gordan decomposition of a tensor product of representation matrices $|R_1,m_1\rangle \otimes |R_2,m_2\rangle = \sum_{R_3,m_3} C^{R_3,m_3}_{R_1,R_2,m_1,m_2} |R_3,m_3\rangle$:

$$\begin{aligned}R_{1,n_1m_1}(g)R_{2,n_2m_2}(g) &\equiv \langle R_1,n_1| \otimes \langle R_2,n_2| \, g \, |R_1,m_1\rangle \otimes |R_2,m_2\rangle\\ &= \sum_{R_3,R_3',n_3,m_3} C^{R_3,n_3}_{R_1,R_2,n_1,n_2} C^{R_3',m_3}_{R_1,R_2,m_1,m_2} \langle R_3,n_3 \mid g \mid R_3',m_3\rangle\\ &\equiv \sum_{R_3,n_3,m_3} C^{R_3,n_3}_{R_1,R_2,n_1,n_2} C^{R_3,m_3}_{R_1,R_2,m_1,m_2} R_{3,n_3m_3}(g)\end{aligned}$$

, using the definition of the representation matrices $\langle R_3,n_3 \mid g \mid R_3',m_3\rangle = R_{3,n_3m_3}(g)\delta_{R_3R_3'}$ in the last line. The group integral over 3 representation matrices can therefore be evaluated by virtue of the grand orthogonality theorem;

$$\begin{aligned}\int df \; R_{1,m_1,n_1}(f^{-1})R_{nm}(f)R_{2,n_2m_2}(f) &= \sum_{R_3,n_3,m_3} C^{R_3,n_3}_{R,R_2,n,n_2} C^{R_3,m_3}_{R,R_2,m,m_2} \int df R_{1,m_1,n_1}(f^{-1})R_{3,n_3,m_3}(f)\\ &= \frac{C^{R_1,n_1}_{R,R_2,n,n_2} C^{R_1,m_1}_{R,R_2,m,m_2}}{\text{dim}(R_1)}.\end{aligned}$$

The Clebsch-Gordan coefficients are alternatively defined in terms of the $3j$-symbols,

$$C^{R_1,n_1}_{R,R_2,n,n_2} \equiv (-)^{R_2-R-n_1}\sqrt{\text{dim}(R_1)} \begin{pmatrix} R & R_2 & R_1 \\ n & n_2 & -n_1\end{pmatrix} \to \sqrt{\text{dim}(R_1)} \begin{pmatrix} R & R_2 & R_1 \\ n & n_2 & n_1\end{pmatrix} \tag{2.123}$$

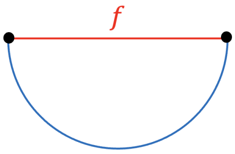
, where we absorbed the minus signs into the definition of the $3j$-symbol to streamline notation. These have a number of attractive symmetry properties, e.g. they are invariant under cyclic permutations of the columns. Alternatively, this serves as a definition for the $3j$-symbols

$$\int df \; R_{1,m_1,n_1}(f^{-1})R_{nm}(f)R_{2,n_2m_2}(f) \equiv \begin{pmatrix} R & R_2 & R_1 \\ n & n_2 & n_1\end{pmatrix}\begin{pmatrix} R & R_2 & R_1 \\ m & m_2 & m_1\end{pmatrix}. \tag{2.124}$$

Note that the order of the indices of the matrix element corresponding to f^{-1} is flipped with respect to the other pair of representation matrices. After imposing a trivial boundary holonomy $h = g = \mathbf{1}$ ($R_{1,n_1 m_1}(\mathbf{1}) = \delta_{n_1 m_1}$, $R_{2,n_2 m_2}(\mathbf{1}) = \delta_{n_2 m_2}$), the Wilson-line amplitude Eq 2.122 may finally be written as:

$$\langle g \mid e^{-\beta H} \mathcal{W}_{R,nm} \mid h\rangle|_{g,h\to\mathbf{1}} = \sum_{R_i,n_i} \text{dim}(R_1)\text{dim}(R_2) \begin{pmatrix} R & R_2 & R_1 \\ n & n_2 & n_1 \end{pmatrix} \begin{pmatrix} R & R_2 & R_1 \\ m & n_2 & n_1 \end{pmatrix} e^{-\beta_1 \mathcal{C}_2(R_1)} e^{-\beta_2 \mathcal{C}_2(R_2)}. \tag{2.125}$$

We represent the procedure above pictorially by gluing two asymptotic patches to the Wilson line along a common group element f. We will take the direction of the Wilson line to define the direction of the group element. For a trivial holonomy along the boundary, the different asymptotic patches are half disks with boundary lengths $\beta' < \beta$:

$Z(f) =$ 

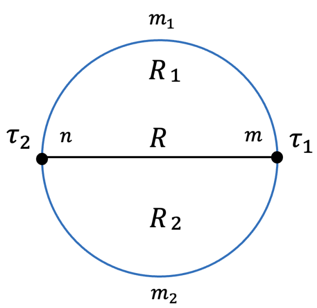
$$= \langle f \mid e^{-\beta' H} \mid \mathbf{1}\rangle = \sum_{R,n} \text{dim}(R) e^{-\beta' \mathcal{C}_2(R)} R_{nn}(f). \tag{2.126}$$

For $\beta' = \beta/2$, this is the thermofield double state (TFD) preparing the vacuum [24]:

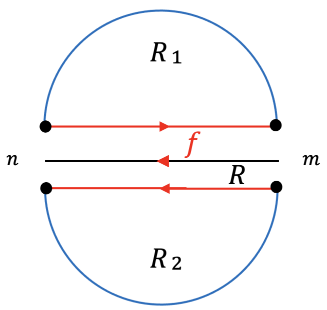
$$|\text{TFD}\rangle = \sum_{R,a} \sqrt{\text{dim}(R)} e^{-\beta/2\, \mathcal{C}_2(R)} \,|R,a,a\rangle = \sum_{R,a,b} e^{-\beta \mathcal{C}_2(R)/2} \,|R,a,b\rangle\, |R,b,a\rangle \tag{2.127}$$

, where using the definition of the representation matrices $\langle g_1 \cdot g_2 \mid R,a,a\rangle = \sum_b \frac{1}{\sqrt{\text{dim}(R)}} \langle g_1 \mid R,a,b\rangle \langle g_2 \mid R,b,a\rangle$ yields the last equality. This demonstrates that the purification of a thermal ensemble of states $|R,a,b\rangle$ can be obtained by cutting a two-sided geometry on the horizon.

Anyway, note that when gluing two asymptotic states together, just like for YM_2, the order between the different patches is important. In particular, a group element f is glued to its inverse f^{-1}, indicated by the direction of the arrows with respect to the Wilson line.



$= \int df$



$$= \int df\; Z_1(f^{-1}) R_{nm}(f) Z_2(f)$$

$$= \delta_{nm} \sum_{R_1,R_2,n_1,n_2} \text{dim}(R_1)\text{dim}(R_2) e^{-\beta_1 \mathcal{C}_2(R_1)} e^{-\beta_2 \mathcal{C}_2(R_2)} \begin{pmatrix} R_1 & R & R_2 \\ n_1 & n & n_2 \end{pmatrix}^2 \tag{2.128}$$

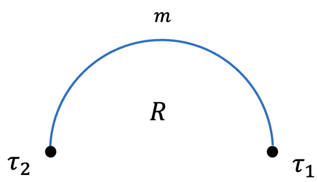
In the last line, I have used the current conservation of the Clebsch-Gordan coefficients, imposing that the indices of the Clebsch-Gordan coefficients $\mathcal{C}^{R,n}_{R_1,R_2,n_1,n_2}$ are additive[5], yielding $n = m = n_1 + n_2$.

Diagrammatic rules

From the previous discussion, we may identify generic diagrammatic rules for calculating the general disk amplitude with any number of *non-intersecting* Wilson line insertions.

We start from the disk that represents the bulk geometry on which the gauge group G lives, and divide it into n different segments cut off by the Wilson lines. These Wilson lines are labeled by a predefined representation R and associated labels m, n. To every bulk sector, we assign an irrep R_i with a corresponding weight factor dim(R_i), which contributes in the final amplitude. At the boundary of each section, we assign a label m_i of the irrep R_i. Eventually both the labels R_i and m_i at the boundary have to be summed over in the final partition function.

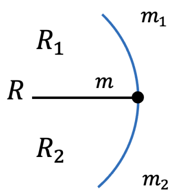
To each boundary segment i corresponding to representation R_i, we assign a Hamiltonian proportional to the quadratic Casimir of the representation $H(R_i) = \mathcal{C}_2(R_i)$. This generates an evolution along a boundary distance $\beta_i = |\tau_{i+1} - \tau_i|$ via the Hamiltonian propagation factor



$$= \exp\left(-\beta_1 \mathcal{C}_2(R)\right). \tag{2.129}$$

Eventually, the final boundary segment n closes the boundary circle with length $\beta_n = \beta - |\tau_1 - \tau_n|$, where β is the total length of the boundary circle.

At every boundary intersection with a Wilson line, a $3j$-symbol quantifies the overlap between the representation matrices along the boundary and the matrix element of the Wilson line;



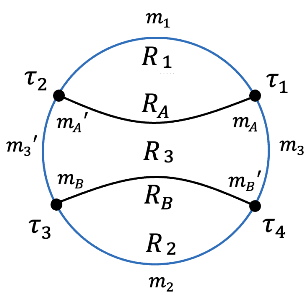
$$= \begin{pmatrix} R_1 & R_2 & R \\ m_1 & m_2 & m \end{pmatrix}. \tag{2.130}$$

These rules are exact to all orders in perturbation theory, unlike the more familiar Feynman rules in ordinary QFT. This is a consequence of the topological nature of the theory at hand.

[5]Note that this answer may be simplified further using the full orthogonality identity Eq 2.141. However, no such factorization appears for higher-point function, and the former is the fundamental answer.

Non-intersecting Wilson lines

The example of two non-intersecting Wilson lines clarifies these rules, where we take $\tau_1 < \tau_2 < \tau_3 < \tau_4$:



$$= \sum_{R_1,R_2,R_3,R_4} \prod_{i=1}^{4} \dim(R_i)\, e^{-\beta_1 \mathcal{C}_2(R_1)} e^{-\beta_2 \mathcal{C}_2(R_3)} e^{-\beta_3 \mathcal{C}_2(R_2)} e^{-(\beta-|\tau_1-\tau_4|)\mathcal{C}_2(R_3)}$$

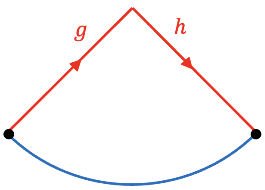
$$\times \sum_{m_1,m_2,m_3,m_3'} \begin{pmatrix} R_1 & R_A & R_3 \\ m_1 & m_A & m_3 \end{pmatrix} \begin{pmatrix} R_1 & R_A & R_3 \\ m_1 & m_A' & m_3' \end{pmatrix} \begin{pmatrix} R_2 & R_B & R_3 \\ m_2 & m_B & m_3' \end{pmatrix} \begin{pmatrix} R_2 & R_B & R_3 \\ m_2 & m_B' & m_3 \end{pmatrix}. \tag{2.131}$$

Note that in general, one usually does not pay much attention to the relative sign factors due to interchanging columns in the $3j$-symbols. These can always be reabsorbed in an appropriate definition of the $3j$-symbols. In accordance with the general discussion in section 1.8.2, the energy along the opposite boundary segments $\tau_2 - \tau_3$ and $\tau_4 - \tau_1$ in the region bounded by the two Wilson lines is the same as a consequence of energy conservation.

Intersecting Wilson lines

The diagrammatic rules above exclude the case of non-intersecting Wilson lines. These may be calculated by splitting the disk into distinct patches, bounded in the interior by the neighboring Wilson lines, and gluing along a common group element. In particular, consider the case of two Wilson lines R_A, R_B, intersecting in the interior. These separate four different patches, which may be glued to a Wilson line segment along the group elements g_1, g_2, g_3, g_4 (as indicated in the figure below). Assuming a trivial holonomy along the boundary, the amplitudes within each patch can again be calculated using the open-slicing approach.
We obtain a pie-shaped region from the half disk Eq 2.126 by splitting the group element f into two group elements $h \cdot g$ at the intersection. Reading from right to left in the matrix element according to the direction of the arrows, and using the definition of representation matrices yields

$$Z(h \cdot g) =$$

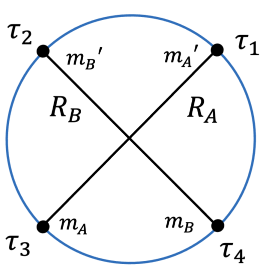


$$= \sum_{R,m} \dim(R)\, e^{-\beta_i \mathcal{C}_2(R)} R_{mm}(h \cdot g)$$

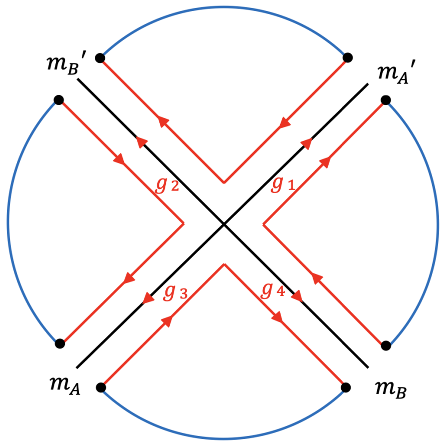
$$= \sum_{R,m,\alpha} \dim(R)\, e^{-\beta_i \mathcal{C}_2(R)} R_{m\alpha}(h) R_{\alpha m}(g). \tag{2.132}$$

m labels the points at the boundary, while α labels the bulk intersection points. We may also split the Wilson lines at the intersection into different parts connected to the boundary:

$R_{A,m_A,m'_A}(g_3g_1^{-1}) = \sum_\gamma R_{A,m_A\gamma}(g_3)R_{A,\gamma m'_A}(g_1^{-1})$, and $R_{B,m_B,m'_B}(g_4g_2^{-1}) = \sum_\gamma R_{B,m_B\gamma}(g_4)R_{B,\gamma m'_B}(g_2^{-1})$, where both Wilson lines point from m'_i to m_i. Note that by making this specific choice for the composition of the group elements, the orientation of the group elements on the figure below is automatically fixed by the direction indicated on the Wilson line. Within this convention, all group elements on the Wilson lines radially point outwards from the bulk intersection point. The final amplitude may be found by gluing the different patches together along the common group element, taking into account the direction relative to the orientation on the Wilson line;



$$= \int \prod_i dg_i \tag{2.133}$$



$$= \int \prod_i dg_i\; Z_4(g_1g_4^{-1})Z_3(g_4g_3^{-1})Z_2(g_3g_2^{-1})Z_1(g_2g_1^{-1})R_{A,m_A,m'_A}(g_3g_1^{-1})R_{B,m_B,m'_B}(g_4g_2^{-1}).$$

This choice is necessary since the gluing procedure should involve one oppositely oriented group element in the group integral Eq 2.124. Inserting the pie-shaped amplitudes of the different patches, we find explicitly

$$\sum_{R_i,m_i,\alpha_i,\gamma_i} \int \prod_i dg_i \left(\dim(R_i)e^{-\beta_i \mathcal{C}_2(R_i)}\right) R_1(g_2)_{m_1\alpha_1}R_1(g_1^{-1})_{\alpha_1 m_1}R_2(g_3)_{m_2\alpha_2}R_2(g_2^{-1})_{\alpha_2 m_2}R_3(g_4)_{m_3\alpha_3}$$
$$\times R_3(g_3^{-1})_{\alpha_3 m_3}R_4(g_1)_{m_4\alpha_4}R_4(g_4^{-1})_{\alpha_4 m_4}R_A(g_3)_{m_A\gamma_1}R_A(g_1^{-1})_{\gamma_1 m'_A}R_B(g_4)_{m_B\gamma_2}R_B(g_2^{-1})_{\gamma_2 m'_B}.$$

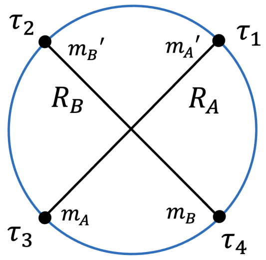
The integrals can be performed using the definition of the $3j$-symbols Eq 2.124, yielding:

$$\sum_{R_i,n_i,\alpha_i,\gamma_i} \prod_i dg_i \left(\dim(R_i)e^{-\beta_i \mathcal{C}_2(R_i)}\right) \begin{pmatrix} R_4 & R_1 & R_A \\ m_4 & m_1 & m'_A \end{pmatrix} \begin{pmatrix} R_4 & R_1 & R_A \\ \alpha_4 & \alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} R_1 & R_2 & R_B \\ m_1 & m_2 & m'_B \end{pmatrix} \begin{pmatrix} R_1 & R_2 & R_B \\ \alpha_1 & \alpha_2 & \gamma_2 \end{pmatrix}$$
$$\times \begin{pmatrix} R_2 & R_3 & R_A \\ m_2 & m_3 & m_A \end{pmatrix} \begin{pmatrix} R_2 & R_3 & R_A \\ \alpha_2 & \alpha_3 & \gamma_1 \end{pmatrix} \begin{pmatrix} R_3 & R_4 & R_B \\ m_3 & m_4 & m_B \end{pmatrix} \begin{pmatrix} R_3 & R_4 & R_B \\ \alpha_3 & \alpha_4 & \gamma_2 \end{pmatrix}.$$

We find that performing the integrals yields eight $3j$-symbols; four of which are labeled at the bulk intersection point, while the other four are related to the physical boundary. Summing over the four $3j$-symbols along the intersection yields the $6j$-symbol, defined in this context as;

$$\begin{Bmatrix} R_B & R_1 & R_4 \\ R_A & R_3 & R_2 \end{Bmatrix} = \sum_{\alpha_i,\gamma_i} \begin{pmatrix} R_4 & R_1 & R_A \\ \alpha_4 & \alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} R_1 & R_2 & R_B \\ \alpha_1 & \alpha_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} R_3 & R_4 & R_B \\ \alpha_3 & \alpha_4 & \gamma_2 \end{pmatrix} \begin{pmatrix} R_2 & R_3 & R_A \\ \alpha_2 & \alpha_3 & \gamma_1 \end{pmatrix} \tag{2.134}$$

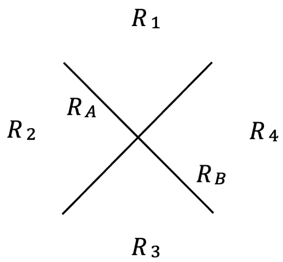
This leads to the final amplitude for the bulk-crossing of two Wilson lines:



$$= \sum_{R_i, m_i} \prod_i dg_i \left(\dim(R_i) e^{-\beta_i \mathcal{C}_2(R_i)}\right) \begin{Bmatrix} R_B & R_1 & R_4 \\ R_A & R_3 & R_2 \end{Bmatrix} \tag{2.135}$$

$$\times \begin{pmatrix} R_4 & R_1 & R_A \\ m_4 & m_1 & m_A' \end{pmatrix} \begin{pmatrix} R_1 & R_2 & R_B \\ m_1 & m_2 & m_B' \end{pmatrix} \begin{pmatrix} R_2 & R_3 & R_A \\ m_2 & m_3 & m_A \end{pmatrix} \begin{pmatrix} R_3 & R_4 & R_B \\ m_3 & m_4 & m_B \end{pmatrix}.$$

Within the diagrammatic rules, one has to account for the presence of bulk crossings by incorporating an additional $6j$-symbol:



$$= \begin{Bmatrix} R_B & R_1 & R_4 \\ R_A & R_3 & R_2 \end{Bmatrix}. \tag{2.136}$$

One can check that with these diagrammatic rules, the Wilson lines can be freely deformed and moved through each other in the bulk, as long as the boundary anchored points remain fixed [22]. This is to be expected from a purely topological theory in the bulk.

2.5.3 Particle-on-a-group theory

Alternatively, one may arrive at these amplitudes directly from the boundary perspective, without resorting to the holographic bulk. In the present context, the holographic dual of BF theory is the 1d particle-on-a-group theory [64] [22]. In particular, one starts from the total BF action, and integrates out $\mathbf{B}$ in the bulk. This renders $\mathbf{A}$ flat ($\mathbf{F} = 0$) and restricts its form to a pure gauge transformation[6]: $\mathbf{A}|_\partial = -dgg^{-1}$. Since the bulk term vanishes in the path integral, the remaining dynamics are completely determined by the boundary term, which reduces to the particle-on-a-group action when inserting the boundary condition $\mathbf{B}|_\partial = \mathbf{A}|_\partial$:

$$I[g] = \frac{1}{2}\int d\tau \, \mathrm{Tr}(\mathbf{A}_\tau^2) = \frac{1}{2}\int d\tau \, \mathrm{Tr}((g\partial_\tau g^{-1})^2). \tag{2.137}$$

Notice that the structure of this reduction is very similar to the way that the Schwarzian theory is obtained from the bulk JT action. In fact, the duality between JT gravity and Schwarzian mechanics is but an application of the duality between BF and particle-on-a-group theory, as we will see shortly. The duality between 2d BF theory and 1d quantum mechanics on a group is an example of holography where correlators in the boundary theory are obtained by path integrating out the bulk fields, leaving an integral over the remaining boundary

[6]Starting from $\mathbf{A} = gdg^{-1} = -dgg^{-1}$, and the definition of the field strength ($\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$), we readily obtain $\mathbf{F} = dg \wedge dg^{-1} + dgg^{-1} \wedge dgg^{-1}$. Using $dgg^{-1} = -gdg^{-1}$ (from $d(gg^{-1}) = 0$), $\mathbf{F} = dg \wedge dg^{-1} - dg \wedge dg^{-1} = 0$.

configurations compatible with some predefined boundary conditions. The degrees of freedom more generally include the inequivalent asymptotic configurations compatible with these boundary conditions. The path integral over bulk fields then essentially prepares an operator insertion in the boundary field theory. The latter should of course still be compatible with the prescribed boundary conditions. This pattern should hold for any holographic duality where the dual theory actually lives at the asymptotic boundary of the higher dimensional theory[7].

In the case at hand, the dynamical boundary variables are the periodic group elements g (with $g(\tau+\beta) \equiv g(\tau)$) that make up the loop group LG. There is, however, still a redundancy in the definition of flat $\mathbf{A}_\tau = g\partial_\tau g^{-1}$ for $g \sim Ug$, with U a constant group element. The integration space of the particle-on-a-group action is therefore defined over the loop group modulo constant functions LG/G. Note that modding by G should give an additional factor of $1/\text{vol}(G)$ in the partition function. This would normally destroy a genuine Hilbert space interpretation of this path integral. However, as noted in [24], we will interpret this factor as a contribution to the zero-temperature entropy, and absorb it in an overall normalization.
The wavefunctions on this group manifold are the square-integrable functions on LG/G. The Hamiltonian of this system is identical to Eq 2.104, and is described in terms of the quadratic Casimir. The measure in the path integral over flat connections is related to the natural BF measure defined in section 2.3.2, which implies the Haar measure for these boundary degrees of freedom. We have seen that this can be related to the natural symplectic measure of the Schwarzian reparametrization modes Eq C.15.

Quantum amplitudes have been studied in [64] by dimensionally reducing the known WZW results. Alternatively, one may arrive at the partition function by simply taking the thermal trace in the quantum-mechanical theory [22]. For example, the disk partition function may be calculated directly along the boundary by taking the trace in configuration space and diagonalizing the Hamiltonian in the representation basis:

$$\begin{aligned}\text{Tr}(e^{-\beta H}) &= \int dg\, \langle g \mid e^{-\beta H} \mid g\rangle = \sum_{R,m,n} e^{-\beta \mathcal{C}_2(R)} \int dg\, R_{mn}(g) R_{nm}(g^{-1}) = \sum_{R,m,n} e^{-\beta \mathcal{C}_2(R)} \\ &= \sum_R \text{dim}(R)^2 e^{-\beta \mathcal{C}_2(R)}.\end{aligned}$$

Particle-on-a-group theory also sheds a new light on the boundary anchored Wilson lines Eq 2.118, which we can describe in terms of bilocal operators at the boundary. In particular, one identifies the Wilson line in the bulk with a bilocal operator at the boundary:

$$\mathcal{W}_{R,mn}(\tau_1,\tau_2) \simeq \mathcal{O}_{R,mn}(\tau_1,\tau_2) \equiv R_{mn}(g(\tau_2)g^{-1}(\tau_1)). \tag{2.138}$$

This again leads to the understanding that, reading the operator identity from right to left, the left label m is associated to the boundary time τ_2, while the right label n is associated to τ_1. One might argue for this identification by noting that in a topological bulk theory, we can freely deform any Wilson line operator while still preserving both endpoints at the boundary. Therefore, we expect them only to depend on the boundary

[7] This should be contrasted to the novel holographic duality between $\mathcal{N}=4$ $D=4$ SUSY Yang-Mills theory and type IIB superstring theory on $AdS_5 \otimes S_5$.

degrees of freedom. Furthermore, both expressions Eqs 2.118 and 2.138 are solutions to the same differential equation (using the definition Eq 2.118 in the first equation and $\mathbf{A}_\tau = -\partial_\tau g g^{-1}$ in the second):

$$\frac{d}{d\tau_2}\mathcal{W}_{R,mn}(\tau_1,\tau_2) = -\sum_\alpha R\left(\mathbf{A}_\tau(\tau_2)\right)_{m\alpha}\mathcal{W}_{R,\alpha n}(\tau_1,\tau_2) \tag{2.139}$$

$$\frac{d}{d\tau_2}\mathcal{O}_{R,mn}(\tau_1,\tau_2) = -\sum_\alpha R\left(\mathbf{A}_\tau(\tau_2)\right)_{m\alpha}\mathcal{O}_{R,\alpha n}(\tau_1,\tau_2) \tag{2.140}$$

The identification is settled by noting that they also satisfy the same boundary conditions $\mathcal{W}(\tau_1,\tau_1) = 1$ and $\mathcal{O}(\tau_1,\tau_1) = 1$.

Boundary anchored Wilson lines in the BF theory are simply bilocal operators in particle-on-a-group theory, whose correlation functions are readily computed using the thermal trace. As an instructive example, consider the expectation value of a single bilocal operator, which should be dual to a single Wilson line insertion in the bulk;

$$\begin{aligned}
&\mathrm{Tr}(e^{-\beta H}R_{mn}(g_2 g_1^{-1})) = \sum_\alpha \mathrm{Tr}(e^{-\beta H}R_{m\alpha}(g_2)R_{\alpha n}(g_1^{-1}))\\
&= \sum_\alpha \int dh_1 dh_2 \langle h_1 \mid e^{-\beta_1 H} \mid h_2\rangle R_{m\alpha}(h_2)\langle h_2 \mid e^{-\beta_2 H} \mid h_1\rangle R_{\alpha n}(h_1^{-1})\\
&= \sum_{R_1,R_2,\alpha,m_i,n_i} \int dh_1 dh_2\, \mathrm{dim}(R_1)\mathrm{dim}(R_2) e^{-\beta_1 \mathcal{C}_2(R_1)} e^{-\beta_2 \mathcal{C}_2(R_2)} R_{1,m_1 n_1}(h_1) R_{1,n_1 m_1}(h_2^{-1}) R_{m\alpha}(h_2)\\
&\quad R_{2,m_2 n_2}(h_2) R_{2,n_2 m_2}(h_1^{-1}) R_{\alpha n}(h_1^{-1})\\
&= \sum_{R_1,R_2,\alpha,m_i,n_i} \mathrm{dim}(R_1)\mathrm{dim}(R_2) e^{-\beta_1 \mathcal{C}_2(R_1)} e^{-\beta_2 \mathcal{C}_2(R_2)} \begin{pmatrix} R & R_2 & R_1 \\ \alpha & n_2 & n_1 \end{pmatrix}^2 \begin{pmatrix} R & R_2 & R_1 \\ n & m_2 & m_1 \end{pmatrix} \begin{pmatrix} R & R_2 & R_1 \\ m & m_2 & m_1 \end{pmatrix}
\end{aligned}$$

Using the following identity on the first $3j$-symbol [22]

$$\mathrm{dim}(R)\sum_{n_1,n_2} \begin{pmatrix} R & R_2 & R_1 \\ \alpha & n_2 & n_1 \end{pmatrix} \begin{pmatrix} R' & R_2 & R_1 \\ \alpha' & n_2 & n_1 \end{pmatrix} = \delta_{RR'}\delta_{\alpha\alpha'} \tag{2.141}$$

, the two-point function of a bilocal operator in particle-on-a-group theory indeed reduces to the result of a boundary-anchored Wilson line Eq 2.125. Non-intersecting Wilson lines in the bulk in Eq 2.131 are dual to time-ordered correlation functions of bilocal operators (for $\tau_1 < \tau_2 < \tau_3 < \tau_4$)

$$\text{Eq } 2.131 = \langle \mathcal{O}_{R_A,m_A,m'_A}(\tau_1,\tau_2)\mathcal{O}_{R_B,m_B,m'_B}(\tau_3,\tau_4)\rangle\,. \tag{2.142}$$

, while bulk crossings of Wilson lines are dual to correlation functions of the form

$$\text{Eq } 2.135 = \langle \mathcal{O}_{R_A,m_A,m'_A}(\tau_1,\tau_3)\mathcal{O}_{R_B,m_B,m'_B}(\tau_2,\tau_4)\rangle\,. \tag{2.143}$$

The exact derivation of the higher order correlation functions directly from particle-on-a-group theory, can be found in appendix D of [22], where the subtleties involving higher order correlation functions are discussed in the context of networks of Wilson lines.

2.6 JT gravity as a constrained BF theory

Having studied the dictionary to describe generic correlators in BF theory, we can now try to use this framework to study quantum correlation functions in JT gravity. This would allow us to quantize JT gravity directly in its bulk description, without resorting to theories related to its holographic Schwarzian dual. In section 2.2, it was argued how JT gravity is equivalent to a topological BF theory in its first order formalism. More precisely, we have argued how the action of the two theories is equivalent on-shell. The gauge algebra at hand was found to locally correspond to (some isometry of) the $\mathfrak{sl}(2,\mathbb{R})$ algebra. However, it was already pointed out at the end of that section that such an equivalence is not readily obvious at a quantum mechanical level. This would require an identification at the level of the Euclidean path integral:

$$\int \mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A}\; e^{-S_{BF}[\mathbf{B},\mathbf{A}]} \;\simeq\; \int \mathcal{D}g\mathcal{D}\Phi\; e^{-S_{JT}[g,\Phi]}. \tag{2.144}$$

Such an identification depends on the contour of choice in each of the integrals, which in turn relies on the global embedding of the group elements in the exponentiation of the algebra.

There exist multiple groups whose linearized generators obey the same algebra. In general, for any Lie algebra $\mathfrak{g}$, the universal covering group of the algebra is a simply connected[8] Lie group $\tilde{G}$, whose Lie algebra is isomorphic with $\mathfrak{g}$ [77]. This is uniquely determined up to local analytic isomorphisms. One can reach the universal covering group by exponentiating vectors in the Lie algebra. In general, other connected Lie groups G with the same algebra (up to possible isomorphisms) can be reached from the universal covering group by a surjective homomorphism that mods out an invariant subgroup D of $\tilde{G}$: $G = \tilde{G}/D$. Both groups share the same algebra upon linearization, but obey another global structure. This story is well known when considering SU(2) versus SO(3). Both groups have an isomorphic algebra $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$. However, only $SU(2)$ is simply connected and serves as the universal covering group. One reaches SO(3) by modding over its center $\mathbb{Z}_2$: $SO(3) = SU(2)/\mathbb{Z}_2$.

Anyway, this implies that there exist different sensible quantum theories whose classical action is given by an $\mathfrak{sl}(2,\mathbb{R})$ BF theory; many of which do not correspond to the correlation functions found for JT gravity in [21]. Different suggestions on the precise group structure of JT gravity have been proposed, including the subsemigroup SL$^+(2,\mathbb{R})$ [24] or a central extension of the universal covering group of $\widetilde{\text{SL}}(2,\mathbb{R})$ by $\mathbb{R}$ [23].
To briefly recapitulate the arguments in section 2.2, many smooth gauge configurations correspond to singular metrics. These should be avoided in the contour of choice for JT gravity. In contrast, the constraint of metric invertibility is not present in a classical first-order rewriting. Furthermore, it is not immediate how to incor-

[8] With simply connected, I mean that each curve between 2 points can be continuously deformed to any other, or equivalently that any connected curve can be continuously shrunk to a point. This is e.g. the case on a 2-sphere, but not on a 2-torus This definition is to be distinguished with connected manifolds, where all points can be continuously deformed to all other points. Of course, for a manifold to be simply connected is a stronger constraint.

porate zero metrics or topological changes in the gravitational path integral. One can prove however [24] [44] that the equivalence between JT gravity and BF theory does hold on a quantum level if one restricts the flat $\mathfrak{sl}(2,\mathbb{R})$ connections to hyperbolic conjugacy class elements of the moduli space, to exclude singular metrics [87]. One should further mod by the mapping class group of the moduli space. One other essential aspect in the identification of JT quantum gravity to BF theory is to show that the natural measure of both theories are related, which we did explicitly in section 2.3.2.

Note that these subtleties do not turn up in perturbative quantum calculations around the classical saddle. These saddles are already characterized by some smooth configurations of the metric with fixed topology. Consequently, we are only considering slight changes away from this smooth saddle.

For now, we consider the full path integral on some fixed disk-shaped topology. However, a sensible theory of 2d quantum gravity is only unitary when we also consider a non-perturbative genus expansion of the spacetime [30].

2.6.1 Particle on SL$(2,\mathbb{R})$

In section 2.3.1, I have reviewed the boundary conditions of the BF theory that eventually lead to the emergence of the Schwarzian theory in the first order perspective. An important takeaway is that additionally to the constraint Eq 2.51, one must impose the asymptotic behaviour of the metric leading to the Schwarzian theory directly on the gauge fields (c.f. Eqs 2.54, 2.58). The remaining gauge degrees of freedom can then be identified with the asymptotic boundary reparametrization modes. In this way, we recover the dynamics on the integration space diff$(S^1)/$SL$(2,\mathbb{R})$.

However, in the context of identifying the holographic dual of $\mathfrak{sl}(2,\mathbb{R})$ BF theory, this is not what we want. Instead, we want to identify the boundary dynamics as a particle on SL$(2,\mathbb{R})$. The integration space in this context is thus over the larger loop group $L($SL$(2,\mathbb{R}))/$SL$(2,\mathbb{R})$. Clearly, pure particle on SL$(2,\mathbb{R})$ dynamics does not correspond to Schwarzian quantum mechanics. To arrive at the latter, we will need an analogous coset constraint on the asymptotic boundary conditions, leading to the fixed behaviour of the gauge field $\mathbf{A}|_\partial$ in Eq 2.58. I will present the coset boundary conditions in a rephrased version of [44] [22] [21].

Let us first consider an unconstrained holographic SL$(2,\mathbb{R})$ particle-on-a-group theory, whose action Eq 2.137 is given in terms of free particles on the thermal disk $g(\tau+\beta)=g(\tau)$:

$$S[g]=\frac{1}{2}\int d\tau \mathrm{Tr}\left((g\partial_\tau g^{-1})^2\right). \tag{2.145}$$

Next, we introduce the following 2×2 realization of the $\mathfrak{sl}(2,\mathbb{R})$ algebra, which we will use throughout from here on

$$iJ_0=\frac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \qquad iJ_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \qquad iJ_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \tag{2.146}$$

, satisfying an isomorphism of the $\mathfrak{sl}(2,\mathbb{R})$ algebra:

$$[J_0,J_\pm]=\pm iJ_\pm, \qquad [J_+,J_-]=2iJ_0. \tag{2.147}$$

The Gauss parametrization covers a section of the $\mathrm{SL}(2,\mathbb{R})$ group manifold in terms of three real parameters ϕ, γ_-, γ_+;

$$g^{-1} = e^{\gamma_- iJ_-}\, e^{\phi 2iJ_0}\, e^{\gamma_+ iJ_+} = \begin{pmatrix} 1 & 0 \\ \gamma_- & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & \gamma_+ \\ 0 & 1 \end{pmatrix}. \tag{2.148}$$

This parametrization covers the Poincaré patch of the $\mathrm{SL}(2,\mathbb{R})$-manifold, with a metric determined by:

$$ds^2 = \frac{1}{2}\mathrm{Tr}((gdg^{-1})^2) = d\phi^2 + e^{-2\phi} d\gamma_- d\gamma_+. \tag{2.149}$$

This is the natural metric inherited on the target space of the particle-on-a-group model Eq 2.145. The Haar measure on the $\mathrm{SL}(2,\mathbb{R})$ group manifold is deduced from the volume form corresponding to this metric:

$$dg = \sqrt{|\det(g)|}\; d\phi d\gamma_- d\gamma_+ = e^{-2\phi}\; d\phi d\gamma_- d\gamma_+. \tag{2.150}$$

More on the representation theory of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}^+(2,\mathbb{R})$ groups can be found in appendix A.

Within this realization, the trace in Eq 2.145 is the sum of the diagonal elements of the 2×2 matrices. Since $g\partial_\tau g^{-1}$ is Lie algebra valued, we can expand it into the generators of Eq 2.146:

$$g\partial_\tau g^{-1} \equiv \mathcal{J} = 2\mathcal{J}^0 iJ_0 + \mathcal{J}^- iJ_- + \mathcal{J}^+ iJ_+ \tag{2.151}$$

, where $\mathcal{J}^a$ are scalar current components. Working out explicitly using the Gauss parameterization, we can check that the components satisfy

$$\mathcal{J}^0 = \phi' + \gamma_+\gamma_-' e^{-2\phi}, \quad \mathcal{J}^- = \gamma_-' e^{-2\phi}, \quad \mathcal{J}^+ = \gamma_+' - 2\gamma_+\phi' - \gamma_+^2\gamma_-' e^{-2\phi}. \tag{2.152}$$

It can be checked that using Eq 2.151 leads to the following particle-on-$\mathrm{SL}(2,\mathbb{R})$ action

$$S[\phi,\gamma_-,\gamma_+] = \int d\tau \left(\phi'^2 + \gamma_-'\gamma_+' e^{-2\phi}\right). \tag{2.153}$$

This is a non-linear sigma model on the target space of the $\mathrm{SL}(2,\mathbb{R})$ manifold. The conjugate momenta are deduced in the usual way from the Lagrangian L: $\pi_q = \frac{\partial L}{\partial q'}$, giving:

$$\pi_\phi = 2\phi', \qquad \pi_+ = \gamma_-' e^{-2\phi}, \qquad \pi_- = \gamma_+' e^{-2\phi}. \tag{2.154}$$

Canonical quantization proceeds by promoting the fields into operators and imposing the commutation relations on the group manifold $[q,\pi_q] = 1$ in Euclidean signature [88].
We now consider a dynamical system where the Hamiltonian is specified by $H = \phi'^2 + \pi_+\pi_- e^{2\phi}$, which leads to the canonical Lagrangian L:

$$L = \pi_\phi\phi' + \pi_+\gamma_+' + \pi_-\gamma_-' - H \tag{2.155}$$

$$\leftrightarrow \quad \boxed{L = \phi'^2 + \pi_+\gamma_+' + \pi_-\gamma_-' - \pi_+\pi_- e^{2\phi}}. \tag{2.156}$$

This reduces to the particle-on-SL$(2,\mathbb{R})$ Lagrangian, by inserting either the explicit identities for the conjugate momenta π_q, or by using their on-shell equations of motion $\gamma_+' - \pi_- e^{2\phi} = 0$, $\gamma_-' - \pi_+ e^{2\phi} = 0$:

$$\boxed{L = \phi'^2 + \gamma_-'\gamma_+' e^{-2\phi}}. \tag{2.157}$$

Likewise, we can rewrite the currents in phase space coordinates (q, π_q) as:

$$\mathcal{J}^0 = \frac{\pi_\phi}{2} + \gamma_+\pi_+, \quad \mathcal{J}^- = \pi_+, \quad \mathcal{J}^+ = e^{2\phi}\pi_- - \gamma_+\pi_\phi - \gamma_+^2\pi_+. \tag{2.158}$$

After quantization ($[q, \pi_q] = 1$), the currents become operators that satisfy an isomorphism of the $\mathfrak{sl}(2,\mathbb{R})$-algebra

$$\begin{aligned}
[\mathcal{J}^0, \mathcal{J}^-] &= \pi_+ = \mathcal{J}^- \\
[\mathcal{J}^0, \mathcal{J}^+] &= \frac{1}{2}[\pi_\phi, e^{2\phi}]\pi_- - \gamma_+[\pi_+, \gamma_+]\pi_\phi - \gamma_+^2[\gamma_+, \pi_+]\pi_+ - \gamma_+[\pi_+, \gamma_+^2]\pi_+ = -\mathcal{J}^+ \\
[\mathcal{J}^-, \mathcal{J}^+] &= \pi_\phi + 2\gamma_+\pi_+ = 2\mathcal{J}^0.
\end{aligned}$$

Compared to the $\mathfrak{sl}(2,\mathbb{R})$ algebra of Eq 2.147, we may identify:

$$i\mathcal{J}^- \leftrightarrow J_+, \qquad i\mathcal{J}^+ \leftrightarrow J_-, \qquad i\mathcal{J}^0 \leftrightarrow J_0. \tag{2.159}$$

Alternatively, the action exhibits another set of generators from $\mathcal{T} = \partial_\tau g g^{-1}$, which commute with $\mathcal{J}^a$ and upon quantization satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Explicitly, these currents are parameterized in phase space by [44]:

$$\mathcal{T}^0 = \frac{\pi_\phi}{2} + \gamma_-\pi_-, \qquad \mathcal{T}^- = \pi_-, \qquad \mathcal{T}^+ = e^{2\phi}\pi_+ - \gamma_-\pi_\phi - \gamma_-^2\pi_-. \tag{2.160}$$

The theory thus exhibits a two-folded symmetry, whose spectrum is organized in unitary representations of SL$(2,\mathbb{R}) \times$ SL$(2,\mathbb{R})$.
The Hamiltonian Eq 2.155 is of course equivalent to the Hamiltonian derived in Eq 2.104 in terms of the quadratic Casimir:

$$H = \text{Tr}(\mathbf{A}|_\partial^2) = \text{Tr}(\mathcal{J}\mathcal{J}) = \text{Tr}(\mathcal{T}\mathcal{T}) = \mathcal{C}_2. \tag{2.161}$$

Since the currents $\mathcal{J}^a$, $\mathcal{T}^a$ become operators working on the group labels $\phi, \gamma_\pm$, the different representation wavefunctions $\psi_i(\phi, \gamma_+, \gamma_-)$ can be solved in terms of a differential equation of the quadratic Casimir $\mathcal{C}_2$ by imposing

$$\mathcal{C}_2\, \psi_j(\phi, \gamma_+, \gamma_-) = -j(j+1)\, \psi_j(\phi, \gamma_+, \gamma_-) \tag{2.162}$$

in either realization of the algebra (see appendix F of [22]). Here j is a (possibly complex) number that labels the representation. To label the representations uniquely, we should consider a maximal set of commuting operators in the set $\{\mathcal{C}_2, \mathcal{J}^a, \mathcal{T}^a\}$. We find that $\mathcal{J}^a$ and $\mathcal{T}^b$ commute for any a, b. Therefore, a proper choice would be to diagonalize both $\mathcal{C}_2$, $\mathcal{J}^-$, $\mathcal{T}^+$, and to label the eigenstates in the Hilbert space in terms of the eigenvalues under these operators $|j, s, r\rangle$. These are simultaneous eigenvectors under

$$\mathcal{C}_2\, |j,s,r\rangle = -j(j+1)\, |j,s,r\rangle\,, \quad \mathcal{J}^+\, |j,s,r\rangle = s\, |j,s,r\rangle\,, \quad \mathcal{T}^-\, |j,s,r\rangle = r\, |j,s,r\rangle\,. \tag{2.163}$$

Another basis is the set of group elements $|g\rangle$, parameterized by the parameters of the group manifold. Since e.g. $\mathcal{J}^+$ works diagonally on $|j,s,r\rangle$ while it also acts as a differential operator on $|g\rangle$ yielding the representation matrices, we can argue for the identification $\langle g \mid j,r,s\rangle \simeq R_{j,r,s}(g)$ by considering the matrix element $\langle g \mid \mathcal{J}^a \mid R,s,r\rangle$ [44]. This immediately confirms the proposed Hilbert space structure in terms of the Peter-Weyl theorem for the case of $\mathrm{SL}(2,\mathbb{R})$.

Minisuperspace Liouville Hamiltonian

To be more concrete, the currents above $\mathcal{T}$, $\mathcal{J}$ are related to the left and right regular realizations of the $\mathfrak{sl}(2,\mathbb{R})$ algebra respectively. Using the identifications Eq 2.159, and canonical quantization $\pi_q \to -\partial_q$, the right regular representation constructed from $\mathcal{J}$ is:

$$iJ_+ = \partial_+, \qquad iJ_- = e^{2\phi}\partial_- - \gamma_+\partial_\phi - \gamma_+^2\partial_+, \qquad iJ_0 = \frac{1}{2}\partial_\phi + \gamma_+\partial_+. \tag{2.164}$$

We compute the quadratic Casimir in this realization following the usual definition (c.f. Eq A.7):

$$\mathcal{C}_2 = (J_0)^2 + \frac{1}{2}(J_+J_- + J_-J_+) = -\frac{1}{4}\partial_\phi^2 + \frac{1}{2}\partial_\phi - e^{2\phi}\partial_+\partial_-. \tag{2.165}$$

Parameterizing $j = -\frac{1}{2} + ik$ for unitary representations (c.f. Eq A.13), the Casimir eigenvalue problem Eq 2.162 is specified by:

$$\mathcal{C}_2\chi(\phi,\gamma_-,\gamma_+) = \left(\frac{1}{4} + k^2\right)\chi(\phi,\gamma_-,\gamma_+). \tag{2.166}$$

We will see shortly that gravitational solutions are constrained under right coset conditions $\pi_+ \equiv 1$ ($\partial_+ = -1$). These are complemented by the left coset boundary condition $\pi_- = -1$ ($\partial_- = 1$), yielding matrix elements of mixed parabolic type. Defining a new function $\chi = e^{\phi}\psi(\phi)$ independent of $\gamma_\pm$, the $\mathfrak{sl}(2,\mathbb{R})$ Casimir eigenvalue equation is equivalent to the minisuperspace limit of the Liouville eigenvalue equation;

$$\left(-\frac{1}{4}\partial_\phi^2 + e^{2\phi}\right)\psi(\phi) = k^2\psi(\phi). \tag{2.167}$$

Its normalized solutions are well-known, and are determined in terms of modified Bessel functions [50] [22]

$$\boxed{R_k(\phi) = e^{\phi}\psi(\phi) = e^{\phi}K_{2is}(e^{\phi})}. \tag{2.168}$$

This directly determines the representation matrices of the principal series representation of $\mathrm{SL}(2,\mathbb{R})$. Given the exponential slope of the Liouville potential, these are the only surviving normalizable solutions for positive energies. This procedure is known as the harmonic analysis of $\mathrm{SL}(2,\mathbb{R})$ representation theory.

2.6.2 Coset boundary conditions

Of course, the two-folded $\mathrm{SL}(2,\mathbb{R})$ particle-on-a-group theory above does not correspond to the Schwarzian theory. To see how the latter emerges, we look back at the Hamiltonian formulation of the Schwarzian theory in terms of a four dimensional phase space (φ,π_φ), (F,π_F), discussed in section 1.6.1. There, we demonstrated that the zero-temperature Schwarzian theory can be obtained in a Hamiltonian formulation by integrating out the auxiliary field π_φ in the total Lagrangian

$$L = \pi_\varphi \partial_t \varphi + \pi_F \partial_t F - H \tag{2.169}$$

, where the Hamiltonian is given in the current convention of $C = 1/2$ as

$$H = \pi_\varphi^2 + e^\varphi \pi_F. \tag{2.170}$$

Plugging in the equation of motion for π_φ: $\pi_\varphi = \frac{\varphi'}{2}$ after path integrating over π_φ, the above can be written as:

$$L = \frac{1}{4}(\varphi')^2 - \pi_F e^\varphi + \pi_F F'. \tag{2.171}$$

Interpreting π_F as a Lagrange multiplier enforcing the constraint $e^\varphi = F'$, we arrive at exactly the Lagrangian of the Schwarzian theory up to a total derivative and a constant prefactor, where

$$\{F,t\} = -\frac{1}{2}(\partial_t\varphi)^2 + \partial_t^2\varphi, \qquad \varphi \equiv \log F'. \tag{2.172}$$

Abusing notations to connect to the current section, we may reformulate the Lagrangian in terms of $\varphi \equiv 2\phi$, $(F,\pi_F) \equiv (\gamma_-,\pi_-)$:

$$\boxed{L = (\phi')^2 + \pi_-\gamma_-' - \pi_- e^{2\phi}}. \tag{2.173}$$

Therefore, integrating out π_- should lead to the Schwarzian derivative in terms of γ_-. In section 1.6.1, we have also seen that integrating out γ_- and setting $\pi_- \equiv \mu$ leads to the 1d Liouville equation

$$\boxed{L = \phi'^2 - \mu e^{2\phi}}. \tag{2.174}$$

On the other hand, the Lagrangian Eq 2.173 itself can be obtained from the Lagrangian Eq 2.156 by integrating out γ_+, and setting $\pi_+ \equiv 1$. Ultimately, integrating out π_+ and π_- in 2.156 leads to the desired particle-on-$\mathrm{SL}(2,\mathbb{R})$ Lagrangian Eq 2.157.
This set of relations is quite involved, so I summarize them in table 2.1.

The conclusion is that in order to obtain Schwarzian quantum mechanics as the holographic dual of a bulk BF theory, we have to impose an additional boundary condition to $\mathbf{A}|_\partial = \mathbf{B}|_\partial$; namely the coset boundary condition:

$$\mathbf{A}|_\partial = \mathbf{B}|_\partial, \qquad \mathcal{J}^- = \pi_+ = 1. \tag{2.175}$$

The coset boundary condition should be fixed on all states of the Hilbert space reaching the boundary where

Particle-on-SL$(2,\mathbb{R})$:	$\boxed{L = \phi'^2 + \pi_+\gamma_+' + \pi_-\gamma_-' - \pi_+\pi_- e^{2\phi}}$	$\xrightarrow{\text{Integrate out } \pi_+,\, \pi_-}$	$\boxed{L = \phi'^2 + \gamma_-'\gamma_+' e^{-2\phi}}$
	$\big\downarrow$ Integrate out $\gamma_+,\ \pi_+ \equiv 1$		
Schwarzian:	$\boxed{L = (\phi')^2 + \pi_-\gamma_-' - \pi_- e^{2\phi}}$	$\xrightarrow{\text{Integrate out } \pi_-}$	$\boxed{L \simeq \{\gamma_-, \tau\}}$

Table 2.1: Overview of different models with underlying SL$(2,\mathbb{R})$ symmetry.

the Schwarzian theory lives: $\pi_+ |\psi\rangle = |\psi\rangle$. This fixes the labels of the representation matrices schematically to $|j,s,r\rangle \equiv |j,s\rangle$. Fixing one of the currents breaks the two-fold SL$(2,\mathbb{R})$ symmetry, leaving only the left-invariant charge algebra $\mathcal{T}^a$ intact, since these generators in general commute with $\mathcal{J}^-$. On the other hand, since the remaining generators in $\mathcal{J}^a$ do not commute with $\mathcal{J}^-$, we cannot additionally constrain them simultaneously.

In terms of the Gauss parametrization Eq 2.152, we can write the coset constraint as fixing

$$\gamma_-'\, e^{-2\phi} \equiv 1. \tag{2.176}$$

Imposing this constraint directly on the action Eq 2.153 yields:

$$S[\gamma_-, \gamma_+] = \int d\tau \left(\frac{1}{4} \frac{\gamma_-''^{\,2}}{\gamma_-'^2} + \gamma_+' \right). \tag{2.177}$$

Writing the Schwarzian as $\{\gamma_-, \tau\} = \left(\frac{\gamma_-''}{\gamma_-'}\right)' - \frac{1}{2}\left(\frac{\gamma_-''}{\gamma_-'}\right)^2$, we can immediately identify the action up to a total derivative with the Schwarzian boundary action:

$$S[\gamma_-, \gamma_+] = -\frac{1}{2} \int d\tau\, \{\gamma_-, \tau\}. \tag{2.178}$$

The field γ_+ is a free field that should be integrated out in the action. Likewise, we can impose a well-chosen gauge constraint. One possibility would be to fix:

$$\gamma_+ = -\phi' = -\frac{1}{2}\frac{\gamma_-''}{\gamma_-'}. \tag{2.179}$$

This choice conveniently puts $\mathcal{J}^0 \equiv 0$, and yields for $\mathcal{J}^+$:

$$\mathcal{J}^+ = \gamma_+' + \gamma_+^2 = -\frac{1}{2}\left(\frac{\gamma_-''}{\gamma_-'}\right)' + \frac{1}{4}\left(\frac{\gamma_-''}{\gamma_-'}\right)^2 \simeq -\frac{1}{2}\{\gamma_-, \tau\}. \tag{2.180}$$

The coset boundary condition $\mathcal{J}^- \equiv 1$, together with this choice of gauge (leading to $\mathcal{J}^0 = 0$, $\mathcal{J}^+ =$

$-\frac{1}{2}\{\gamma_-, \tau\}$), allows us to write the total current as:

$$\mathbf{A}|_\partial = \mathcal{J}^- iJ_- + \mathcal{J}^+ iJ_+ \equiv iJ_- - \frac{1}{2}T(\tau)iJ_+ = \begin{pmatrix} 0 & -\frac{1}{2}T(\tau) \\ 1 & 0 \end{pmatrix} \tag{2.181}$$

, with $T(\tau) \equiv \{\gamma_-, \tau\}$. Inserted in the particle-on-a-group action $I = \frac{1}{2}\int d\tau \text{Tr}(\mathcal{J}^2)$ directly leads to the Schwarzian boundary action:

$$I = -\frac{1}{2}\int d\tau T(\tau) = -\frac{1}{2}\int d\tau \,\{\gamma_-, \tau\}. \tag{2.182}$$

Of course, the choice of gauge for γ_+ is arbitrary, and any choice will lead to this conclusion. However, it will turn out to be useful to fix this gauge immediately to deduce the holographic interpretation of boundary anchored Wilson lines later.
This settles the interpretation of the Schwarzian boundary action as a constrained particle on the $\text{SL}(2,\mathbb{R})$ manifold[9]. Being the holographic dual of bulk JT gravity, we may also interpret JT gravity as a constrained version of $\mathfrak{sl}(2,\mathbb{R})$ BF theory.
Imposing the coset boundary condition on the natural $\text{SL}(2,\mathbb{R})$ volume form Eq 2.150 also leads to the Schwarzian symplectic volume form Eq C.25. Indeed, the functional volume form at every τ is:

$$\boxed{\mathcal{D}g = \prod_\tau e^{-2\phi} d\phi d\gamma_+ d\gamma_- = \prod_\tau \frac{d\gamma_-}{\gamma_-'}}$$

, after writing ϕ in terms of γ_- via $e^{-2\phi}\gamma_-' = 1$, and gauge-fixing γ_+, thereby leaving only dependence on $d\gamma_-$. This provides a holographic perspective on the bulk BF measure of section 2.3.2.

2.7 Generalized BF theory

The embedding of JT amplitudes within the quantum mechanical BF framework requires making two obvious modifications to the latter. First of all, as we have seen in the previous section, we need to fine-tune the boundary conditions to account for the dynamics on cosets.
Secondly, the discussion of BF quantization in the last section has been restricted to the well-known compact case. Since JT gravity is ultimately described by (some modification) of $\text{SL}(2,\mathbb{R})$, we need to extend this discussion to include non-compact groups as well. For a Lie-group to be compact means that any infinite series of points contains a part that converges to a point on the manifold [77]. Note that the underlying Lie algebra of any compact group G is automatically a compact real form $\mathfrak{g}$, described soley in terms of anti-Hermitian generators. On the contrary, only if a Lie algebra is both semisimple and compact, then we known that its universal covering group is a compact manifold. Concretely, the parameters of a compact Lie group span only a finite range[10], while some (sub)parameters of non-compact groups span an infinite range. The main difference in the Peter-Weyl decomposition to compact groups is that there might appear irreps with continuous irrep labels, as well as infinite-dimensional modules of discrete representations.

[9] [24] finetunes this interpretation later for the subsemigroup $\text{SL}^+(2,\mathbb{R})$.
[10] The textbook example being the compact $(S)U(N)$, $(S)O(N)$ groups, as part of the unitary and orthogonal groups, parameterized in terms of compact parameters describing rotations.

2.7.1 Quantum amplitudes on the coset G/H

The extension of BF amplitudes to cosets over G/H was rigorously established in [24]. Here, the standard boundary conditions $\mathbf{B}|_\partial = \mathbf{A}|_\partial$ are generalized to:

$$A^a|_\partial = B^a|_\partial, \qquad A^b|_\partial = B^b|_\partial \equiv 0 \tag{2.183}$$

for some subset of generators labeled by b, spanning a proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In fact, the standard boundary conditions are in a sense the least restrictive ones. In the quantum theory, additionally constraining $A^b_\tau = (g\partial_\tau g^{-1})^b = \mathcal{J}^b \equiv 0$ leads to a constrained particle moving on the right coset G/H:

$$I[g] = \frac{1}{2}\int d\tau \, \mathrm{Tr}(g\partial_\tau g^{-1})^2|_{G/H}. \tag{2.184}$$

At the extremal end, when $H = G$, this leads to $\mathbf{A}|_\partial = \mathbf{B}|_\partial \equiv 0$. Since the boundary term is linear in $\mathbf{B}$, this destroys any dynamics on the boundary, and the net-result is a topological BF theory inside the bulk, for which the only natural observables are knots and Wilson loops.
In the Peter-Weyl theorem, functions on the coset G/H are restricted by right-invariance under H: $f(g) = f(g \cdot H)$. These may be expanded in the representation basis $R_{a0}(g)$

$$R_{a0}(g) = \langle R, a \mid g \mid R, 0\rangle \equiv \langle R, a \mid g \cdot H \mid R, 0\rangle \tag{2.185}$$

, where the label 0 implies right-invariance under H:

$$h\,|R, 0\rangle = |R, 0\rangle\,, \qquad h \in H. \tag{2.186}$$

Expanding the group elements in terms of infinitesimal generators implies that all right-invariant basis states $|R, a\rangle$ are annihilated by the generators under $\mathfrak{h}$:

$$\mathcal{J}^b\,|R, 0\rangle = 0. \tag{2.187}$$

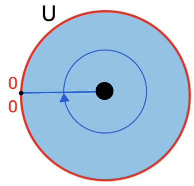
This is exactly the content of the coset boundary restriction Eq 2.183, written in terms of the quantized generators on the boundary $A^b_\tau = g\partial_\tau g^{-1} = \mathcal{J}^b$. It can be shown [24] that for homogeneous spaces (which any sensible modification of $\mathrm{SL}(2,\mathbb{R})$ falls under), there is only one basis vector $|R, 0\rangle$ that is invariant under H for each irrep R. The Hilbert space of right-invariant functions on G/H is thus spanned by so-called *spherical* functions

$$\phi^R_{0a}(g) = \langle R, a \mid g \mid R, 0\rangle = \langle g \mid R, a, 0\rangle \tag{2.188}$$

$$= \sqrt{\dim(R)}\; R_{a0}(g). \tag{2.189}$$

Since the generators are only restricted at the boundary, the states $|R, 0\rangle$ label points at the boundary, while $|R, a\rangle$ label points in the bulk. The natural slicing to cover the disk partition function on G/H is, in this case, the angular slicing of Eq 2.117. Concretely,

$$\langle U \mid e^{-\beta H} \mid \mathbf{1} \rangle = \sum_{R,a} \phi^R_{0a}(U)\phi^R_{a0}(\mathbf{1})e^{-\beta \mathcal{C}_2(R)} = \sum_{R,a} \dim(R)\ R_{0a}(U)R_{a0}(\mathbf{1})e^{-\beta \mathcal{C}_2(R)} =$$



. (2.190)

The red boundary indicates the locus where the coset boundary restriction holds. Restricting to a trivial holonomy along the disk $U \equiv \mathbf{1}$, the novel difference compared to the generic disk amplitudes Eq 2.115 is that there is only one free parameter a in the sum, leading to a different exponent of the measure $\dim(R)$:

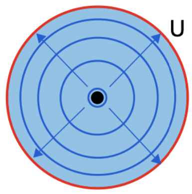
$$Z_{\text{disk}}|_{G/H} = \sum_R \dim(R) e^{-\beta \mathcal{C}_2(R)}. \tag{2.191}$$

The remaining measure factor $\dim(R)$ is reminiscent of the normalization of the complete set of spherical wavefunctions. a denote the free labels in the bulk that are not affected by the coset condition on the boundary. On the contrary, when we consider a Hilbert space on an interval that ends at either side on the physical boundary, one should expand in doubly constrained *zonal* wavefunctions

$$\phi^R(g) = \langle g \mid R, 0, 0 \rangle = \sqrt{\dim(R)} R_{00}(g) \tag{2.192}$$

, where $R_{00}(g) \equiv \langle R, 0 \mid g \mid R, 0 \rangle$. This restricts the Hilbert space along the interval to be both left-and-right invariant under H. Concretely, since the particle moves on the holonomy of $g = g_1^{-1} \cdot g_2$ (with g_1 and g_2 the location of the boundary particle on a group at two different time instances), the restriction above indicates that it actually travels on the right coset G/H ($g_1 \simeq g_1 \cdot h_1,\ \ g_2 \simeq g_2 \cdot h_2$): $R_{00}(g) = R_{00}(h_1^{-1} \cdot g \cdot h_2)$. Thus it should be both left and right invariant.
Evaluating the disk partition function with a radial slicing restricts the character on the boundary to the a predefined weight $\chi(U) = R_{00}(U)$, yielding manifestly the same amplitude as before [24]:

$$\langle U \mid e^{-\beta H} \mid \mathbf{1} \rangle = \sum_R \chi_R(\mathbf{1})\ \chi_R|_{00}(U) e^{-\beta \mathcal{C}_2(R)} = \sum_R \dim(R)\ R_{00}(U) e^{-\beta \mathcal{C}_2(R)} =$$



. (2.193)

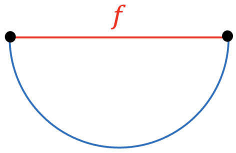
Imposing the coset restriction on both endpoints of the Hilbert space also allows an open slice covering of the disk partition function in terms of the zonal wavefunctions:

$$Z_{\text{disk}}|_{G/H} = \langle g \mid e^{-\beta H} \mid h \rangle = \sum_R \phi^R(g)\phi^R(h^{-1})e^{-\beta \mathcal{C}_2(H)} = \sum_R \dim(R)\ R_{00}(g)R_{00}(h^{-1})e^{-\beta \mathcal{C}_2(H)}. \tag{2.194}$$

This is manifestly equal to the radial and angular slicing on the coset boundaries since $R_{00}(U) = R_{00}(g)R_{00}(h^{-1})$. In any case, restricting the physical boundaries to $g = h = \mathbf{1}$ yields up to some R-independent prefactor: $R_{00}(\mathbf{1}) \to 1$. In all slicings, the dimension of the weight $\dim(R)$ is lowered by one compared to the free particle.

Using the zonal matrix elements, the Hilbert space expansion of the asymptotic state Eq 2.126 does not involve a summation over the indices:

$Z(f) =$



$$= \langle f \mid e^{-\beta H} \mid \mathbf{1}\rangle = \sum_R \dim(R) e^{-\beta' \mathcal{C}_2(R)} R_{00}(f). \tag{2.195}$$

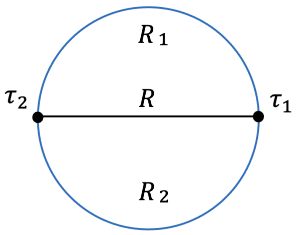
The coset boundary restriction at the physical boundary also restricts the free indices on the Wilson line matrix element $\mathcal{W}_{R,nm}$. Indeed, we saw in section 2.5.2 that gluing two asymptotic half-disks along a Wilson line involves the Clebsch-Gordan coefficients

$$\int df\ R_{1,m_1,n_1}(f^{-1}) R_{nm}(f) R_{2,n_2 m_2}(f) \sim C^{R_1,n_1}_{R,R_2,n,n_2} C^{R_1,m_1}_{R,R_2,m,m_2}. \tag{2.196}$$

It is a standard property of the Clebsch-Gordan coefficients $C^{R_1,m_1}_{R,R_2;m,m_2} = \langle R_1, m_1 \mid R, R_2; m, m_2\rangle$ that they are only non-zero if the currents are conserved $\mathcal{J}^1 = \mathcal{J} + \mathcal{J}^2$. Therefore, if both currents of the asymptotic states $\mathcal{J}^1$ and $\mathcal{J}^2$ are constrained to a predefined weight, the weight of the Wilson line under $\mathcal{J}$ is constrained to zero. The matrix elements of the latter should therefore be evaluated in the lowest-weight states:

$$\mathcal{W}_R \equiv \mathcal{W}_{R,00}. \tag{2.197}$$

Gluing the asymptotic states together along a Wilson line $\int df\ Z_1(f^{-1}) R_{00}(f) Z_2(f)$ now leads to the modified boundary correlation function:



$$= \sum_{R_1,R_2} \dim(R_1)\dim(R_2) e^{-\beta_1 \mathcal{C}_2(R_1)} e^{-\beta_2 \mathcal{C}_2(R_2)} \begin{pmatrix} R_1 & R & R_2 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{2.198}$$

The bulk crossing of two Wilson lines involves $6j$-symbols, which are obtained as a sum over $3j$ symbols in the bulk. We have come to this conclusion by splitting the asymptotic half-disk amplitude into pie-shaped amplitudes at the bulk intersection point. In this case, we can likewise split the zonal matrix elements into spherical matrix elements along a free parameter a at the bulk intersection point:

$$R_{00}(g \cdot h) = \sum_a R_{0a}(g) R_{a0}(g). \tag{2.199}$$

This decomposition makes it clear that a is still a free index, unconstrained by the coset conditions at the boundary. Therefore, the sum over $3j$-symbols at the intersection still produces the same $6j$-symbol of the ambient G space.

2.7.2 Non-compact generalization

As opposed to the well-studied compact BF theory, non-compact generalizations have remained relatively unexplored. In fact, by matching the amplitudes in the BF description of JT gravity with the previously established results in e.g. [21], Blommaert et al. [22] not only provided a Wilson line perspective on Schwarzian correlators, but also proved that calculations in constrained $\mathrm{SL}^+(2,\mathbb{R})$ BF theory are structurally equivalent to those of their compact relatives.

The novel difference is to include both infinite-dimensional and continuous labeled representations in the Peter-Weyl theorem. We assume that the square-integrable functions on the group manifold $L^2(G)$ can still be decomposed in the representation basis

$$f(g) = \sum_{k,a,b} c_{k,ab}\, R_{k,ab}(g), \qquad f \in L^2(G), \quad g \in G.$$

In this notation, the sum represents an integral if k, a, b assume continuous values[11]: "$\sum_{k,a,b} \to \int dkdadb$". In the case of the latter, one usually speaks of the *Plancherel decomposition* in the representation basis. The delta-regularized orthogonality relation for continuous representations is:

$$\int dg R_{k,ab}(g) R_{k',cd}(g^{-1}) = \frac{\delta(k-k')}{\rho(k)} \delta_{ad}\delta_{bc} \tag{2.200}$$

, where $dg = e^{-2\phi} d\gamma_+ d\gamma_- d\phi$ is the natural Haar measure. This defines the Plancherel measure $\rho(k)$ of the representation k, which takes over the role of $\dim(k)$ for continuous irreps. The relation above defines an orthonormal basis of representation wavefunctions:

$$\psi^k_{sr}(g) = \langle g \mid k, s, r\rangle \equiv \rho(k)^{1/2} R_{k,sr}(g), \qquad \psi^k_{rs}(g^{-1}) = \langle k, s, r \mid g\rangle \equiv \rho(k)^{1/2} R_{k,rs}(g^{-1}). \tag{2.201}$$

The Hilbert space structure $\langle g \mid R, s, r\rangle^* = \langle R, s, r \mid g\rangle$ again constrains the representation matrices to the class of unitary irreducible representations. Note that the Plancherel measure derived from the orthogonality theorem, is independent of the choice of basis. Restricting to continuous representations, the disk partition function is readily generalized to:

$$\langle g \mid e^{-\beta H} \mid h\rangle = \int ds \int dr \int dk\; \psi^k_{sr}(g)\psi_{rs}(h^{-1}) e^{-\beta \mathcal{C}_2(k)} = \int dk\; \rho(k)\chi_k(g\cdot h^{-1}) e^{-\beta \mathcal{C}_2(k)} \tag{2.202}$$

, with an obvious definition of the character evaluated in continuous representations $\chi_k(g) = \int ds\;\, R_{k,ss}(g)$. This in turn provides an alternative definition of the Plancherel measure $\chi_k(\mathbf{1}) = \rho(k)$, such that on a physical boundary with $U = g = h = \mathbf{1}$:

$$Z(\beta) = \int dk\; \rho(k)^2 e^{-\beta \mathcal{C}_2(k)}. \tag{2.203}$$

[11] One should be a little more careful in the overall normalization of this relation. In particular, summation over irreps has to transform contragrediently to the Kronecker delta, leading to an overall volume factor on the space of conjugacy classes. I will graciously ignore these subtleties. See appendix C of [24].

Restricting to coset boundary conditions of course strips off the integral over the labels r, s, resulting in the disk amplitude

$$\boxed{Z(\beta)|_{G/H} = \int dk\ \rho(k) e^{-\beta \mathcal{C}_2(k)}}. \tag{2.204}$$

In the case of $\mathrm{SL}(2,\mathbb{R})$, the unitary irreps have all been thoroughly studied. As always, states are labeled by the irreps and the eigenvalues of one of the generators. In the basis $\{\mathcal{J}^0, \mathcal{J}^\pm = \mathcal{J}^1 \pm i\mathcal{J}^2\}$, $\mathcal{J}^2$ generates a compact subgroup. Diagonalizing $\mathcal{J}^2$ defines the elliptic basis [22]. Its eigenvalues are denoted by integer values m. Diagonalizing either $\mathcal{J}^\pm$ defines the (mixed) parabolic basis with continuous eigenvalues $\nu_\pm$. The (continuous or discrete) label j is related to the eigenvalue of the quadratic Casimir $\mathcal{C}_2(j) = -j(j+1)$ and labels the representation. This classifies the collection of square-integrable unitary irreducible representations:

- Principal Continuous series representation $\mathcal{C}_k$ with $j = -\frac{1}{2} + ik$ for $k \in \mathbb{R}$
- (highest and lowest-weight) Discrete series representation $\mathcal{D}_j^\pm$ with $j = \mathbb{N}$ and $m = \pm j, \pm(j+1), \pm(j+2), \dots$

All other unitary irreps are not in $L^2(\mathrm{SL}(2,\mathbb{R}))$. The general Plancherel decomposition of any function $f(g) \in L^2(\mathrm{SL}(2,\mathbb{R}))$ in the elliptic basis is then [22]:

$$f(g) = \int_0^\infty dk\ k\tanh(\pi k) \sum_{m,n\in\mathbb{Z}} c_{k,m,n}\ R_{-\frac{1}{2}+ik,mn}(g) + \sum_{l=0}^{+\infty} \sum_{m,n=\pm l}^{\pm\infty} \left(l + \frac{1}{2}\right) c_{l,m,n} R^\pm_{l,mn}(g). \tag{2.205}$$

From the first term, we read off the Plancherel measure of the principal continuous representation $\rho(k) \sim k\tanh(\pi k)$.

2.8 Gravitational amplitudes of JT gravity

To proceed, we need a fair amount of the specific representation theory of $\mathrm{SL}(2,\mathbb{R})$ and of $\mathrm{SL}^+(2,\mathbb{R})$. Since the original work of this thesis builds on this representation theory, I have included an extensive appendix A covering these two topics. I have mostly summarized the approaches of [22], [24] and [40] while working out much of the calculational details explicitly. A convenient summary is given hereunder.

2.8.1 Representation theory of SL$(2,\mathbb{R})$

This section serves as a summary of the appendix A to which the reader is heartily redirected for more concrete derivations.

We start from the defining realization of $\mathrm{SL}(2,\mathbb{R})$ in terms of 2×2 matrices $g \in \mathrm{SL}(2,\mathbb{R})$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \equiv 1. \tag{2.206}$$

Upon imposing the determinant constraint $ad - bc = 1$, this leaves a three-dimensional group manifold.
As demonstrated in Eq A.3, the conjugacy classes of this matrix are categorized in terms of the value of its trace, according to:
elliptic: $\mathrm{Tr}(g) < 2$, *parabolic*: $\mathrm{Tr}(g) = 2$, and *hyperbolic*: $\mathrm{Tr} > 2$.
Expanding the group elements into generators $g = 1 + i\epsilon^a J_a$, the constraint $\det g \equiv 1$ is translated to the ambient $\mathfrak{sl}(2,\mathbb{R})$ algebra as the vanishing of the trace of the 2×2 matrices $\mathrm{Tr}(iJ_a) \equiv 0$.
This leaves a three dimensional algebra, spanned in terms of the fundamental generators Eq 2.146:

$$iJ_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad iJ_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad iJ_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.207}$$

These satisfy an isomorphism of the $\mathfrak{sl}(2,\mathbb{R})$ algebra:

$$\boxed{[J_0, J_\pm] = \pm i J_\pm, \qquad [J_+, J_-] = 2iJ_0}. \tag{2.208}$$

The Cartan-Killing metric is defined from the normalization of the generators:

$$\kappa_{ab} = 2\mathrm{Tr}[(iJ_a)(iJ_b)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.209}$$

Generator indices are (raised) and lowered using this (inverse) metric. The inverse Cartan-Killing metric defines the quadratic Casimir:

$$\boxed{\mathcal{C}_2 = -\kappa^{ab} iJ_a iJ_b = J_0^2 + \frac{1}{2}\{J_+, J_-\} \equiv -j(j+1)}. \tag{2.210}$$

This is seen to commute with all generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra, and labels the representation unambiguously by its eigenvalue $\mathcal{C}_2(j) = -j(j+1)$. The possibly complex number j is called the spin of the representation. It is readily seen that the fundamental generators Eq 2.207 span a spin-1/2 representation

To go beyond the defining spin-1/2 representation, more general spin-j representations can be projected on the real number line in terms of a square integrable function $f^j_\nu(x) = \langle x \mid j, \nu\rangle \in L^2(\mathbb{R})$, where $|x\rangle$ is introduced as a complete set of states in configuration space, defined in terms of the inner product on $L^2(\mathbb{R})$:

$$\langle j\nu \mid l\mu\rangle = \int_{\mathbb{R}} dx\, \langle j\nu \mid x\rangle\langle x \mid l\mu\rangle = \delta_{jl}\int_{\mathbb{R}} f^j_\nu(x)^* f^l_\mu(x). \tag{2.211}$$

On the space of square integrable functions, one introduces the principal series action of $\mathrm{SL}(2,\mathbb{R})$ Eq A.9:

$$\boxed{\langle x \mid g \mid j\nu\rangle = (g\cdot f^j_\nu)(x) = ((\hat{T}_j(g))f^j_\nu)(x) = |bx+d|^{2j} f^j_\nu\left(\frac{ax+c}{bx+d}\right)}. \tag{2.212}$$

The action on $L^2(\mathbb{R})$ is denoted as

$$\langle j\nu \mid g \mid l\nu\rangle = \delta_{jl}\int_{\mathbb{R}} f^j_\nu(x)^*(g\cdot f^l_\mu)(x). \tag{2.213}$$

Infinitesimally expanding into generators $g = \mathbf{1}+i\epsilon^a J_a$ defined in Eq 2.207, leads to the Borel-Weil realization of the $\mathfrak{sl}(2,\mathbb{R})$ algebra:

$$\boxed{iJ_- = \partial_x, \qquad iJ_0 = -x\partial_x + j, \qquad iJ_+ = -x^2\partial_x + 2jx} \tag{2.214}$$

, for which the quadratic Casimir yields explicitly $\mathcal{C}_2(j) = -j(j+1)$ (c.f. Eq A.12).
The generators J_a are hermitian with respect to the inner product on $L^2(\mathbb{R})$ if we constrain the value of the representation label to Eq A.13:

$$\boxed{j = -\frac{1}{2}+ik, \qquad k\in\mathbb{R}}. \tag{2.215}$$

This defines the unitary *principal continuous series representation*, labeled by the real parameter k.

The generators $iJ_0, iJ_\pm$ may be exponentiated to elements of $\mathrm{SL}(2,\mathbb{R})$ according to:

$$e^{2\phi iH} = \begin{pmatrix} e^{-\phi} & 0\\ 0 & e^{\phi}\end{pmatrix}, \qquad h_-(\gamma_-) = e^{iJ^-\gamma_-} = \begin{pmatrix} 1 & 0\\ \gamma_- & 1\end{pmatrix}, \qquad h_+(t) = e^{iJ^+\gamma_+} = \begin{pmatrix} 1 & \gamma_+\\ 0 & 1\end{pmatrix}. \tag{2.216}$$

From the general inequality $e^\phi + e^{-\phi} \geqslant 2$, we see that $e^{2\phi iH}$ parameterizes the hyperbolic group elements. On the other hand, $h_\pm$ label parabolic group elements. The Gauss parameterization covers the Poincaré patch of the $\mathrm{SL}(2,\mathbb{R})$ manifold:

$$g = e^{\gamma_- iJ_-}\, e^{\phi 2iJ_0}\, e^{\gamma_+ iJ_+} = \begin{pmatrix} 1 & 0\\ \gamma_- & 1\end{pmatrix}\begin{pmatrix} e^{-\phi} & 0\\ 0 & e^{\phi}\end{pmatrix}\begin{pmatrix} 1 & \gamma_+\\ 0 & 1\end{pmatrix}. \tag{2.217}$$

The Haar measure on this group manifold is given by the natural volume form Eq 2.150:

$$dg = e^{-2\phi}d\phi d\gamma_+ d\gamma_-. \tag{2.218}$$

The action of the left-parabolic group element h_-

$$\langle x \mid h_-(t) \mid f^k_{\nu_-}\rangle = (h_-(t)\cdot f^k_{\nu_-})(x) = f^k_{\nu_-}(x+t) \tag{2.219}$$

is readily diagonalized by the plane wave basis. We denote this state on $L^2(\mathbb{R})$ as:

$$\boxed{\langle x \mid \nu_- \rangle = e^{i\nu_- x}} \tag{2.220}$$

, which has the associated continuous eigenvalue $J_- = \nu_-$: $(h_-(t) \cdot f^k_{\nu_-})(x) = e^{i\nu_- t} f^k_{\nu_-}(x)$. The action of $h_+(t)$ on the carrier space is given by:

$$\langle x \mid h_+(t) \mid f^k_\nu \rangle = (h_+(t) \cdot f^k_{\nu_+})(x) = |tx+1|^{2j} f^j_{\nu_+}\left(\frac{x}{tx+1}\right) \tag{2.221}$$

, which is diagonalized by the right parabolic eigenvectors

$$\boxed{\langle x \mid \nu_+ \rangle = |x|^{2j} e^{i\nu_+/x} = |x|^{2ik-1} e^{i\nu_+/x}}. \tag{2.222}$$

Its associated continuous eigenvalue is labeled by $J_+ = \nu_+$:

$$(h_+(t) \cdot f^k_{\nu_+})(x) = e^{i\nu_+ t}(|x|^{2ik-1} e^{i\nu_+/x}). \tag{2.223}$$

The eigenvectors are transformed into each other by application of $\omega \in \mathrm{SL}(2,\mathbb{R})$:

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rightarrow \quad \omega\,|\nu_-\rangle = |-\nu_+\rangle\,. \tag{2.224}$$

We can consider the matrix element of a general Gauss-parameterized group element

$$g(\phi, \gamma_-, \gamma_+) = e^{\gamma_- i J_-}\ e^{\phi 2i J_0}\ e^{\gamma_+ i J_+} \tag{2.225}$$

evaluated between a left and right parabolic eigenvector. Being eigenvectors of respectively $J_\pm$, we can write directly (using the hermiticity of J_-)

$$\langle \nu_- \mid g(\phi, \gamma_-, \gamma_+) \mid \nu_+ \rangle = e^{i\gamma_- \nu_-} e^{i\gamma_+ \nu_+} \langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+ \rangle\,. \tag{2.226}$$

The matrix element diagonalizes up to the hyperbolic matrix element $\langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+ \rangle$, which we call the *Whittaker function*. The left- and right parabolic eigenstates are called *Whittaker vectors*. We calculate the total matrix element in terms of the modified Bessel functions of the second kind (c.f. Eq A.28):

$$\boxed{\langle \nu_- \mid g(\phi, \gamma_-, \gamma_+) \mid \nu_+ \rangle = e^{i\gamma_- \nu_-} e^{i\gamma_+ \nu_+}\ e^{\phi} \cosh(\pi k) \left(\frac{\nu_+}{\nu_-}\right)^{ik} K_{2ik}\left(\sqrt{\nu_- \nu_+} e^{\phi}\right)}. \tag{2.227}$$

From the orthogonality relation Eq A.29

$$\int_0^\infty \frac{dx}{x} K_{2ik}(x) K_{2ik'}(x)^* = \int_{-\infty}^{+\infty} d\phi K_{2ik}(e^\phi) K_{2ik'}(e^\phi)^* = \frac{\pi^2}{8k\sinh(2\pi k)} \delta(k-k') \tag{2.228}$$

, the Plancherel measure of the continuous series representation is given by Eq A.31:

$$\boxed{\rho(k) = k\tanh(\pi k)}. \tag{2.229}$$

We could also have chosen to diagonalize the hyperbolic group element $g(t) = e^{2itJ_0}$. The eigenvectors in this case are Eq A.33:

$$\boxed{f_s^k(x) = \langle x \mid s, \pm\rangle = \frac{1}{\sqrt{2\pi}}\,(\pm x)^{is-1/2}, \qquad \langle s, \pm \mid x\rangle = \frac{1}{\sqrt{2\pi}}(\pm x)^{-is-1/2}, \quad \pm x > 0.} \tag{2.230}$$

, with eigenvalue $iJ_0 = i(k-s)$. This hyperbolic basis naturally decomposes into a basis of states associated to either $\mathbb{R}^{\pm}$. Furthermore, the states $|s, \pm\rangle$ are complete and orthogonal on both half-lines $\mathbb{R}^{\pm}$. This allows us to write generic matrix elements as a 2×2 matrix in terms of the components $K^{\pm\pm}_{s_1 s_2}(g) \equiv \langle s_1, \pm \mid g \mid s_2, \pm\rangle$:

$$\mathbf{K}(g) = \begin{pmatrix} K^{++} & K^{+-} \\ K^{-+} & K^{--} \end{pmatrix} \tag{2.231}$$

This matrix composes under group multiplication in terms of matrix multiplication $\mathbf{K}(g_1 \cdot g_2) = \mathbf{K}(g_1)\mathbf{K}(g_2)$ [22], and has as its inverse $\mathbf{K}(g^{-1}) = \mathbf{K}(g)^{-1}$ [24].

2.8.2 Representation theory of $\mathrm{SL}^+(2,\mathbb{R})$

Group elements $g \in \mathrm{SL}^+(2,\mathbb{R})$ are still represented by sets of 2×2 matrices with unit determinant Eq 2.206 with the additional restriction that all matrix entries are strictly positive $a, b, c, d > 0$. This satisfies both closure, the existence of the identity element and associativity, but has no proper inverse. This defines a semi-group. The inverse is however well defined from the parent $\mathrm{SL}(2,\mathbb{R})$ manifold, such that we refer to it as a subsemigroup.

The carrier space of the principal series representation is now over the square integrable functions on the positive half line $L^2(\mathbb{R}^+)$, whose inner product is constrained to $x > 0$:

$$\langle j\nu \mid l\mu\rangle = \int_{\mathbb{R}^+} dx\ \langle j\nu \mid x\rangle\langle x \mid l\mu\rangle = \delta_{jl}\int_{\mathbb{R}^+} f_\nu^j(x)^* f_\mu^l(x). \tag{2.232}$$

We again define the *principal continuous series representation* of $\mathrm{SL}^+(2,\mathbb{R})$ on $L^2(\mathbb{R}^+)$ in terms of the *Borel-Weil* action, but restricted to positive $x > 0$:

$$\langle x \mid g \mid j\nu\rangle = (g \cdot f_\nu^j)(x) = |bx+d|^{2j} f_\nu^j\left(\frac{ax+c}{bx+d}\right), \qquad x > 0. \tag{2.233}$$

The corresponding infinitesimal generators $iJ_0, iJ_\pm$ of $\mathfrak{sl}(2,\mathbb{R})$ are still given by the Borel-Weil generators

Eq 2.214. The Gauss parameterization now has the convenient property that it covers the entire $\mathrm{SL}^+(2,\mathbb{R})$ subsemigroup manifold by limiting $\gamma_+, \gamma_- > 0$.

The left and right parabolic states on $\mathbb{R}^+$, corresponding to eigenfunctions of iJ_- and iJ_+ with the complex eigenvalues $J_- = i\nu_-$ and $J_+ = i\nu_+$, are given by Eqs 2.220, 2.222 in the region $x > 0$, by shifting the eigenvalues under $J_\pm$: $\nu_- \to i\nu_-$, $\nu_+ \to i\nu_+$

$$\langle x \mid \nu_- \rangle = f^j_{\nu_-}(x) = e^{-\nu_- x}, \qquad \langle x \mid \nu_+ \rangle = f^j_{\nu_+}(x) = x^{2ik-1} e^{-\nu_+/x}. \tag{2.234}$$

The imaginary shift to exponentially damped basis states is a more natural choice than the plane wave basis, since the latter is strictly speaking not even in $L^2(\mathbb{R}^+)$.
Using the adjoint of J_-, the mixed parabolic matrix elements are readily computed using the Gauss parametrization. This matrix element again diagonalizes up to the Whittaker function Eq A.49

$$R_{k,\nu_-\nu_+}(g) = \langle \nu_- \mid g(\phi, \gamma_+, \gamma_-) \mid \nu_+ \rangle = e^{\nu_-\gamma_-} e^{-\nu_+\gamma_+} \langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+ \rangle \tag{2.235}$$

, which is calculated explicitly as Eq A.51:

$$\boxed{R_{k,\nu_-\nu_+}(g) = e^{\nu_-\gamma_-} e^{-\nu_+\gamma_+} e^{\phi} \left(\frac{\nu_+}{\nu_-}\right)^{ik} K_{2ik}(\sqrt{\nu_-\nu_+} e^{\phi})}. \tag{2.236}$$

Normalized with respect to the Haar measure $dg = e^{-2\phi} d\phi$, the Plancherel measure is calculated as Eq A.52:

$$\boxed{\rho(k) \simeq k \sinh(2\pi k)}. \tag{2.237}$$

An additional subtlety is that the parabolic states do not constitute a complete delta-normalizable basis on $\mathrm{SL}^+(2,\mathbb{R})$. This property is reserved for the hyperbolic basis Eq A.41:

$$f^k_s(x) = \langle x \mid s \rangle = \frac{1}{\sqrt{2\pi}} x^{is-1/2}, \qquad f^k_s(x)^* = \langle s \mid x \rangle = \frac{1}{\sqrt{2\pi}} x^{-is-1/2}. \tag{2.238}$$

The corresponding matrix elements on $\mathbb{R}^+$ are now restricted to $K^{++}_{s_1,s_2}(g)$:

$$K^{++}_{s_1 s_2}(g) = \langle s_1 \mid g \mid s_2 \rangle = \frac{1}{2\pi} \int_0^{+\infty} dx\, x^{-is_1-1/2} (g \cdot x^{is_2-1/2}) \tag{2.239}$$

since for positive $g > 0$, the action of g on $f \in L^2(\mathbb{R}^2)$ cannot change the sign in the integral. As a consequence, the overlaps vanish in this case $\langle s_1, \pm \mid g \mid s_2, \mp \rangle \equiv 0$ [44].
The matrix composition law $\mathbf{K}(g_1 \cdot g_2) = \mathbf{K}(g_1)\mathbf{K}(g_2)$ is now constrained to K^{++}, and yields a proper representation of irreps on $\mathrm{SL}^+(2,\mathbb{R})$:

$$K^{++}_{ab}(g_1 \cdot g_2) = \int_{-\infty}^{+\infty} ds\, K^{++}_{as}(g_1) K^{++}_{sb}(g_2).$$

2.8.3 SL$^+(2,\mathbb{R})$ subsemigroup structure of JT gravity

An obvious issue regarding a (constrained) SL$(2,\mathbb{R})$ BF description of JT quantum gravity is the fact that the partition function over the coset Eq 2.204 with Plancherel measure given in Eq A.31 $\rho(k) = k\tanh(\pi k)$ does not match the direct calculation in the Schwarzian theory perspective Eq 1.140. In particular, by defining a momentum variable $E \equiv k^2/2C$, we would expect the density of states $\rho(E)dE = \rho(k)dk$ to be given by

$$\rho(E) = \sinh(2\pi\sqrt{2CE}) \tag{2.240}$$

, while SL$(2,\mathbb{R})$ BF theory predicts an asymptotically flattening slope $\rho(E) = \tanh(\pi\sqrt{2CE})$. At high energies $E \gg 1/C$, the latter violates the Cardy-scaling of the classical solution Eq 1.107 $\rho(E) \propto e^{2\pi\sqrt{2CE}}$. Therefore, an SL$(2,\mathbb{R})$ description cannot predict the black hole entropy due to a lack of microstates. With the classical regime, one means the regime where the relevant energies are much larger than the Newton's constant.

The specific exponentiation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra is not a priori obvious. The subsemigroup SL$^+(2,\mathbb{R})$ was first proposed in [22], by noting that the corresponding Plancherel measure Eq A.52 $\rho(k) = k\sinh(2\pi k)$ matches precisely with the JT density of states. However, different exponentiations might lead to the same desired Plancherel measure. In particular, [23] investigated the same amplitudes using a suitable limit of the universal covering group of SL$(2,\mathbb{R})$.
An insightful argument showcasing the discrepancy between BF gauge theory and gravity was given in [40]. Here, it was argued that the transition from SL$(2,\mathbb{R})$ BF theory to SL$^+(2,\mathbb{R})$ stems from another *global* constraint on the physical gravity solutions. In particular, constrained SL$(2,\mathbb{R})$ BF theory describes Schwarzian mechanics on the gravitational coset

$$I = -\frac{1}{2}\int_0^\beta d\tau\, \{F(\tau),\tau\} \tag{2.241}$$

, where we restrict F to satisfy the trivial monodromy constraint $F(\tau+\beta) = F(\tau)$ in terms of a boundary reparametrization mode $f(\tau)$: $F(\tau) = \tan\frac{\pi}{\beta}f(\tau)$. However, there exists an infinite family of solutions $f(\tau+\beta) = f(\tau) + n\beta$ $(n \in \mathbb{N})$ satisfying this constraint, leading to the over-complete partition function:

$$Z(\beta) = \sum_{n=1}^{\infty}\int_{f(\tau+\beta)=f(\tau)+n\beta} \mathcal{D}f\, e^{\frac{1}{2}\int_0^\beta d\tau\,\left\{\tan\frac{\pi}{\beta}f(\tau),\tau\right\}} = \sum_{n=1}^{\infty}(-)^{n-1}n\left(\frac{\pi}{\beta}\right)^{3/2} e^{\frac{\pi^2}{\beta}n^2}.$$

The negative modes are taken into account in the one-loop evaluations of the Schwarzian partition functions. Performing an inverse Laplace transform leads to the density of states[12]

$$Z(\beta) = \int_0^\infty dk\left(2\sum_{n=1}^{\infty}(-)^{n-1}k\sinh(2\pi nk)\right)e^{-\beta k^2} = \int dk(k\tanh\pi k)e^{-\beta k^2} \tag{2.242}$$

, where we indeed recognize the Plancherel measure of pure SL$(2,\mathbb{R})$ Eq A.31. However, this cannot be the physical gravitational answer, since every $n > 1$ corresponds to a replicated solution with a conical singularity of $2\pi n$. Restricting to smooth geometries thereby additionally constrains the global structure to $n \equiv 1$, whose

[12] Making use of the regularized $q \to 0$ limit of $2\sum_{n=1}^{\infty}(-)^{n-1}e^{-2\pi qn}k\sinh(2\pi nk) = \frac{k\sinh(2\pi k)}{\cosh(2\pi q)+\cosh(2\pi k)}$.

dynamics are that of the gravitational coset on $\mathrm{SL}^+(2,\mathbb{R})$. The constraint $n = 1$ naturally implements the restriction in the integration range of the path integral to smooth geometries in Teichmüller space.
In [24], the argument in favour for the subsemigroup $SL^+(2,\mathbb{R})$ was fine-tuned further by noting that quantum mechanics on $\mathrm{SL}^+(2,\mathbb{R})$ automatically excludes non-singular metrics. This choice is again not a priori imposed in the path integral, but should hold for any sensible theory of quantum gravity. Thereby, restricting to $\mathrm{SL}^+(2,\mathbb{R})$ directly leads to the correct consistency requirements.
Let me elaborate more on this fact; a non-trivial holonomy in the closed slicing might persist due to defect insertions. On the gravity side, these defects are related to non-trivial monodromies when traveling around the thermal boundary circle. These have all been studied and classified in [37], where it was found that the three different classes of character insertions *hyperbolic* ($\mathrm{Tr}(U) > 2$), *parabolic* ($\mathrm{Tr}(U) = 2$) and *elliptic* ($\mathrm{Tr}(U) < 2$) (c.f. Eq A.3) lead to wormhole geometries, cusp- and conical singularities respectively. Only the former has a smooth behaviour in the metric. Therefore, if we are only interested in smooth invertible geometries, we have to exclude both parabolic and elliptic conjugacy classes. The restriction to the subsemigroup, where all matrix entries a, b, c, d in the group element g Eq A.1

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \qquad a, b, c, d > 0 \tag{2.243}$$

are strictly positive naturally implements this restriction, since the constraint $ad - bc = 1$ leads to $d > 1/a$ due to $b, c > 0$. Therefore, $\mathrm{Tr}(U) = a + d > a + 1/a \geqslant 2$ automatically constrains to the regular hyperbolic conjugacy class elements.

From the BF perspective, integrating over $\mathbf{B}$ reduces the path integral to the volume of the moduli space of flat connections $\mathbf{F} = 0$. Since each flat connection is specified by its holonomy around a non-trivial cycle, the moduli space is divided by the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. The restriction to smooth configurations (hyperbolic component) specializes the moduli space to the Teichmüler space. A choice which as we just saw is naturally implemented by the subsemigroup. This result implies that the subsemigroup description is a sufficient condition to exclude geometries that contain conical singularities (elliptic) or cusps (parabolic). However, this does not necessarily imply that this restriction is also necessary in the sense that all smooth geometries are captured by flat $\mathrm{SL}^+(2,\mathbb{R})$ connections. In [24], argumentation in favour of this claim was provided by looking at the three-holed sphere.

A structural argument in favour of the subsemigroup concerns the dimensional reduction of 3d gravity, which itself is governed by the representation theory of the quantum group $\mathrm{SL}_q^+(2,\mathbb{R})$ [24] [89]. [24] checked that the Plancherel decomposition of the latter in the classical limit indeed reduces to the one corresponding to $\mathrm{SL}^+(2,\mathbb{R})$ JT gravity. A particularly attractive feature of restricting to the subsemigroup is that only the principal continuous series representation matrices P_k appear in the Plancherel decomposition:

$$\mathcal{L}^2(\mathrm{SL}^+(2,\mathbb{R})) = \int_{\oplus} dk k \sinh(2\pi k)\; P_k \otimes P_k. \tag{2.244}$$

This is unlike the argument of the central extension of the universal covering group $\tilde{\mathrm{SL}}(2,\mathbb{R})$ by $\mathbb{R}$ of [23], where the discrete series representation matrices also appear in the Plancherel decomposition. To effectively localize on the continuous series, the authors had to consider an analytically continued limit of the discrete series representation label to suppress the spurious contribution of the latter.

In the following, I will consider the approach of the subsemigroup. This structure naturally generalizes to the $\mathfrak{osp}(1|2,\mathbb{R})$ BF description of JT supergravity, whose quantum amplitudes can also be described by restricting to the positive subsemisupergroup $\mathrm{OSp}^+(1|2,\mathbb{R})$ [40].

2.8.4 Gravitational coset boundary conditions

With the representation theory presented in sections A.1, A.2, we are equipped to study quantum mechanics on the $\mathrm{SL}^+(2,\mathbb{R})$ subsemigroup. In particular, using the generalized set-up of section 2.7.2, we may write the propagator between two asymptotic states:

$$\langle g \mid e^{-\beta H} \mid h\rangle = \int ds_1 \int ds_2 \int dk \; \psi^k_{s_1 s_2}(g)\psi_{s_2 s_1}(h^{-1})e^{-\beta \mathcal{C}_2(k)}$$

, with the normalized continuous series matrix elements in the *hyperbolic* basis defined in Eq A.42,

$$\psi^k_{sr}(g) = \langle g \mid k, s, r\rangle \equiv \sqrt{k\sinh(2\pi k)}K^{++}_{k,s_1 s_2}(g). \tag{2.245}$$

However, we run into difficulties when considering the inverse $\psi_{s_2 s_1}(h^{-1})$, since $h \in \mathrm{SL}^+(2,\mathbb{R})$ has no natural inverse on $\mathrm{SL}^+(2,\mathbb{R})$. However, we should think of $\mathrm{SL}^+(2,\mathbb{R})$ as a subsemigroup of $\mathrm{SL}(2,\mathbb{R})$. Thus, although the inverse is not contained in $\mathrm{SL}^+(2,\mathbb{R})$, it is well-defined.

More precisely, Eq A.36 states that the hyperbolic basis of $\mathrm{SL}(2,\mathbb{R})$ naturally decomposes into a basis of states associated to either the positive or negative half-line $\mathbb{R}^\pm$, which we represent by a 2×2 matrix

$$\mathbf{K}(g) = \begin{pmatrix} K^{++} & K^{+-} \\ K^{-+} & K^{--} \end{pmatrix}.$$

This matrix composes under matrix multiplication $\mathbf{K}(g_1\cdot g_2) = \mathbf{K}(g_1)\mathbf{K}(g_2)$, and has as its inverse $\mathbf{K}(g^{-1}) = \mathbf{K}(g)^{-1}$.

On $\mathrm{SL}^+(2,\mathbb{R})$, $|s,+\rangle$ forms a complete set of basis states, and the matrix element is restricted to $K^{++}_{s_1 s_2}(g) = \langle s_1, + \mid g \mid s_2, +\rangle$ (c.f. Eq A.42). Using the fact that $K^{\pm\mp}_{s_1 s_2}(g) = 0$ for positive g [90], $\mathbf{K}$ is a diagonal matrix whose inverse is the inverse of the diagonal entries. Specifically,

$$K^{++}_{s_1 s_2}(g)^{-1} = K^{++}_{s_1 s_2}(g^{-1}). \tag{2.246}$$

Thus, the inverse representation matrix is naturally defined on $\mathrm{SL}^+(2,\mathbb{R})$ from the parent $\mathrm{SL}(2,\mathbb{R})$ manifold. Since the matrix elements can be shown to be unitary $K^{++}_{\beta\alpha}(g)^* = K^{++}_{\alpha\beta}(g)^{-1}$ [24], this leads to

$$K^{++}_{\alpha\beta}(g)^* = K^{++}_{\alpha\beta}(g^{-1}). \tag{2.247}$$

Furthermore using $K^{\pm\mp}_{s_1s_2}(g) = 0$, matrix multiplication of $\mathbf{K}$ directly yields the composition law:

$$\int ds_1 ds_2 K^{++}_{s_1s_2}(g) K^{++}_{s_2s_1}(h^{-1}) = \int ds K^{++}_{ss}(gh^{-1}) = \chi_k(g \cdot h^{-1}). \tag{2.248}$$

More generally, we have the composition between SL$(2,\mathbb{R})$ and SL$^+(2,\mathbb{R})$ valued elements [24]:

$$\int ds K^{++}_{s_1s}(g) K^{++}_{ss_2}(h) = K^{++}_{s_1s_2}(g \cdot h), \qquad h \in SL^+(2,\mathbb{R}), \;\; g \in SL(2,\mathbb{R}). \tag{2.249}$$

This leads to the final disk partition function

$$\langle g \mid e^{-\beta H} \mid h\rangle = \int dk\, \rho(k) \chi_k(g \cdot h^{-1}) e^{-\beta \mathcal{C}_2(k)} \tag{2.250}$$

, with the continuous hyperbolic character determined below in Eq 2.323.
However, this is not the end of the story, since we have argued that JT gravity is in fact described as a coset on SL$^+(2,\mathbb{R})$ with constrained boundary conditions Eq 2.175 $\mathcal{J}^- = \pi_+ \equiv 1$. Using the identifications Eq 2.159, they are part of the parabolic eigenbasis with eigenvalue $J_+ \equiv i$,

$$J_+ = i\mathcal{J}^- \equiv i. \tag{2.251}$$

The eigenstates of J_+ are given in Eq A.48 by the right Whittaker vector (with eigenvalue $J_+ = i\nu_+$):

$$\langle x \mid \nu_+\rangle = f^j_{\nu_+}(x) = x^{2ik-1} e^{-\nu_+/x}.$$

The constraints on the conjugate state readily follow from the adjoint action of J_+ (deduced from the parent SL$(2,\mathbb{R})$):

$$\langle 1_+ | \, J_+ = (J_+ \, |1_+\rangle)^\dagger = (i \, |1_+\rangle)^\dagger = -i \, \langle 1_+ | \, .$$

Thereby, the adjoint state is constrained by the eigenvalue of $J_+ = -i$ acting on the right. Following the discussion in section 2.7.1, states on an interval reaching the boundary on either side should be expanded in terms of wavefunctions with the fixed labels ingrained:

$$R_{k,-1_+1_+}(g) = \langle k, -1_+ \mid g(\phi, \gamma_+, \gamma_-) \mid k, 1_+\rangle \, . \tag{2.252}$$

Using the Gauss decomposition of g Eq A.40, it rewards to work in a mixed parabolic basis by writing the group element as $g \equiv \omega \cdot g'$, with ω defined as in Eq A.20. This is always possible in the integral, since the Haar measure is invariant under shifts $g \to \omega g$. Since ω acts on the states as $\omega \, |\nu_+\rangle = |-\nu_-\rangle$, this mixes the right parabolic basis into the left parabolic basis. We hence consider matrix elements in the *mixed* parabolic basis. Combined with the natural coset boundary conditions, the matrix elements under consideration are:

$$R_{k,1_-1_+}(g) = \langle k, 1_- \mid g \mid k, 1_+\rangle \, . \tag{2.253}$$

Matrix elements in the mixed parabolic basis with eigenvalues determined by the coset constraint transform covariantly under both $e^{\gamma_- iJ_-}$ and $e^{\gamma_+ iJ_+}$ (c.f. Eq A.49):

$$R_{k,1_-1_+}(g) = \langle k, 1_- \mid g \mid k, 1_+ \rangle = e^{\gamma_- - \gamma_+} \langle k, 1_- \mid e^{2i\phi J_0} \mid k, 1_+ \rangle . \tag{2.254}$$

The term within brackets is called the Whittaker function and is determined in Eq A.51:

$$R_{k,1_-1_+}(g) = e^{\gamma_- - \gamma_+}\, e^{\phi} K_{2ik}(e^{\phi}) \tag{2.255}$$

, normalized with Plancherel measure Eq A.52:

$$\rho(k) = k \sinh(2\pi k). \tag{2.256}$$

The answer for the continuous series matrix elements should be compared to the normalizable Liouville minisuperspace solutions of the Casimir eigenvalue equation with constrained coset boundary conditions in Eq 2.168. Since the matrix elements are diagonal in $\gamma_\pm$, and the Haar measure Eq 2.4 $dg = e^{-2\phi} d\phi d\gamma_+ d\gamma_-$ is parametrically independent of their values, their dependence will drop in the total integral over $d\gamma_\pm$, yielding an irrelevant constant. We can therefore strip off their dependence immediately and consider the properly normalized wavefunctions in the Hilbert space determined by the Peter-Weyl theorem:

$$\boxed{\langle \phi \mid k \rangle = \sqrt{k \sinh(2\pi k)}\; R_{k,1_-1_+}(\phi) = \sqrt{k \sinh(2\pi k)}\; e^{\phi} K_{2ik}(e^{\phi})}. \tag{2.257}$$

The vectors $|k\rangle$ constitute the representation basis in the Peter-Weyl theorem (c.f. Eq 2.201), while the hyperbolic group elements $|\phi\rangle$ take the role of the configurational group elements.

2.8.5 Thermal partition function

Having established JT gravity as a constrained $\mathrm{SL}^+(2,\mathbb{R})$ BF theory, we may finally calculate some explicit amplitudes, along the lines of section 2.5. To avoid repeating myself, I will not go over the details again.
We write the disk partition function as the Hamiltonian propagation between two asymptotic states:

$$Z_{\text{disk}}(\phi_i, \phi_f) = \langle \phi_f \mid e^{-\beta H} \mid \phi_i \rangle . \tag{2.258}$$

We diagonalize the Hamiltonian in the coset formalism by inserting a complete set of representation states in the mixed parabolic basis:

$$Z_{\text{disk}}(\phi_i, \phi_f) = \int_0^\infty dk\; \langle \phi_f \mid k \rangle \langle k \mid \phi_i \rangle e^{-\beta \mathcal{C}_2(k)} = \int_0^\infty dk \rho(k)\; R_{k,1_-1_+}(\phi_f) R_{k,1_-1_+}(\phi_i)^* e^{-\beta \mathcal{C}_2(k)}.$$

The quadratic Casimir is defined in terms of the quantum number j in Eq A.7. Inserting $j = -\frac{1}{2} + ik$ for unitarity (c.f. Eq A.13) yields:

$$\mathcal{C}_2(k) = -j(j+1) = \frac{1}{4} + k^2.$$

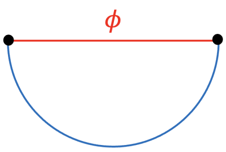
The factor $1/4$ may be dropped in the expectation values when normalizing the partition function [22]. Of course, the fact that we identify the Hamiltonian of our system with the quadratic Casimir in the context of JT is no coincidence, since this was already established at the classical level in Eq 1.103.

Physical boundary segments are ultimately characterized by a trivial holonomy. In the mixed parabolic basis, we cannot simply set $g = \mathbf{1}$ in the expectation values since $\langle k, 1_- \mid \mathbf{1} \mid k, + \rangle = \langle k, 1_- \mid k, 1_+ \rangle$ yields a non-trivial overlap Eq A.26. Instead, the identity element is given by the matrix ω^{-1} that interchanges left and right eigenstates (c.f. Eq A.20). In terms of the Gauss decomposition, its locus is parameterized by $\phi \to +\infty$, $\gamma_- = -\gamma_+ = e^{\phi}$, as can readily be checked using the explicit parameterization Eq A.40. Using the asymptotics of the modified Bessel function [90] in this limit $R_{k,1_-1_+}(\phi) \to 1$ (up to some immaterial constant), the final disk amplitude is given by:

$$\boxed{Z_{\text{disk}}(\beta) = \int_0^{\infty} dk\, k \sinh(2\pi k) e^{-\beta k^2}}. \tag{2.259}$$

This not only matches with the classical large c-limit of the torus partition function of the Virasoro algebra [21], but also matches with the one-loop exact partition function obtained in Eq 1.138, up to some irrelevant constant that depends on the scheme of renormalization.
Alternatively, we end up with this result by gluing two asymptotic half disks (c.f. Eq 2.126) along a common group element. In the gravitational context, these are often refered to as *Hartle-Hawking* states preparing the vacuum:

$$Z_{\text{Hartle}}(\phi, \beta') =$$



$$= \langle \phi \mid e^{-\beta' H} \mid \mathbf{1} \rangle = \int_0^{\infty} dk\, k \sinh 2\pi k\; e^{\phi} K_{2ik}(e^{\phi}) e^{-\beta' k^2}. \tag{2.260}$$

Due to the orthogonality theorem

$$\int_{-\infty}^{+\infty} d\phi e^{-2\phi} R_{k_1,1_-1_+}(\phi) R_{k_2,1_-1+}(\phi)^* = \frac{1}{k_1 \sinh 2\pi k_1} \delta(k_1 - k_2) \tag{2.261}$$

, we obtain the disk partition function by gluing along ϕ with the Haar measure Eq 2.150 $dg = e^{-2\phi} d\phi$:

$$Z_{\text{disk}}(\beta_1 + \beta_2) = \int_{-\infty}^{+\infty} d\phi\; e^{-2\phi} Z_{\text{Hartle}}(\phi, \beta_1) Z_{\text{Hartle}}(\phi, \beta_2)^*. \tag{2.262}$$

This is equivalent to inserting a complete set of group elements $\mathbf{1} = \int d\phi\; e^{-2\phi} \left|\phi\right\rangle\left\langle\phi\right|$ in Eq 2.258.

2.8.6 Wilson line insertion

Wilson lines in JT gravity are evaluated in the lowest-weight discrete series representation of $\mathrm{SL}^+(2,\mathbb{R})$, where the object $\mathcal{W}_{R,nm}(\tau_1,\tau_2)$ is defined as Eq 2.118:

$$\mathcal{W}_{R,nm}(\tau_1,\tau_2) = \mathcal{P}\exp\left(-\int_{\tau_1}^{\tau_2} d\tau\ R(\mathbf{A})\right)_{nm}. \tag{2.263}$$

In section 2.5.3, we have interpreted these objects in holography as the bilocal operators Eq 2.138 $\mathcal{O}_{R,nm}(\tau_1,\tau_2) = R_{nm}(g(\tau_2)g^{-1}(\tau_1))$ in the particle-on-a group theory.
Using the constrained set-up of section 2.6.2, we can show that by choosing the discrete series representation with $j=-\ell$, the bilocal operator in constrained $\mathrm{SL}^+(2,\mathbb{R})$ particle-on-group can be identified with the holographic Schwarzian bilocal operator determined in Eq 1.166.
The usual coset boundary restriction $\mathcal{J}^- \equiv 1$ can be rephrased in coordinates as Eq 2.176 $e^{-2\phi}\gamma_-' = 1$. As elaborated in section 2.183, a convenient way to obtain the Schwarzian derivative directly from particle-on-a-group is to additionally gauge fix $\gamma_+ = -\frac{1}{2}\frac{\gamma_-''}{\gamma_-'}$, leading to Eq 2.181:

$$\mathbf{A}_\tau = g\partial_\tau g^{-1} = \mathcal{J} \equiv iJ_- - \frac{1}{2}T(\tau)iJ_+ = \begin{pmatrix} 0 & -\frac{1}{2}T(\tau) \\ 1 & 0 \end{pmatrix} \tag{2.264}$$

with $T = \{\gamma_-,\tau\}$ the Schwarzian derivative. Inserted in the particle-on-a-group action automatically yields the Schwarzian derivative:

$$I = \frac{1}{2}\int \mathrm{Tr}(\mathcal{J}\mathcal{J}) = -\frac{1}{2}\int d\tau T(\tau) = -\frac{1}{2}\int d\tau\ \{\gamma_-,\tau\}. \tag{2.265}$$

This leads to an interpretation of the Schwarzian reparametrization mode F in terms of the $\mathrm{SL}^+(2,\mathbb{R})$ group coordinate $\gamma_- \equiv F$. In hindsight, we may rephrase the coset and gauge constraints as:

$$\boxed{\gamma_- = F, \qquad e^{-\phi} = \frac{1}{\sqrt{F'}}, \qquad \gamma_+ = -\frac{1}{2}\frac{F''}{F'}}. \tag{2.266}$$

Alternatively, this identification may be found directly by considering the ansatz Eq 2.264, and parameterizing $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, leading to the Hill's equations:

$$\begin{aligned} A'' + \frac{1}{2}T(\tau)A = 0, \quad B = A' \\ C'' + \frac{1}{2}T(\tau)C = 0, \quad D = C' \end{aligned} \tag{2.267}$$

, along with the $\mathrm{SL}^{(+)}(2,\mathbb{R})$ constraint $AC' - A'C = 1$. The unique solutions, up to Möbius transformations, are:

$$A = \frac{1}{\sqrt{F'}}, \qquad C = \frac{F}{\sqrt{F'}}. \tag{2.268}$$

with F the solution to $\{F,\tau\} = T(\tau)$, as can readily be checked. Identifying these parameters with the Gauss parametrization of

$$g^{-1} = e^{\gamma_- iJ_-}\ e^{\phi 2iJ_0}\ e^{\gamma_+ iJ_+} = \begin{pmatrix} e^{-\phi} & e^{-\phi}\gamma_+ \\ e^{-\phi}\gamma_- & e^{\phi} + e^{-\phi}\gamma_+\gamma_- \end{pmatrix} \tag{2.269}$$

leads to the same solutions Eq 2.266.

A lowest-weight discrete series module is constructed by acting on the lowest-weight state defined as

$$\langle x \mid \ell, 0\rangle = \frac{1}{x^{2\ell}}, \qquad \langle -\ell, 0 \mid x\rangle = \delta(x). \tag{2.270}$$

The weight under iJ_0 defined from the Borel-Weil realization of $\mathfrak{sl}(2,\mathbb{R})$ Eq A.11

$$iJ_- = \partial_x, \qquad iJ_0 = -x\partial_x + j, \qquad iJ_+ = -x^2\partial_x + 2jx \tag{2.271}$$

reads $j = -\ell$. A lowest-weight module is created by acting on the lowest-weight state with the raising[13] operators iJ_-. I will comment on the discrete series representation theory in more detail in the next chapter. From the discussion on Wilson lines in the context of cosets in section 2.7.1, current conservation imposes that the only relevant states that reach the boundary are in fact the lowest-weight states. Indeed, sandwiched between two asymptotic states with constrained eigenvalue under $\mathcal{J}^-$, the Clebsch-Gordan coefficients $C^{R_1,1}_{R,R_2;m,1} = \langle R_1, 1 \mid R, R_2; m, 1\rangle$ appearing in integrals like Eq 2.196 only yield a non-vanishing result for $m, n \equiv 0$. Evaluated in a mixed parabolic basis, this is $\mathcal{W}_R \equiv \mathcal{W}_{R,00}$. In the holographic interpretation, the bilocal operators are evaluated by inserting a complete set of states in configuration space, and using the specified lowest-weight configurations

$$\mathcal{O}^\ell(\tau_1,\tau_2) = \langle -\ell, 0 \mid g(\tau_2)g^{-1}(\tau_i) \mid \ell, 0\rangle = \int dx\ \delta(x) \left(g(\tau_2)g^{-1}(\tau_1) \cdot \frac{1}{x^{2\ell}}\right). \tag{2.272}$$

The eigenvalue of the adjoint lowest-weight state is $-\ell$, as dictated by the anti-hermiticity of iJ_0. This matrix element can be solved by writing out explicitly the Gauss decomposition Eq 2.148 $g^{-1} = e^{\gamma_- iJ_-}\ e^{\phi 2iJ_0}\ e^{\gamma_+ iJ_+}$ in terms of the reparametrization mode F:

$$g^{-1} = \begin{pmatrix} e^{-\phi} & e^{-\phi}\gamma_+ \\ e^{-\phi}\gamma_- & e^{\phi} + e^{-\phi}\gamma_+\gamma_- \end{pmatrix} = \frac{1}{\sqrt{F'}} \begin{pmatrix} 1 & -\frac{1}{2}\frac{F''}{F'} \\ F & F' - \frac{1}{2}\frac{F''}{F'}F \end{pmatrix} \tag{2.273}$$

$$g = \begin{pmatrix} e^{\phi} + e^{-\phi}\gamma_-\gamma_+ & -e^{-\phi}\gamma_+ \\ -e^{-\phi}\gamma_- & e^{-\phi} \end{pmatrix} = \begin{pmatrix} \sqrt{F'} - \frac{1}{2}\frac{F''}{(F')^{3/2}}F & \frac{1}{2}\frac{F''}{F'^{3/2}} \\ -\frac{F}{\sqrt{F'}} & \frac{1}{\sqrt{F'}} \end{pmatrix}. \tag{2.274}$$

The spin $j = -\ell$ principal series action Eq A.9 generated by the $\mathfrak{sl}(2,\mathbb{R})$ Borel-Weil generators Eq A.11 yields:

$$\frac{1}{x^{2\ell}} \xrightarrow{g^{-1}(\tau_1)} \frac{F'(\tau_1)^\ell}{(x + F(\tau_1))^{2\ell}} \xrightarrow{g(\tau_2)} \frac{F'(\tau_1)^\ell F'(\tau_2)^\ell}{\left(F(\tau_1) - F(\tau_2) + \left(F'(\tau_2) + \frac{F(\tau_1)}{2}\frac{F''(\tau_2)}{F'(\tau_2)} - \frac{F(\tau_2)}{2}\frac{F''(\tau_2)}{F'(\tau_2)}\right)x\right)^{2\ell}}.$$

Inserted in Eq 2.272 sets $x = 0$, yielding exactly the Schwarzian bilocal operator Eq 1.166 with the character-

[13] This is not a typo. Contrary to intuition, the $\mathfrak{sl}(2,\mathbb{R})$ algebra encountered in Eq 2.147 treats iJ_- as the raising operator and iJ_+ as the lowering operator.

istic $1d$ conformal scaling:

$$\boxed{\mathcal{O}^{\ell}(\tau_1,\tau_2) = \frac{F'(\tau_1)^{\ell}F'(\tau_2)^{\ell}}{(F(\tau_1)-F(\tau_2))^{2\ell}}}. \tag{2.275}$$

The discrete series label $\ell = -j$ is identified with the conformal scaling dimension Δ of the bilocal operator Eq 1.166 when we parameterize the uniformizing coordinate F in terms of the reparameterization mode of the thermal circle $f \in \text{diff}(S_1)$: $F(\tau) = \tan\left(\frac{\pi}{\beta}f(\tau)\right)$.
Moreover, since there exist no infinite energy configurations $T(\tau) < \infty$, F' cannot change sign. Choosing $F' > 0$ identifies the integration space with diff$(S_1)/\text{SL}(2,\mathbb{R})$. This proves the complete holographic equivalence between Schwarzian bilocal operators and Wilson lines in the holographic bulk evaluated in the discrete series representation of SL$^+(2,\mathbb{R})$. The group theoretic perspective now allows for a quantization of the latter directly in the bulk BF description.

Correlation functions may be calculated along the lines of section 2.5.2. In particular, starting from a Hamiltonian propagation between two asymptotic states with coset boundary conditions $J_+ = i$ in the presence of a Wilson line, and inserting a complete set of representation states to diagonalize the Hamiltonian, the calculation is equivalent to gluing two Hartle Hawking states and the Wilson line insertion $\mathcal{W}_\ell(\phi) = R_{\ell,00}(\phi)$ along a common group element

$$\langle \mathcal{W}_\ell(\tau_1,\tau_2)\rangle = \langle \mathbf{1} \mid e^{-\beta H}\mathcal{W}_{\ell,00} \mid \mathbf{1}\rangle = \int d\phi\, e^{-2\phi} Z_{\text{Hartle}}(\phi,\beta_1)^* Z_{\text{Hartle}}(\phi,\beta_2) R_{\ell,00}(\phi) \tag{2.276}$$

, with $\beta_1 \equiv |\tau_2-\tau_1|$, $\beta_2 = \beta - |\tau_2-\tau_1|$. This involves an integral of the type

$$\int_{-\infty}^{+\infty} d\phi\, e^{-2\phi} R_{k_1 1_+ 1_+}(\phi) R_{\ell,0_+0_+}(\phi) R_{k_2 1_+ 1_+}(\phi)^* = \begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix}^2$$

, in terms of the $3j$-symbols defined Eq 2.123. It again rewards switching to a mixed parabolic basis by shifting $g \to \omega \cdot g$. The integral itself remains unchanged since the Haar measure is invariant under shifts in ω;

$$\int_{-\infty}^{+\infty} d\phi\, e^{-2\phi} R_{k_1 1_- 1_+}(\phi) R_{\ell,0_-0_+}(\phi) R_{k_2 1_- 1_+}(\phi)^* = \begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix}^2. \tag{2.277}$$

To proceed, we need an explicit expression for the lowest-weight matrix element in the discrete series representation of SL$^+(2,\mathbb{R})$, evaluated in the mixed parabolic basis. It turns out that the diagonal states are given by modified Bessel functions of the first kind [90] [22], given by ($\nu > 0$):

$$R_{\ell,\nu_-\nu_+}(\phi) = e^{\phi} I_{2\ell-1}(\nu e^{\phi}). \tag{2.278}$$

Using the asymptotics of the Bessel function $I_\alpha(x) \sim x^{|\alpha|}$ as $x \to 0$, the lowest-weight is found in the limit of $\nu \to 0$ for $\ell > 1/2$ to be,

$$\boxed{R_{\ell,00}(\phi) = e^{2\ell\phi}}. \tag{2.279}$$

To evaluate the $3j$ symbol, we need the asymptotic states Eq 2.255, and the integral identity [22]

$$\int_{-\infty}^{+\infty} d\phi\, e^{2\ell\phi} K_{2ik_1}(e^{\phi}) K_{2ik_2}(e^{\phi})^* = \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)} \tag{2.280}$$

, where we understand $\Gamma(\ell \pm ik_1 \pm ik_2)$ to be a product of six gamma functions of all possible signs. The $3j$-symbol is then determined as:

$$\boxed{\begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix} = \sqrt{\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}}}. \tag{2.281}$$

Using the asymptotics of the physical boundary states ($\phi \to \infty : e^{\phi} K_{2ik}(e^{\phi}) \to 1$), the correlation function is readily obtained as the constrained continuous analogue of Eq 2.128

$$\boxed{\langle \mathcal{W}_\ell(\tau_1, \tau_2) \rangle = \int_0^\infty dk_1 k_1 \sinh(2\pi k_1) e^{-\beta_1 k_1^2} \int_0^\infty dk_2 k_2 \sinh(2\pi k_2) e^{-\beta_2 k_2^2} \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}} \tag{2.282}$$

, with $\beta_1 + \beta_2 \equiv \beta$. This is the structurally anticipated result in Eq 1.169, where we have now obtained the explicit matrix elements $|\langle E_1 \mid \mathcal{O} | E_2 \rangle|$ from first principles.
We can see how this result is consistent by taking the $\ell \to 0$ limit, which should correspond to the disk partition function. Splitting $\Gamma(\ell \pm ik_1 \pm ik_2) = \Gamma(\ell \pm (ik_1 + k_2))\Gamma(\ell \pm (ik_1 - k_2))$, and using the identity $\lim_{\ell \to 0} \frac{\Gamma(\ell \pm i(k_1 - k_2)))}{\Gamma(2\ell)} = 2\pi\delta(k_1 - k_2)$ on the second gamma factor leads to a single k_1-integral over $\Gamma(\pm 2ik_1)$. Using the identity $\Gamma(\pm 2ik_1) = \frac{\pi}{2k \sinh 2\pi k}$ removes one of the Plancherel measures, and recovers Eq 2.259.

As a final example, we should consider the bulk crossing of two Wilson lines. The set-up is essentially the same as in section 2.5.2, where we split the Hartle-Hawking states at the intersection point using the defining feature of representation matrices (c.f. $R_{A,m_A,m'_A}(g_3 g_1^{-1}) = \sum_\beta R_{A,m_A\beta}(g_3) R_{A,\beta m'_A}(g_1^{-1})$, and $R_{B,m_B,m'_B}(g_4 g_2^{-1}) = \sum_\beta R_{B,m_B\beta}(g_4) R_{B,\beta m'_B}(g_2^{-1})$) to obtain pie-shaped amplitudes. In the constrained set-up of $\mathrm{SL}^+(2,\mathbb{R})$, the intermediate bulk labels are free from the boundary constraints and can take any arbitrary value. A subtlety is that they are necessarily evaluated in the hyperbolic basis since this is the only complete set of states on $\mathrm{SL}^+(2,\mathbb{R})$, as elaborated in section A.2. For example, we decompose the mixed parabolic states along the hyperbolic parameter s:

$$R_{k,1_-1_+}(g_1 \cdot g_2) = \int_{-\infty}^{+\infty} ds\, R_{k,1_-s}(g_1) R_{k,s1_+}(g_2). \tag{2.283}$$

Gluing along four group elements in Eq 2.133 again leads to four $3j$-symbols at the bulk intersection, and four $3j$-symbols labeled at the constrained boundary. Integrating along the continuous hyperbolic labels at the bulk intersection leads to the Schwarzian $6j$-symbol [22]

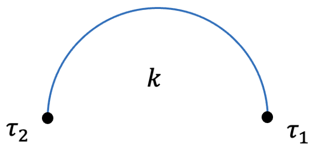
$$\begin{Bmatrix} R_B & R_1 & R_4 \\ R_A & R_3 & R_2 \end{Bmatrix} = \int \prod_i ds_i \begin{pmatrix} R_1 & R_2 & R_A \\ s_1 & s_2 & s_A \end{pmatrix} \begin{pmatrix} R_2 & R_3 & R_B \\ s_2 & s_3 & s_B \end{pmatrix} \begin{pmatrix} R_3 & R_4 & R_A \\ s_3 & s_4 & s_A \end{pmatrix} \begin{pmatrix} R_4 & R_1 & R_B \\ s_1 & s_2 & s_B \end{pmatrix}.$$

Using the classical limit of the q-deformed result by Ponsot and Teschner [89], the relevant $6j$-symbol was recovered in [22] in terms of the Wilson function defined in [21]. Barring the details, the explicit expression

can be found in the aforementioned papers.
This concludes the discussion on the exact amplitudes of bilocal operators in JT gravity in terms of the exact diagrammatic expressions defined in section 2.5.2. The latter only needs trivial modifications for the constrained continuous case at hand, and I summarize them here below.

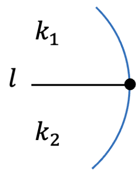
Diagrammatic expressions

- To each region of the disk, we assign a label k_i of the continuous series representation of $\mathrm{SL}^+(2,\mathbb{R})$, where each region contributes a weight $\rho(k) = k\sinh(2\pi k)$. This may be interpreted as the momentum flow along the boundary of length $\beta_1 = |\tau_2 - \tau_1|$, generated by the quadratic Casimir $\mathcal{C}_2(k) = k^2$:



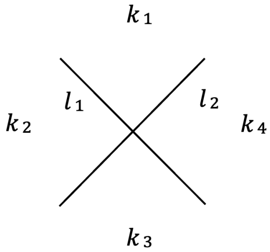
$$= e^{-\beta_1 k^2} \tag{2.284}$$

- A Wilson line is represented in the lowest-weight discrete series representation of $\mathrm{SL}^+(2,\mathbb{R})$ with $j = -\ell$. Here, ℓ is the conformal scaling of the associated bilocal operator. Each intersection of the Wilson line with the boundary constitutes a $3j$-symbol between two continuous series representation matrices and a discrete series representation matrix in the mixed parabolic basis:



$$= \sqrt{\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}} \tag{2.285}$$

- To each bulk crossing of two Wilson lines, we associate a $6j$-symbol over $\mathrm{SL}^+(2,\mathbb{R})$ hyperbolic states



$$= \begin{Bmatrix} k_1 & \ell_1 & k_2 \\ k_3 & \ell_2 & k_4 \end{Bmatrix} \tag{2.286}$$

- Eventually, all momentum labels should be integrated over.

2.9 Defects in JT gravity

Before moving on, we need one additional rule on how to deal with defects insertions in the bulk geometry. These will be essential to describe higher topological solutions, which form the core of the next few chapters. In the bulk BF perspective, these correspond to "magnetic monopoles" that implement a non-trivial monodromy of the gauge field around it. In the second order JT perspective, these may either be microscopic punctures corresponding to conical defects, or macroscopic tubes which open up to Euclidean wormholes. In the Schwarzian perspective, this has the effect of changing the natural integration manifold from diff$(S_1)/\mathrm{SL}(2,\mathbb{R})$ to other coadjoint orbits of the Virasoro algebra diff$(S_1)/H$, labeled by a distinct stabilizer subgroup H. These have all been classified and thoroughly studied in [37].

Let us first investigate the insertion of defects in a gauge-theoretical perspective. In a BF theory with compact gauge group G, these are labeled by punctures with non-trivial holonomy $z = e^{-2\pi\boldsymbol{\lambda}}$. $\boldsymbol{\lambda} = \lambda^a iJ_a$ represents an element in the Cartan subalgebra, where we restrict iJ_a to the set of maximally commuting generators ($a = 1, \ldots, \text{rank}$). Using a closed slicing propagation, we imagine propagating from holonomy z in the bulk to holonomy U at the boundary, yielding the general cylinder partition function Eq 2.114. Setting $U \equiv 1$ at the physical boundary readily yields:

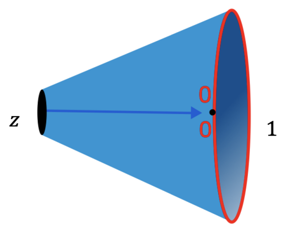
$$Z_\lambda(\beta) = \sum_R \chi_R(\mathbf{1})\chi_R(z)e^{-\beta\mathcal{C}_2(R)} = \sum_R \text{dim}(R)\chi_R(z)e^{-\beta\mathcal{C}_2(R)}. \tag{2.287}$$

Compared to the disk partition function Eq 2.115, we see that the effect of this deformation is to insert a suitably normalized character of the group element z in the region of the disk with representation R,

$$D_\mu = \frac{\chi_R(z)}{\text{dim}(R)}, \quad z = e^{-2\pi\boldsymbol{\lambda}}, \quad \boldsymbol{\lambda} = \lambda^a iJ_a \tag{2.288}$$

, where z is defined in terms of the holonomy of a Wilson loop $z = \exp(-\oint A)$. Within the coset boundary set-up relevant for JT, we have seen that the additional boundary restriction sets the character at the boundary to its lowest-weight matrix element $\chi_R(\mathbf{1})|_{00} = R_{00}(\mathbf{1})$. Properly normalizing the representation matrix elements sets $R_{00}(\mathbf{1}) = 1$. On the contrary, the interior is not restricted by the coset boundary conditions. Therefore, the effect of the coset set-up is to essentially strip off an additional factor of dim(R) in the partition function, yielding [24]:

$$Z_\lambda(\beta) = \sum_R \chi_R(z)e^{-\beta\mathcal{C}_2(R)} = \sum_R \text{dim}(R)D_\lambda(z)e^{-\beta\mathcal{C}_2(R)} = \tag{2.289}$$



.

Within the BF path integral, a z-holonomy is created by inserting a local puncture evaluated in the representation labeled by the weight vectors (λ^a) of $\boldsymbol{\lambda}$ [37]:

$$\mathcal{P}_\lambda(y) = \text{Tr}_\lambda(e^{2\pi\mathbf{B}(y)}). \tag{2.290}$$

Inserted in the path integral, we can bring the trace into the action by integrating out microscopic "color" degrees of freedom in the path integral [37]:

$$\int \mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A}\mathrm{Tr}_\lambda(e^{2\pi \mathbf{B}(y)})\, e^{\int \mathrm{Tr}(\mathbf{BF})} = \int \mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A} \int dw\, e^{-2\pi \mathrm{Tr}(\boldsymbol{\lambda} w^{-1}\mathbf{B}w)} e^{\int \mathrm{Tr}(\mathbf{BF})}. \tag{2.291}$$

It is possible to absorb the w-dependence completely by making a suitable global G-transformation $\mathbf{B} \to w\mathbf{B}w^{-1}$, $\mathbf{F} \to w\mathbf{F}w^{-1}$. Including the boundary term, we end up with an action of the form

$$I = -\int_{\mathcal{M}} [\mathrm{Tr}(\mathbf{BF}) - 2\pi\delta(x-y)\sqrt{g}d^2x\mathrm{Tr}(\boldsymbol{\lambda}\mathbf{B})] + \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{Tr}(\mathbf{BA}). \tag{2.292}$$

Integrating out $\mathbf{B}$ renders $\mathbf{F} = 0$, up to a local puncture at location y:

$$\mathbf{F}(x) = 2\pi\boldsymbol{\lambda}\delta(x-y)\sqrt{g}d^2x. \tag{2.293}$$

Since the bulk is independent of the metric, the precise location of y is arbitrary, and we may set $y \equiv 0$ w.l.o.g. Taking the Cartan algebra along P_2 defined in Eq 2.23, we may perform a G-transformation to put the defect parallel to the curvature F^2: $\boldsymbol{\lambda} = \lambda P_2$, leading to

$$F^2 = d\omega + e^0 \wedge e^1 = 2\pi\lambda\delta(x)e^0 \wedge e^1 \;\; \to \;\; R = -2 + 4\pi(1-\theta)\delta(x) \tag{2.294}$$

, for some constant angle $\theta = 1 - \lambda$ that is related to a conical singularity in the gravity solution. To reach the second equality, we have used the general relation between the first and second order forms Eqs 2.15, 2.16.
To make a more insightful argument, we can show that the puncture indeed creates a holonomy when we restrict to the Abelian case, for which $\mathbf{F} = d\mathbf{A}$. Integrating along some region Σ that encloses the point y, and using the divergence theorem, we see that the holonomy along the boundary curve $\partial\Sigma$ is given by:

$$\int_\Sigma \mathbf{F} = \oint_{\partial\Sigma} \mathbf{A} = 2\pi\boldsymbol{\lambda}. \tag{2.295}$$

The second equality follows from the puncture Eq 2.293. Therefore, the insertion of a puncture fixes the holonomy around it as $z = e^{-\oint \mathbf{A}} = e^{-2\pi\boldsymbol{\lambda}}$, confirming the claim that the puncture $\mathcal{P}_\lambda$ Eq 2.290 creates a holonomy z around it. In the radial slicing, these are the eigenstates of the Hilbert space whose character turns up in the partition function. Writing the flat field ansatz $\mathbf{A} = gdg^{-1} = -dgg^{-1}$, the integral Eq 2.295 is readily performed for group elements satisfying

$$g(\tau+\beta) = g(\tau)\, e^{-2\pi\boldsymbol{\lambda}}. \tag{2.296}$$

We can argue for this in the Abelian case by diagonalizing the matrix g, leading to the informal manipulation:

$$\oint_{\partial\Sigma} \mathbf{A} = -\oint_{\partial\Sigma} dgg^{-1} \simeq -\ln g|_0^\beta = 2\pi\boldsymbol{\lambda}. \tag{2.297}$$

The generalization to the non-Abelian case is technically not straightforward and requires knot-theoretic considerations [80]. However, the qualitative features remain the same. That is, parameterizing $\mathbf{A}_\tau = g\partial_\tau g^{-1}$, and using the Hill's equations Eq 2.267, the most general solution has a non-trivial monodromy depending on

$M = e^{-2\pi\boldsymbol{\lambda}}$ as [37]:

$$g(\tau+\beta) = g(\tau)\cdot M. \tag{2.298}$$

Since the equation $\mathbf{A}_\tau = g\partial_\tau g^{-1}$ is redundant under constant shifts $g \sim g \cdot S$ with $S \in \mathrm{SL}(2,\mathbb{R})$, this leads to dependence only on the conjugacy classes of the monodromies;

$$M \sim S\cdot M \cdot S^{-1}. \tag{2.299}$$

Writing $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the equation $g\partial_\tau g^{-1} = \mathbf{A}_\tau$ with constrained asymptotic behaviour for $\mathbf{A}_\tau$ Eq 2.264 leads to the Hill's equations Eq 2.267. Since the equations are symmetric under $A \leftrightarrow C$ and $B \leftrightarrow D$, we may characterize the general solution Eq 2.268 as:

$$A = \frac{F}{\sqrt{F'}}, \qquad C = \frac{1}{\sqrt{F'}}. \tag{2.300}$$

Here, F defines the Schwarzian reparameterization mode as the projective coordinate:

$$F(\tau) = \frac{A}{C}. \tag{2.301}$$

Parameterizing the (inverse) monodromy matrix as $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the monodromy of the group elements Eq 2.298 is realized as $g^{-1}(\tau+\beta) = M^{-1}g^{-1}(\tau)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\tau+\beta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\tau}. \tag{2.302}$$

This acts projectively on the uniformizing coordinate $F(\tau) = A/C$ as

$$\boxed{F(\tau+\beta) = \frac{aF(\tau)+b}{cF(\tau)+d}}. \tag{2.303}$$

Note that it is, in general, not possible to reach every $\mathrm{SL}(2,\mathbb{R})$-valued matrix M by tuning S, since all S satisfying $[S, M] = 0$ preserve this matrix M. These are elements of the stabilizer subgroup $S \in H$ of M, and label the conjugacy classes unambiguously. Therefore, the natural integration space diff$(S_1)/SL(2,\mathbb{R})$ should be modified to diff$(S_1)/H$.
According to Eq A.3, the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ are labeled as *elliptic* (Tr$(M) < 2$), *parabolic* (Tr$(M) = 2$), and *hyperbolic* (Tr$(M) > 2$), depending on the value of the trace.

Elliptic orbits

Taking $\boldsymbol{\lambda} = -\lambda P_2$ along the Cartan element defined in 2.23, the monodromy $M = e^{-2\pi\boldsymbol{\lambda}}$ is readily exponentiated to:

$$M^{-1} = \begin{pmatrix} \cos\pi\lambda & \sin\pi\lambda \\ -\sin\pi\lambda & \cos\pi\lambda \end{pmatrix} \in SL(2,\mathbb{R}). \tag{2.304}$$

We verify that this indeed corresponds to the elliptic monodromy class by taking the trace and noting $|\cos\pi\lambda| < 1$. Since this is just a rotation matrix, the stabilizer subgroup in this case is another $SO(2) \simeq U(1)_\lambda$ transformation. The integration space is therefore naturally realized as diff$(S_1)/U(1)_\lambda$.
The monodromy induces the following transformations on F:

$$F(\tau+\beta) = \frac{F(\tau) + \tan\pi\lambda}{1 - F(\tau)\tan\pi\lambda}. \tag{2.305}$$

In terms of the reparametrization mode of the thermal circle $f(\tau+\beta) = f(\tau)+\beta$, the monodromy is realized by:

$$F(\tau) = (F \circ_\lambda f)(\tau) \equiv \tan\left(\frac{\pi}{\beta}\lambda f(\tau)\right) \tag{2.306}$$

, as a consequence of the tangent addition rule

$$\tan\left(\frac{\pi}{\beta}\lambda f(\tau+\beta)\right) = \tan\left(\frac{\pi\lambda}{\beta}f(\tau) + \pi\lambda\right) = \frac{\tan\left(\frac{\pi\lambda}{\beta}f(\tau)\right) + \tan(\pi\lambda)}{1 - \tan\left(\frac{\pi\lambda}{\beta}f(\tau)\right)\tan(\pi\lambda)}.$$

We can extrapolate the boundary description into the bulk by choosing the conformal gauge Eq 1.53, and continuing the reparametrization mode into $U(u) = F(u)$, $V(v) = F(v)$. This leads to the metric[14]

$$ds^2 = \frac{4\pi^2\lambda^2}{\beta^2}\frac{d\tau^2 + dz^2}{\sinh^2\frac{2\pi}{\beta}\lambda z}. \tag{2.307}$$

Note that for $\lambda = 1$, the transformation of $F(\tau)$ becomes the usual thermal reparametrization mode $F(\tau) = \tan(\frac{\pi}{\beta}f(\tau))$. The corresponding monodromy $M \simeq 1$ commutes with every SL$(2,\mathbb{R})$ transformation, and we end up with the usual integration space diff$(S_1)/$SL$(2,\mathbb{R})$. Close to the horizon $z \to \infty$, we may approximate $\sinh^2\frac{2\pi}{\beta}\lambda z \to \frac{e^{4\pi\lambda z/\beta}}{4}$. Defining a new variable through $dx = \frac{4\pi\lambda}{\beta}\frac{dz}{e^{2\pi\theta z/\beta}}$ $(x = -\frac{2}{e^{2\pi\lambda z/\beta}})$, we can approximate the metric as:

$$ds^2 = x^2\left(d\left(\frac{2\pi\lambda\tau}{\beta}\right)\right)^2 + dx^2.$$

Interpreting this patch in polar coordinates, we readily see that one revolution in $\tau \to \tau+\beta$ corresponds to a rotation in $2\pi\lambda \equiv 2\pi(1-\alpha)$. Therefore, this type of defect is holographically described as a conical singularity in the bulk geometry. This was already anticipated from the Ricci curvature Eq 2.294. The singularity only seizes in pure JT gravity, for which $\lambda = 1$.

[14] A subtlety; the solution Eq 1.53 is written in real-time coordinates. Continuing the monodromy to real-time coordinates, we have $U(u) = \tanh\frac{\pi}{\beta}\lambda u$, $V(v) = \tanh\frac{\pi}{\beta}\lambda v$, yielding $ds^2 = -4\frac{\partial_u U \partial_v V}{(U-V)^2}dudv = -\frac{4\pi^2\lambda^2}{\beta^2}\frac{dt^2 - dz^2}{\sinh^2\left(\frac{\pi\lambda}{\beta}(u-v)\right)}$. Identifying $u - v = 2z$, and going back to Euclidean time leads to Eq 2.307.

Hyperbolic orbits

The previous example has little value in JT gravity. In fact, the whole premise of choosing the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ was to avoid the singular geometries in the path integral. A structurally more interesting class of observables are the hyperbolic defects, obtained by choosing the Cartan element in the hyperbolic conjugacy class. An efficient trick to obtain the corresponding monodromy transformation is by shifting $\lambda \to i\lambda$, leading to:

$$F \circ_\lambda f = \tanh\left(\frac{\pi}{\beta}\lambda f\right), \qquad M^{-1} = \begin{pmatrix} \cosh(\pi\lambda) & \sinh(\pi\lambda) \\ \sinh(\pi\lambda) & \cosh(\pi\lambda) \end{pmatrix} \in SL(2,\mathbb{R}) \tag{2.308}$$

, for which indeed $\mathrm{Tr}(M) > 2$. The geometry Eq 2.307 is shifted to:

$$ds^2 = 4\left(\frac{\pi\lambda}{\beta}\right)^2 \frac{d\tau^2 + dz^2}{\sin^2 \frac{2\pi}{\beta}\lambda z}. \tag{2.309}$$

This is a non-contractible Euclidean geometry, hence there is no horizon. Instead, at its minimal extend (for which $\sin\frac{2\pi}{\beta}\lambda z = 1$ at $z = \frac{\beta}{4\lambda}$), it has a geodesic of circumference $b = \int ds = \frac{2\pi\lambda}{\beta}\int d\tau = 2\pi\lambda$.

As argued around Eq 2.289, the insertion of a defect in the partition function is achieved by inserting a suitably normalized character of the holonomy $z = e^{-2\pi\boldsymbol{\lambda}}$ around the puncture, evaluated in the appropriate representation that characterizes the region of the disk. In the case of JT gravity, we use the constrained result Eq 2.289, with the continuous series representation labeled by k:

$$Z(\beta, b) = \sum_R \chi_R(z) e^{-\beta\mathcal{C}_2(R)} \;\to\; \int_0^{+\infty} dk\; \chi_k(z) e^{-\beta k^2}. \tag{2.310}$$

For hyperbolic defects, we choose the Cartan element to lie along the hyperbolic generator iJ_0: $z = e^{-2\pi\lambda(iJ_0)}$. The appropriate hyperbolic character for $\mathrm{SL}(2,\mathbb{R})$ is calculated in the next subsection (Eq 2.323) for a general class element $g = e^{2i\phi J_0}$:

$$\chi_k(g) = \cos(-2k\phi). \tag{2.311}$$

In [24], it was noted that this is also the appropriate character to use in the case of $\mathrm{SL}^+(2,\mathbb{R})$, since any $\mathrm{SL}(2,\mathbb{R})$ character can be expressed in the complete hyperbolic basis (c.f. Eq A.36) according to:

$$\chi_j(U) = \mathrm{Tr}(K^{++}(U)) + \mathrm{Tr}(K^{--}(U)). \tag{2.312}$$

It can be proven that $K^{++}(U) = K^{--}(U)$ [90], thereby identifying the characters of $\mathrm{SL}^+(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$, up to an overall prefactor.
ϕ now takes over the role of λ up to some fixed normalization. Written in terms of the geodesic length $b = 2\pi\lambda$, identifying $z = e^{-2\pi\lambda(iJ_0)}$ and $z = e^{2i\phi J_0}$ leads to a relation between the hyperbolic group parameter and the geodesic length:

$$2\phi = -b \tag{2.313}$$

, which allows us to write the partition function as:

$$Z(\beta, b) =$$

$$= \int_0^{+\infty} dk \; \cos(kb) e^{-\beta k^2} = \frac{1}{2}\left(\frac{\pi}{\beta}\right)^{1/2} e^{-\frac{b^2}{4\beta}}. \tag{2.314}$$

One usually refers to this integral as the *trumpet* partition function.
A few notes perhaps before I continue. First of all, compared to the one-loop exact disk partition function Eq 1.138, the exponent in the coupling (π/β) is lowered from $3/2$ to $1/2$. This is a consequence of the reduction of the number of zero modes in the stabilizer subgroup of the integration manifold (3 for $\mathrm{SL}(2,\mathbb{R})$ vs 1 for $\mathrm{U}(1)$).
Secondly, I have stripped off the denominator of the character insertion compared to Eq 2.323. The reason is that the characters are class functions; i.e. they are only orthogonal if we mod out the redundancy under conjugacy transformations $g \sim cgc^{-1}$, where $c \in \mathrm{SL}(2,\mathbb{R})$. This is achieved by modifying the usual Haar measure dg to the Weyl integration measure $dg\Delta$, where Δ is the Jacobian under this transformation. [40] finds that the general solution for bosonic groups is given in terms of its root vectors α:

$$\Delta = \prod_\alpha |e^\alpha - 1|. \tag{2.315}$$

Exponentiating the algebra to $\mathrm{Ad}(t^{-1})X^\alpha = t^{-1}X^\alpha t = e^{-\alpha(t)}X^\alpha$ defines these root vectors $\alpha(t)$. Within the hyperbolic conjugacy classes, the relevant group elements are $X^\alpha \equiv e^{2iH\phi}$. From the algebra relation $[iH, iE_\pm] = \mp iE_\pm$, the roots are given by:

$$e^{\alpha(iE_\pm)} = e^{\mp 2\phi}.$$

This yields the Weyl integration measure (where in the computation of the character we restricted to positive lengths $\phi < 0$):

$$\Delta = (e^{-2\phi} - 1)(1 - e^{2\phi}) = 4\sinh^2(-\phi). \tag{2.316}$$

We readily see that the hyperbolic characters Eq 2.323 are orthogonal with respect to this measure. Therefore, we can directly take the numerator of the character Eq 2.311 orthogonal with respect to the flat integration measure $d\phi$. Also, it was argued in [85] that the character without the denominator is in fact the essential object required for gravity, where we choose the measure obtained from the classical limit of 3d Chern-Simons theory [44]. In this case, the measure on the space of conjugacy class elements is essentially the flat one.
Another argument in favour of stripping off the denominator is by thinking from the Schwarzian theory perspective. [20] argues that the Schwarzian path integral of $F(\tau) = \tanh\left(\frac{\pi}{\beta}\lambda f(\tau)\right)$ over the modified integration space $\mathrm{diff}(S_1)/U(1)$ is again one-loop exact. We may therefore immediately write down the answer by considering the number of zero modes (one for $U(1)$) and the on-shell action. An inverse Laplace transform consequently yields the hyperbolic character, up to the denominator:

$$\int_{\mathrm{diff}(S_1)/U(1)} \mathcal{D}f \; e^{\frac{1}{2}\int_0^\beta d\tau \, \{\tanh \frac{\pi}{\beta}\lambda f(\tau), \tau\}} = \frac{1}{2\pi}\left(\frac{\pi}{\beta}\right)^{1/2} e^{-\frac{b^2}{4\beta}} = \int_0^{+\infty} dk \, \frac{\cos(2\pi\lambda k)}{\pi} e^{-\beta k^2}. \tag{2.317}$$

2.9.1 Evaluation of the hyperbolic character

We have argued that hyperbolic characters of the continuous series representation of $\mathrm{SL}(2,\mathbb{R})$ are used to insert operational defects in the gravitational amplitudes, and to create higher-genus surfaces when gluing along the class elements in Techmüller space. The calculation for $\mathrm{SL}(2,\mathbb{R})$ closely parallels section 7.8.2 of [90], and [40] for the $\mathrm{OSp}(1|2,\mathbb{R})$ supergroup case.
The character of a finite-dimensional representation is simply given by the trace of the representation matrices. For representations with continuous labels, this sum is in general divergent. In order to regularize this result, let us represent the action of a spin-j representation on $f \in L^2(\mathrm{SL}(2,\mathbb{R}))$ in the form of an integral operator with a kernel:

$$(\hat{T}_j(g)f)(x) = \int_{-\infty}^{+\infty} K(x,y;g,j)f(y)dy. \tag{2.318}$$

In terms of the principal series action of $\mathrm{SL}(2,\mathbb{R})$ on the space of square integrable functions $L^2(\mathbb{R})$ Eq A.9, the kernel is given by:

$$K(x,y;g,j) = |bx+d|^{2j}\delta\left(\frac{ax+c}{bx+d} - y\right). \tag{2.319}$$

The regularized character is now given by the trace of the kernel in configuration space

$$\chi_j(g) \equiv \int dx\, K(x,x;g,j) = \int dx\, |bx+d|^{2j}\delta\left(\frac{ax+c}{bx+d} - x\right). \tag{2.320}$$

The character evaluates to the sum of all fixed points of the group action on the real number line. We may simplify the calculation by noting that the character is a class function, thereby depending only on the conjugacy class elements of $\mathrm{SL}(2,\mathbb{R})$. For the latter, these split into *elliptic* $\mathrm{Tr}(g) < 2$, *parabolic* $\mathrm{Tr}(g) = 2$, and *hyperbolic* $\mathrm{Tr} > 2$ class elements. The relevant case in gravitational applications will be the hyperbolic class elements. The matrix elements with unit determinant in the hyperbolic conjugacy class are all parameterized in terms of a single parameter ϕ (c.f. Eq A.23):

$$g = \begin{pmatrix} e^{-\phi} & \epsilon \\ 0 & e^{\phi} \end{pmatrix} \simeq e^{2i\phi J_0} \tag{2.321}$$

, where $\mathrm{Tr}(g) = e^{-\phi} + e^{\phi} > 2$ automatically holds for any ϕ. The parabolic class is only included with measure zero in the hyperbolic class, and is therefore irrelevant in the current discussion. ϵ is introduced as an infinitesimal regulator along the lines of [40], since the number of fixed points changes discontinuously near $\epsilon \approx 0$. Inserted in the kernel, the regularized character is worked out as:

$$\chi_j(\phi) = \int dx\, |\epsilon x + e^{\phi}|^{2j}\, \delta\left(\frac{e^{-\phi}x}{\epsilon x + e^{\phi}} - x\right).$$

As always, the delta function can be written in terms of an expansion in its zeros: $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$. In this case, the zeros are given by $x = 0, (e^{-\phi} - e^{\phi})/\epsilon$. Assuming $\phi < 0$ in the following, we find:

$$\delta\left(\frac{e^{-\phi}x}{\epsilon x + e^{\phi}} - x\right) = \frac{\delta(x)}{e^{-2\phi} - 1} + \frac{\delta(x - \frac{e^{-\phi}-e^{\phi}}{\epsilon})}{1 - e^{2\phi}}.$$

The character is now readily evaluated as:

$$\chi_j(\phi) = \frac{e^{2j\phi}}{e^{-2\phi}-1} + \frac{e^{-2j\phi}}{1-e^{2\phi}} = \frac{e^{(2j+1)\phi}}{2\sinh(-\phi)} + \frac{e^{-(2j+1)\phi}}{2\sinh(-\phi)}$$
$$= \frac{\cosh(2j+1)\phi}{\sinh(-\phi)}. \tag{2.322}$$

Parameterized for unitary representations $j = -\frac{1}{2} + ik$ (c.f. Eq A.13), we finally obtain:

$$\boxed{\chi_k(\phi) = \frac{\cos(-2k\phi)}{\sinh(-\phi)}}. \tag{2.323}$$

Note that the character of elliptic elements vanishes as there turn out to be no fixed points on the real number line [40]. A formal definition of the elliptic character is to analytically continue the result for the hyperbolic case to imaginary $\phi \to i\phi$. Parabolic characters are of no relevance since they have associated zero measure.

2.9.2 From monodromies to coadjoint orbits

This section aims to materialize the connection between monodromies, and the coadjoint orbits of the Virasoro algebra defined in the appendix C, along the lines of [37]. There, we have labelled the symplectic manifold $\text{diff}(S_1)/\text{SL}(2,\mathbb{R})$ as the coadjoint orbit of a particular coadjoint identity element in Eq C.12. In particular, denoting $T(\tau) = \{\tan\frac{\pi}{\beta}f(\tau), \tau\}$ as the Schwarzian derivative, we have found that this action is generated by the coadjoint orbit of the representative element $\phi_0 = -\frac{b}{48\pi}$ under variations of f:

$$\text{Ad}^*_{f^{-1}}(-\frac{b}{48\pi}) \;=\; -\frac{b}{24\pi}\left\{\tan\frac{f(\tau)}{2}, \tau\right\}.$$

Since the Schwarzian derivative is invariant under the projective action of $\text{SL}(2,\mathbb{R})$ on f, this orbit is stabilized by the zero modes of $\text{SL}(2,\mathbb{R})$.

More generally, the requirement for a coadjoint element $T(\tau)d\tau^2$ to remain invariant under infinitesimal transformations $f(\tau) = \tau + \epsilon(\tau)$ is given by Eq C.9:

$$\delta_\epsilon T = 2(\partial_\tau\epsilon)T + \epsilon\partial_\tau T - \frac{b}{24\pi}\partial_\tau^3\epsilon = 0.$$

The stabilizers of $T(\tau)$ in turn label the orbit under ϵ. For $T(\tau)$ a more general Schwarzian action, the solutions above label the possible stabilizers $\epsilon(\tau) \in H$, and thereby the coadjoint orbit $\text{diff}(S_1)/H$. One can readily check that if $\psi_1(\tau)$ and $\psi_2(\tau)$ are two independent solutions of the Hill's equations $\partial_\tau^2\psi_i = \frac{12\pi}{b}T(\tau)\psi_i$, the most general solution $\epsilon(\tau) \in H$ is given by a linear combination of $\psi_i(\tau)\psi_j(\tau)$, $i,j = 1,2$. As mentioned in the previous subsection, the conformally non-equivalent solutions of Hill's equation are labeled by the conjugacy classes of the monodromy of ψ. Therefore, the problem of labeling coadjoint orbits is one-to-one related to the possible monodromy solutions of the Hill's equation.

For example, the constant identity representative $\phi = -\frac{b}{48\pi}\lambda^2$ generates a coadjoint orbit that is (in general)

only invariant under constant shifts $f(\tau) = \tau + c$. This generates the elliptic orbits diff$(S_1)/U(1)$. The corresponding monodromy is given in Eq 2.304. For $\lambda = 1$, this becomes the representative of the Schwarzian derivative, enhancing the symmetry to diff$(S_1)/\mathrm{SL}(2,\mathbb{R})$. For general $\lambda \equiv n \in \mathbb{Z}$, the representative $\phi = -\frac{b}{48\pi}n^2$ generates the orbits diff$(S_1)/\mathrm{SL}^n(2,\mathbb{R})$.
Shifting $\lambda \to i\lambda$, the representative becomes $\phi = \frac{b}{48\pi}\lambda^2$, leading to hyperbolic orbits diff$(S_1)/U(1)$ with monodromy Eq 2.308.
An intermediate case is for the representative $\phi = 0$, labeling parabolic orbits. The parabolic monodromy with Tr$(M) = 2$, is given by:

$$M = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \qquad F \circ_0 f \equiv f.$$

for $q = \pm 1$ [37]. Gravitationally, it corresponds to a cusp-like singularity, identifiable as thermal AdS_2 in Poincaré coordinates [25].
We could also consider coadjoint orbits without constant representative elements. However, it can be proven that such orbits admit no solutions to the classical equation of motion [40] (c.f. Eq 1.95):

$$\partial_\tau \{F(\tau), \tau\} = 0. \tag{2.324}$$

Therefore, the restriction to constant representative orbits only spans those defects that have a classical saddle interpretation.

Chapter 3

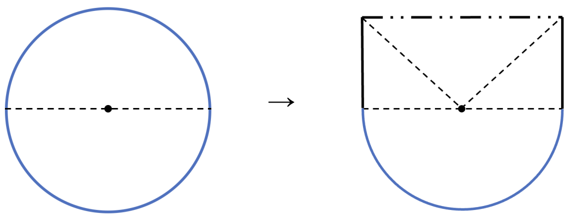
EOW branes in JT gravity

"Imagination is more important than knowledge."

Einstein, Albert

3.1 Motivation: pure states and end-of-the-world branes

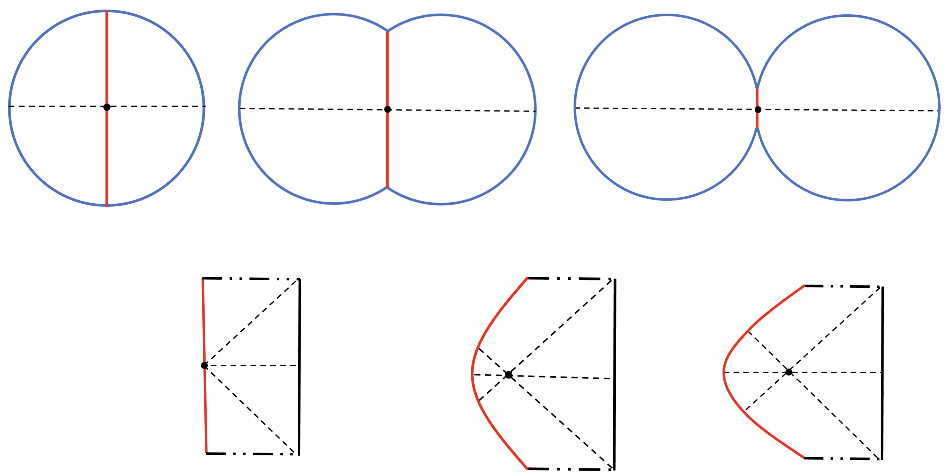
So far, we have studied JT gravity on the Euclidean hyperbolic disk. In [9], Maldacena proposed that the Thermofield double state $|\mathrm{TFD}\rangle = \sum_n e^{-\beta E_n/2} |E_n\rangle_L \otimes |E_n\rangle_R$ is dual to an eternal black hole in Lorentzian signature. This is in fact a maximally entangled state between two copies of the CFTs dual to each side of the eternal Lorentzian black hole. Formulated in the operator formalism of AdS/CFT, the gravity partition function on a manifold M depicting the eternal black hole equals the CFT partition function living on the boundary $\partial M = \Sigma$: $Z_{\text{grav}}[M] = Z_{CFT}[\partial M = \Sigma]$. The Euclidean path integral that prepares a TFD is a path integral[1] on an interval $\Sigma = \text{Interval}_{\beta/2} \times S_{d-1}$, where d is the number of spatial dimensions.
Gravitationally, the boundary conditions of Σ are precisely those of the Euclidean disk with boundary length β cut in half. Therefore, starting from half of the Euclidean disk at time $t = \tau = 0$, and continuing in real time prepares the holographic dual of the TFD state. Pictorially;



[1] Explicitly, this path integral has two open cuts at both sides of the interval, representing pure states. Projecting onto them yields [62] $\langle\phi_1| e^{-\beta H/2} |\phi_2^*\rangle$, where $|\phi_2^*\rangle$ is the adjoint of $\langle\phi_2|$. Inserting a complete set of energy eigenstates $\cdots = \sum_n e^{-\beta E_n/2} \langle\phi_1 \mid E_n\rangle\langle\phi_2 \mid E_n\rangle$ yields precisely a projection on both sides of the TFD state $\cdots = \langle\phi_1|\langle\phi_2 \mid \mathrm{TFD}\rangle$.

In the Lorentzian continuation, the horizontal direction indicates Euclidean time, while the vertical direction is the analytically continued Lorentzian time direction. Each point on the diagram represents a S_{d-1}-sphere. Thus, propagating $\beta/2$ in Euclidean time brings you to the other side of the Lorentzian-signature black hole. The entanglement between the two copies of the CFT is materialized in gravity by an Einstein-Rosen bridge (the black hole interior itself) connecting the two causally disconnected sides. This is the idea of the ER=EPR conjecture raised by Maldacena and Susskind [91].

Pure states on the right subsystem are created by left projecting on a particular state. In [92], this was modelled by inserting a local heavy operator acting on the left subsystem in the context of SYK. In the low-energy limit, they considered the dual nearly-AdS_2 gravity configurations, where the operator insertion creates an end-of-the-world (EOW) brane. Intuitively, these branes are massive particles of mass μ on which the spacetime ends. These are generated by performing a $\mathbb{Z}_2$-quotient along a line in the Penrose diagram. This mods out one side of the solution; effectively purifying the system. The massive particle is constrained to sit along the $\mathbb{Z}_2$-invariant points. As the mass of this particle increases, the end-of-the-world brane is moved deeper into the left side of the purified black hole geometry.
In the boundary particle formalism, we can describe the wiggly boundary curve as the motion of a particle in rigid AdS_2. This boundary particle emits a massive EOW particle and reabsorbs it at some later time. The classical solutions are found from energy conservation, where each of the particle's trajectory is specified in terms of their $\mathrm{SL}(2,\mathbb{R})$ charge. Energy-momentum conservation then requires the total charge to be zero. See [92] for the explicit calculations.



Qualitatively, the boundary particle moves along trajectories that look like circles in Euclidean space. The centers of these circles are the positions of the bifurcation point of the Lorentzian black hole after modding by $\mathbb{Z}_2$. The massive particle on the other hand moves along a geodesic trajectory. At each scattering vertex, the boundary particle recoils, displacing the natural circular trajectory, as shown in the figures above. The mass of the EOW branes is imagined to increase from left to right in these figures. The heavier the EOW brane particle, the further the two boundary trajectories are displaced. Demanding that the two circles are not disjoint constrains the mass of the EOW particle to $\mu < 2m$, where m is the mass of the boundary particle. After

performing a $\mathbb{Z}_2$-operation along the geodesic EOW brane trajectory in red, we generate the gravity dual of a pure state. The analytic continuation to Lorentzian time consequently yields a geometry with a single horizon that shields the EOW brane for every value of $\mu > 0$. The horizontally dotted line indicates the moment of time reflection symmetry that is used to connect the Euclidean and Lorentzian solutions. The EOW brane trajectory still gives a geodesic in the ambient Lorentzian AdS_2 spacetime. For increasing values of μ, the geodesic approaches the left boundary.
At the point where $\mu \to 2m$, the two Euclidean circles are almost tangential, inducing an almost localized EOW particle in the Euclidean picture. In the Lorentzian picture, it behaves like a shockwave behind the horizon. In [92], the localized point particle in this limit is interpreted as a UV-like boundary changing defect that acts as a projection operator onto pure states.

More recently, EOW branes have been used in [34] to describe pure states in a simple toy model of the black hole evaporation process, leading to a unitary Page curve. Their setup is very general, but only in the context of JT gravity coupled to EOW branes are they able to make real quantitative calculations. These are based on the exact quantum EOW brane amplitudes that have been calculated before in the boundary particle formalism [93] [94]. The same formalism has also been used to model closed loops of EOW branes in [36]. Although this perspective provides a nice gravitational interpretation of the physical Hilbert space, it should be noted that this technique is in general cumbersome to perform explicit calculations with.
The original work in this chapter aims to provide an alternative perspective on the quantization of JT gravity in the presence of EOW branes, using the methods of an $\mathrm{SL}^+(2,\mathbb{R})$ BF gauge theory, defined in the previous chapter. In particular, we will see that this framework naturally describes the Euclidean disk ending on an EOW brane studied in [34], and the higher topological solutions of [36]. The framework also generalizes naturally to the supergroup structure of JT supergravity studied in [40], which will be the content of the next chapter.

3.2 Setup of the model

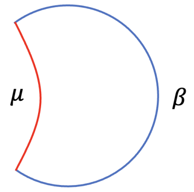
EOW branes are defined starting from the action (written here in Lorentzian signature) [36]:

$$\boxed{S = \frac{1}{2}\int \Phi\sqrt{|g|}(R+2) + \int_{AdS} du\ \Phi\sqrt{-g_{uu}}(K-1) + \int_{EOW} dv\sqrt{-g_{vv}}(\Phi K - \mu)}. \tag{3.1}$$

$\frac{1}{8\pi G} \equiv 1$ by convention. u and v represent two affine parameters along the AdS_2 boundary and the EOW brane respectively. μ denotes the mass tension along the EOW brane, while K denotes the extrinsic curvature along the EOW brane.
Various quantum amplitudes have been obtained using the boundary-particle formalism. The results greatly depend on the topology. [34] obtained quantum amplitudes of EOW branes attached to the disk partition

function. The result can be written as:

$$Z_{EOW}(\beta) =$$

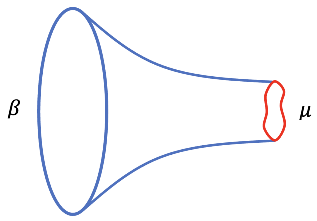


$$= \int_{-\infty}^{+\infty} db \; Z_{\text{Hartle}}(\beta, b) e^{-\mu b} \tag{3.2}$$

, where $Z_{\text{Hartle}}(\beta, b)$ denotes the Hartle-Hawking state preparing the vacuum with a geodesic of length b and an asymptotic boundary of length β. We may already guess the appearance of the EOW brane wavefunction $e^{-\mu b}$ from the classical on-shell approximation of the action Eq 3.1. These are precisely the pure state amplitudes obtained by performing a $\mathbb{Z}_2$-quotient along the fixed points of an EOW brane in the Euclidean black hole depicted above.

More recently, [36] have obtained the quantum amplitude of an EOW loop attached to the neck of a single trumpet:

$$Z_{EOW}(\beta) =$$



$$= \int_0^{\infty} db \; Z_{\text{trumpet}}(\beta, b) \frac{e^{-\mu b}}{2\sinh(b/2)}. \tag{3.3}$$

As opposed to the disk partition function, this result exhibits an interesting correction to the classical saddle in the denominator of the EOW brane wavefunction. In particular, there appears to be a spurious UV divergence of this amplitude near $b \to 0$. The form of this denominator is not readily obvious from the boundary particle formulation in [36]. It was noted briefly afterwards in [38] that the form of this wavefunction coincides with a discrete series character of $\text{SL}(2, \mathbb{R})$.
In this chapter, we aim to make this argument more precise, and obtain a generic method to arrive at both amplitudes within the framework of the BF formulation of JT gravity. The main motivation will be to extend this framework to EOW branes in theories of JT supergravity in the next chapter. As far as I known, this has not been considered so far, and an analogous action of Eq 3.1 for EOW branes in $\mathcal{N} = 1$ JT supergravity has yet to be defined.

3.3 Wilson lines as probe particles

A crucial identity to interpret EOW branes in a gauge theoretic formulation is the equivalence between Wilson loops (lines) and point-probe particles in the second order metric formulation. Intuitively, we can understand this relation by realizing that Wilson lines are the gravitational duals to Schwarzian bilocal operators, which in turn descend from the free field generating functional of JT gravity coupled to matter. This was demonstrated very explicitly in section 1.8.1. According to the diagrammatic rules established in the previous chapter, each insertion of a bilocal operator changes the energy between the two adjacent boundary sectors separated by the

bilocal operator, conform energy conservation.
Therefore, we can interpret Wilson lines holographically as an energy injection in the bulk that propagates between the two boundary points. Each time a bilocal insertion is crossed, the Wilson line acts as a probe particle that carries energy away from the boundary into the gravitational bulk at one point, only to be removed again at the second point of the bilocal operator. This can be visualised according to the example given in section 1.4.2, which discusses the injection of matter pulses from the boundary into the bulk.
We might therefore be tempted to propose the following equivalence between a Wilson line in representation $j = -\ell$ along the path $\mathcal{C}_{\tau_1\tau_2}$ (Eq 2.118), and the path integral of a massive probe particle along all paths $x(s)$ diffeomorphic to $\mathcal{C}_{\tau_1\tau_2}$, weighted by the standard point particle action:

$$\mathcal{W}_{j,nm}(\mathcal{C}_{\tau_1,\tau_2}) \simeq \int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}x\; e^{-m\int_{\mathcal{C}_{\tau_1\tau_2}} dv\,\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}. \tag{3.4}$$

To each endpoint τ_1, τ_2, we associate a label m, respectively n of the representation matrix labeled by j.

Remember that Wilson lines in the context of JT gravity are evaluated in a lowest-weight $j = -\ell$ discrete series representation of $\mathrm{SL}^+(2,\mathbb{R})$, where $\ell \in \mathbb{N}$ is identified with the conformal weight of the bilocal operator in Eq 2.275. From generic AdS/CFT argumentations, the conformal weight of a quasi-primary operator is related to the mass of the dual scalar field by Eq 1.146,

$$\ell = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}, \qquad \rightarrow \qquad m^2 = \ell(\ell-1). \tag{3.5}$$

In terms of the representation label $j = -\ell$, this is related to the eigenvalue of the quadratic Casimir Eq A.7

$$m^2 = j(j+1) = -\mathcal{C}_2. \tag{3.6}$$

As intuitive as this argumentation might seem, there are still some subtleties that go along with it. For example, it is not immediately clear where the dependence on the weight labels resides on the right-hand side of the proposed equivalence Eq 3.4. Furthermore, the right-hand side is only dependent on the diffeomorphism class of $\mathcal{C}_{\tau_1\tau_2}$, while the left-hand side depends on its exact trajectory.

3.3.1 Formal identification

The exact relation has been proven rigorously in appendix E of [23], which itself was based on earlier considerations in the context of writing 3D gravity as an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory, see e.g. [95] [96]. Let me outline the proof given in [23], while reformulating it slightly in the present context.
As an intermediate step, we rewrite the Wilson line, as a path integral over maps $g(v) : \mathbb{R} \rightarrow \mathrm{SL}(2,\mathbb{R})$ in a theory that is minimally coupled to the gauge fields. The gauge fields $\mathbf{A}_v(v)$ take values in $\mathfrak{sl}(2,\mathbb{R})$, and are parameterized along the curve $\mathcal{C}_{\tau_1\tau_2}$: $\mathbf{A}_v(v) = \mathbf{A}_\mu(x(v))\dot{x}^\mu(v)$ (with $\dot{x}^\mu = \frac{dx^\mu}{dv}$). We can think of v as the affine parameter labeling the path $\mathcal{C}_{\tau_1\tau_2}$ of the free particle.
We first solve an intermediate problem and rewrite a Wilson loop over all closed paths $\mathcal{C}$, evaluated in a lowest-

weight representation $j = -\ell$, as a path integral over paths $g(s) \in \mathrm{SL}(2,\mathbb{R})$ after [23]:

$$\mathrm{Tr}_j \left[\mathcal{P} \exp \left(-\int_{\mathcal{C}} \mathbf{A} \right) \right] = \int_{\mathcal{C}} \mathcal{D} g_\alpha \; e^{-S_\alpha[g,\mathbf{A}]}. \tag{3.7}$$

Here, $S_\alpha[g, A]$ denotes the coadjoint orbit action of the representation j coupled to $\mathbf{A}$, and is given by:

$$S_\alpha[g, \mathbf{A}] = -\int_{\mathcal{C}_{\tau_1\tau_2}} dv \; \mathrm{Tr}(\alpha g^{-1} D_A g) = \int_{\mathcal{C}_{\tau_1\tau_2}} dv \; \left(-\mathrm{Tr}(\alpha g^{-1} \partial_v g) - \mathrm{Tr}(\alpha g^{-1} \mathbf{A}_v g) \right). \tag{3.8}$$

D_A is the covariant derivative $D = d + \mathbf{A}$ defined in Eq 2.38, and transforms in such a way that the action $S_\alpha[g, \mathbf{A}]$ is gauge-invariant under left multiplication of g by elements of $\mathrm{SL}(2,\mathbb{R})$. α is a vector in $\mathfrak{sl}(2,\mathbb{R})$ and can be expanded into generators. To choose a specific basis, we opt for the familiar $\mathfrak{sl}(2,\mathbb{R})$ generators (P_a, $a \in \{0, 1, 2\}$) defined in Eq 2.23:

$$[P_0, P_1] = P_2, \qquad [P_0, P_2] = P_1, \qquad [P_1, P_2] = -P_0. \tag{3.9}$$

This choice allows us to make contact with the metric formulation of JT gravity as demonstrated in section 2.2. The gauge field in particular is expanded into generators in terms of the frame fields and spin one-forms (c.f. Eq 2.18):

$$\mathbf{A} = A^a P_a, \qquad A^a = (e^0, e^1, \omega). \tag{3.10}$$

We take the length of the vector $\alpha = \alpha^a P_a$ to be constrained by the eigenvalue of the quadratic Casimir:

$$\boxed{-\frac{1}{2}\mathrm{Tr}(\alpha^2) = -\frac{1}{4}\kappa_{ab}\alpha^a\alpha^b \equiv \mathcal{C}_2(j)}. \tag{3.11}$$

$\kappa_{ab} = 2\mathrm{Tr}[(P_a)(P_b)]$ denotes the Cartan-Killing metric following the usual conventions (c.f. Eq A.6). For the case at hand, the Cartan-Killing metric is in fact the flat space metric (c.f. Eq 2.19)):

$$\mathrm{Tr}(P_a P_b) = \frac{\eta_{ab}}{2}, \quad \text{with} \quad \eta_{ab} = \mathrm{diag}(1, 1, -1). \tag{3.12}$$

As usual, indices are raised and lowered with respect to this metric. The restriction to $a = 0, 1$ coincides with the local Lorentz metric in Euclidean signature δ_{ab}. Without loss of generality, I will continue to denote the metric as κ_{ab} since the following discussion is a priori independent of the choice of generators.

It looks as though the integration space is over the entire $\mathrm{SL}(2,\mathbb{R})$ group manifold. However, a closer look at the integrand in Eq 3.8 shows that all group elements obtained by right-multiplying g with a constant group element that belong in the same orbit of $g\alpha g^{-1}$, share the same action. Therefore, the natural phase space is over the coadjoint orbits of the Lie-algebra valued element α instead [23];

$$\mathcal{C} \to \{ g\alpha g^{-1} \mid g \in \mathrm{SL}(2,\mathbb{R}) \}.$$

This redundantly parameterizes the orbit of α under the adjoint action of $\mathrm{SL}(2,\mathbb{R})$ as the gauge group modulo the stabilizer subgroup. This is the meaning of the measure $\mathcal{D}_\alpha g$. The Hilbert space of the theory is thus to be identified with functions that are invariant under the right group actions that stabilize α. One such function is the quadratic Casimir Eq 3.11 that labels the representation. Therefore, the Hilbert space itself forms an irreducible representation labeled by $j = -\ell$.

To prove the equivalence Eq 3.7, we resort to the standard Gibbons-Hawking prescription. Concretely, we aim to interpret the path integral as an operator that generates the Hamiltonian evolution along the path $\mathcal{C}_{\tau_1\tau_2}$.
To this end, we should identify the conjugate momenta along the path x^a: $\pi_a = \frac{\delta L}{\delta \dot{x}^a}$ and write the Lagrangian in Eq 3.8 as $L = \pi_a \dot{x}^a - H$. Parameterizing the curve $g(v) \in \mathrm{SL}(2,\mathbb{R})$ in terms of the left action of $g(v) = e^{-x^a(v)P_a} g(v_0) \equiv U(v)g(v_0)$, we readily deduce the conjugate momenta

$$\pi_a = \mathrm{Tr}(P_a g\alpha g^{-1}) = (g\alpha g^{-1})^b \mathrm{Tr}(P_a P_b) = (g\alpha g^{-1})^b \frac{\kappa_{ab}}{2} = \frac{(g\alpha g^{-1})_a}{2}. \tag{3.13}$$

In the quantum theory, these become the generators of the left $\mathrm{SL}(2,\mathbb{R})$ action on $g \to U(v)g$ along the path $x(v)$. After expanding $\mathbf{A}_v$ into generators $\mathbf{A}_v = A_v^a P_a$, we may identity:

$$H = \mathrm{Tr}(\alpha g^{-1} \mathbf{A}_v g) = A_\tau^a \mathrm{Tr}(P_a g\alpha g^{-1}) = A_v^a \pi_a. \tag{3.14}$$

In the quantum theory, the conjugate momenta become operators that statisfy the (Euclidean) commutation relations

$$[\hat{\pi}_a(s), x^b(t)] = -\delta_a^b \delta(s-t) \tag{3.15}$$

, and are realized as $\hat{\pi}_a(s) = -\frac{\delta}{\delta x^a(s)}$. These act on the Hilbert space of functions that are invariant under the global right action that stabilizes the orbit of α, and are left-parameterized by $g(s) = e^{-x^a(s)P_a} g(s_0)$. The Hamiltonian is thus diagonalized on the elements of $g(s)$ by the expansion of $\mathbf{A}$ in terms of the matrix generators P_a;

$$H\ g = A_v^a P_a\ g. \tag{3.16}$$

The Casimir associated to these operators is found from the inverse of the Cartan-Killing metric κ^{ab}:

$$\begin{aligned} \mathcal{C}_2 = -\kappa^{ab}\pi_a\pi_b &= -\frac{\kappa^{ab}}{4}(g\alpha g^{-1})_a (g\alpha g^{-1})_b = -\frac{\kappa_{ab}}{4}(g\alpha g^{-1})^a (g\alpha g^{-1})^b \\ &= -\frac{1}{2}\mathrm{Tr}(P_a P_b)(g\alpha g^{-1})^a (g\alpha g^{-1})^b = -\frac{1}{2}\mathrm{Tr}(g\alpha^2 g^{-1}) = -\frac{1}{2}\mathrm{Tr}(\alpha^2) \end{aligned} \tag{3.17}$$

, where the indices are lowered and raised using the metric and inverse metric respectively. This is of course compatible with the initial constraint Eq 3.11.
Wilson loops that do not reach the boundary are effectively unconstrained by the the coset boundary conditions. Moreover, functionally integrating over a closed contour $g(v) : \mathcal{C} \to \mathrm{SL}(2,\mathbb{R})$ results in a trace over the Hilbert space, yielding a Wilson loop operator in the interior (c.f. Eq 3.7) according to the general Gibbons-Hawking

prescription Eq D.26:

$$\mathcal{W}_j(\mathcal{C}) \;=\; \mathrm{Tr}_j\left(\mathcal{P}\exp - \oint_{\mathcal{C}} \mathbf{A}\right) \;=\; \oint_{\mathcal{C}} \mathcal{D}g_\alpha\, e^{-S_\alpha[g,\mathbf{A}]}. \tag{3.18}$$

On the other hand, for paths $\mathcal{C}_{\tau_1\tau_2}$ that reach the boundary, the path integral calculates an evolution from one asymptotic state to another. The Hilbert space that spans the representation $j = -\ell$ on the boundary is constrained by the coset boundary condition Eq 2.251 $\mathcal{J}^- = 1$ on each asymptotic state. As argued in section 2.8.4 (more precisely Eq 2.253), this effectively constrains the wavefunctions on either end to (mixed) parabolic type $R_{j,1_-1_+}(g) = \langle j, 1_- \mid g \mid j, 1_+\rangle$. By current conservation of the Clebsch-Gordan coefficients $C^{R_1,1}_{R,R_2;m,1} = \langle R_1, 1 \mid R, R_2; m, 1\rangle$, the Wilson line itself is constrained to its lowest-weight state $\mathcal{W}_{j,00}$. The Hamiltonian evolution in the physical Hilbert space can thus be identified as (c.f. Eq D.23):

$$\mathcal{W}_j(\mathcal{C}_{\tau_1\tau_2}) \;=\; \langle j, 0_- \mid \mathcal{P}\exp\left(-\int_{\mathcal{C}_{\tau_1\tau_2}} \mathbf{A}\right) \mid j, 0_+\rangle \;=\; \int_{\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}g_\alpha\, e^{-S_\alpha[g,\mathbf{A}]} \tag{3.19}$$

, proportional to the Wilson line $\mathcal{W}_j(\mathcal{C}_{\tau_1\tau_2})$. The novelty features are again the coset conditions that define JT gravity, which constrain the Wilson line in the j-representation to a predefined weight. On the other hand, while these boundary conditions feel like a natural choice in the constrained setup of JT gravity, it is not clear by now how this information on the weight states is incorporated in the evaluation of the path integral. Presumably the boundary conditions of the probe $g(s)$ should be compatible with these sates. [96] sheds further light on this in the context of rotated Ishibashi states in asymptotic negatively curved 3d gravity. Since this procedure makes explicit use of the two copies of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory, it is not clear whether this procedure is readily applicable in our constrained BF setup. Note that this difficulty does not turn up for closed loops $g(s)$ that prepare a Wilson loop instead. The only requirement in this case is that the probe $g(s)$ should be single valued around the circle.

Since the adjoint action of $\mathrm{SL}(2,\mathbb{R})$ is transitive on all Lie algebra elements of a given length, we can include in the definition a functional integral over all elements of the form $\alpha(v) = \alpha^a(v)(P_a)$. Furthermore, to establish the equivalence between the free particle action and a Wilson line operator, we impose the additional constraint

$$\boxed{\frac{1}{2}\mathrm{Tr}(\alpha^2) = \frac{\kappa_{ab}\alpha^a\alpha^b}{4} \equiv m^2}. \tag{3.20}$$

This fixes the Hilbert space to a predefined representation according to Eq 3.11. We may expand the vector α without loss of generality [40]

$$\alpha(v) = \alpha^0 P_0 + \alpha^1(v) P_1 \tag{3.21}$$

, without writing the component related to the spin ω in the P_2-direction. Indeed, the bulk path integral imposes flatness of the gauge field, which implements the no-torsion constraints. This expresses ω as a function of the frame fields e^a_μ, such that the direction along P_2 is not independent from the other directions. The fact that we include an integral over a sub vector space of the algebra does not pose a problem, as long as its total length is still constrained by Eq 3.20.

We implement this constraint by functionally integrating over a Lagrange multiplier Θ:

$$\mathcal{W}_j(\mathcal{C}_{\tau_1\tau_2}) \;\sim\; \int \mathcal{D}\alpha_a \mathcal{D}g_\alpha \mathcal{D}\Theta \; e^{-S_\alpha[g,\mathbf{A},\Theta]}. \tag{3.22}$$

The total action is then given by:

$$S_\alpha[g,\Theta,\mathbf{A}] = \int_{\mathcal{C}_{\tau_1\tau_2}} dv \; [-\mathrm{Tr}(\alpha g^{-1} D_A g) + \frac{i}{4}\Theta(\kappa_{ab}\alpha^a\alpha^b - 4m^2)]. \tag{3.23}$$

Θ and α appear Gaussian in the integrand. Denoting $\mathrm{Tr}(\alpha g^{-1} D_A g) = \frac{\kappa_{ab}}{2}\alpha^a(g^{-1}D_A g)^b$, we can perform the Gaussian integral over α by substituting the on-shell solution. Exploiting the symmetry of the Cartan-Killing metric, this is found to be equal to:

$$\begin{aligned} \delta\left[-\frac{\kappa_{ab}}{2}\alpha^a(g^{-1}D_A g)^b + \frac{i}{4}\Theta\kappa_{ab}\alpha^a\alpha^b - i\Theta m^2\right] &= 0 \\ \leftrightarrow \quad \left[-\frac{\kappa_{ab}}{2}(g^{-1}D_A g)^b + \frac{i}{2}\Theta\kappa_{ab}\alpha^b\right]\delta\alpha^a &= 0 \\ -\frac{i}{\Theta}(g^{-1}D_A g)^b &= \alpha^b. \end{aligned}$$

Inserted in the action readily gives:

$$\begin{aligned} S_\alpha[g,\Theta,\mathbf{A}] &= \int_{\mathcal{C}_{\tau_1\tau_2}} dv \; \left[\frac{i}{2\Theta}\kappa_{ab}(g^{-1}D_A g)^a(g^{-1}D_A g)^b - \frac{i}{4\Theta}\kappa_{ab}(g^{-1}D_A g)^a(g^{-1}D_A g)^b - i\Theta m^2\right] \\ &= i\int_{\mathcal{C}_{\tau_1\tau_2}} dv \; \left[\frac{\kappa_{ab}}{4\Theta}(g^{-1}D_A g)^a(g^{-1}D_A g)^b - \Theta m^2\right]. \end{aligned} \tag{3.24}$$

Choosing the saddle

$$\Theta = \frac{i}{2m}\sqrt{\kappa_{ab}(g^{-1}D_A g)^a(g^{-1}D_A g)^b}$$

ultimately yields:

$$S_\alpha[g,\Theta,\mathbf{A}] = m\int_{\mathcal{C}_{\tau_1\tau_2}} dv \; \sqrt{\kappa_{ab}(g^{-1}D_A g)^a(g^{-1}D_A g)^b} \tag{3.25}$$

, where we have switched the real contour of Θ into the positive imaginary plane. Note that while this equivalence is exact on-shell, one should modify the natural integration measure to make the action in Eq 3.24 Gaussian in Θ [97].

In the remaining directions $P_{0,1}$, the Cartan-Killing metric $\kappa_{ab} = \eta_{ab}$ is identified with the Kronecker delta: $\kappa_{ab} = \delta_{ab}$, $a, b = 0, 1$, and we may write

$$S_\alpha[g,\Theta,\mathbf{A}] = m\int_{\mathcal{C}_{\tau_1\tau_2}} dv \; \sqrt{\delta_{ab}(g^{-1}D_A g)^a(g^{-1}D_A g)^b}. \tag{3.26}$$

The action is manifestly invariant under gauge transformations that act on the left $g \to U(s)g$ when the gauge field transforms accordingly. However, such gauge transformations in general mix the components of the field strength and spin connection. To partially gauge fix the choice Eq 3.21, we set $g \equiv \mathbf{1}$ along the curve $\mathcal{C}_{\tau_1\tau_2}$ by applying a gauge transformation $U(v) = g^{-1}(v)$ at every point along the curve. This gauge can be smoothly extended into the entire bulk. Expanding the gauge fields into their components along the curve $\mathcal{C}_{\tau_1\tau_2}$:

$$\mathbf{A}_v(v) \equiv e^0_\mu(v)\dot{x}^\mu(v)P_0 + e^1_\mu(v)\dot{x}^\mu(v)P_1 \tag{3.27}$$

, the action simply becomes the action of a free particle coupled to gravity by the definition of the covariant derivative $D_A = d + \mathbf{A}$:

$$S_\alpha[g,\Theta,\mathbf{A}] = m\int_{\mathcal{C}_{\tau_1\tau_2}} dv\ \sqrt{\delta_{ab}e^a_\mu e^b_\nu \dot{x}^\mu\dot{x}^\nu}. \tag{3.28}$$

In the second order formalism, we may write the metric tensor in terms of the frame fields and the local Lorentz metric as $g_{\mu\nu} = \delta_{ab}e^a e^b$ along the lines of section 2.1. This demonstrates the on-shell equivalence:

$$\boxed{S_\alpha[g,\Theta,\mathbf{A}] = m\int_{\mathcal{C}_{\tau_1\tau_2}} dv\ \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}. \tag{3.29}$$

To finish the proof, we ought to show that the functional integral over $\mathbf{A}$ in the presence of the bulk BF action is equivalent to integrating over all paths diffeomorphic to $\mathcal{C}_{\tau_1\tau_2}$. First of all, remember that the bulk BF integral over the auxiliary field $\mathbf{B}$ renders $\mathbf{A}$ flat. As noted before in section 2.2, infinitesimal gauge transformations on flat gauge fields are equivalent to infinitesimal diffeomorphisms in the gravity theory. Therefore, inside the path integral, gauge transformations on the frame fields $e^a_\mu \to \tilde{e}^a_\mu$ that leave the gauge connection flat are equivalent to diffeomorphisms of the worldline $\mathcal{C}_{\tau_1\tau_2} \to \tilde{\mathcal{C}}_{\tau_1\tau_2}$, in the sense that the action simply transforms as:

$$m\int_{\mathcal{C}_{\tau_1\tau_2}} dv\ \sqrt{\delta_{ab}\tilde{e}^a_\mu\tilde{e}^b_\nu\dot{x}^\mu\dot{x}^\nu} \quad \to \quad m\int_{\tilde{\mathcal{C}}_{\tau_1\tau_2}} dv\ \sqrt{\delta_{ab}e^a_\mu e^b_\nu\dot{x}^\mu\dot{x}^\nu}.$$

This proves the identification Eq 3.4 between the path integral of a free particle in the metric formulation of JT gravity and a Wilson line operator *inside* the path integral of BF theory. More precisely, over closed paths $\mathcal{C}$, the free particle path integral is equivalent to a Wilson loop insertion:

$$\boxed{\int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{B}\ \mathcal{W}_j(\mathcal{C})e^{-I_{BF}} \;=\; \int \mathcal{D}g\mathcal{D}\Phi \oint_{\text{paths}\sim\mathcal{C}} \mathcal{D}x\ e^{-m\oint_{\mathcal{C}} dv\ \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}e^{-I_{JT}}} \tag{3.30}$$

, while the free particle path integral over a curve anchored at the proper boundary points τ_1, τ_2 is equivalent to a Wilson line insertion evaluated in its lowest-weight state:

$$\boxed{\int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{B}\ \mathcal{W}_{j,0_-0_+}(\mathcal{C}_{\tau_1\tau_2})e^{-I_{BF}} \;=\; \int \mathcal{D}g\mathcal{D}\Phi \int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}x\ e^{-m\int_{\mathcal{C}_{\tau_1\tau_2}} dv\ \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}e^{-I_{JT}}}. \tag{3.31}$$

The equality in Eq 3.4 should therefore be thought of as an operator identity that only holds as an operator insertion in a path integral over flat gauge connections.

The Wilson operator insertion is evaluated in the lowest-weight discrete representation j, determined by the eigenvalue of the quadratic Casimir. In order for the action of the Wilson line and the free particle to coincide, the mass m of the particle should be related to the length Tr(α^2) according to Eq 3.20, which itself is constrained by the eigenvalue of the quadratic Casimir by Eq 3.11. Combining both constraints relates the mass m to the Casimir eigenvalue;

$$m^2 = -\mathcal{C}_2. \tag{3.32}$$

We can determine an explicit expression for the quadratic Casimir in terms of the generators P_a from the general definition of the inverse Cartan-Killing metric $\kappa^{ab} = \kappa_{ab} = \eta_{ab} = \text{diag}(1,1,-1)$. The general definition that has been used throughout is given by Eq 2.93:

$$\mathcal{C}_2 = -\kappa^{ab} P_a P_b = -P_0^2 - P_1^2 + P_2^2. \tag{3.33}$$

In the previous chapter, we have labeled the irreducible representations of SL$^{(+)}(2,\mathbb{R})$ in terms of the eigenvalue of the quadratic Casimir acting on the generators in the Borel-Weil realization Eq A.11: iJ_a $(a = 0,+,-)$. We may relate both realizations of the algebra by the transformation

$$P_0 = iJ_0, \qquad P_1 = \frac{1}{2}(iJ_- + iJ_+), \qquad P_2 = \frac{1}{2}(iJ_- - iJ_+). \tag{3.34}$$

The quadratic Casimir corresponding to the P_a-generators can be expressed in terms of the Borel-Weil generators iJ_a

$$\mathcal{C}_2 = -P_0^2 - P_1^2 + P_2^2 = J_0^2 + \frac{1}{2}(J_- J_+ + J_+ J_-) = -j(j+1) \tag{3.35}$$

, whose eigenvalue j is determined in Eq A.12. Accordingly, the mass of the free particle is related to the label j as

$$\boxed{m^2 = j(j+1)}. \tag{3.36}$$

This is the expected relation between the mass and the conformal dimension from generic AdS/CFT considerations in Eq 3.6.

3.3.2 Geodesic description of EOW branes

According to the previous discussion along the lines of [23], the free part of the EOW brane action defined in Eq 3.1 is related to a Wilson line insertion in the BF path integral. Written in Euclidean signature, the total action of the EOW particle is:

$$I = \int_{\mathcal{C}} dv \sqrt{g_{vv}} (\mu - \Phi K). \tag{3.37}$$

We relate the induced metric g_{vv} of the affine parameter v along the boundary curve in terms of the metric in the bulk $g_{\alpha\beta}$ according to:

$$ds^2|_{\mathcal{C}} = g_{vv} dv^2 = g_{\alpha\beta} \frac{dx^\alpha}{dv} \frac{dx^\beta}{dv} dv^2.$$

The first term in the action therefore corresponds to the free particle action in the path integral Eq 3.4. In this context, the mass μ is often denoted as the tension along the brane.
To deal with the second term involving the extrinsic curvature K, we consider its classical equation of motion.

Crucially, the dilaton field is unconstrained by Dirichlet boundary conditions. Inserted in the path integral, it acts as a Lagrange multiplier, enforcing its classical equation motion as an off-shell constraint on the particle's trajectory

$$\boxed{K = 0}. \tag{3.38}$$

We will now show that the vanishing of the extrinsic curvature along the trajectory severely restricts its degrees of freedom. In particular, it is well known that all geodesic trajectories satisfy $K = 0$. On the other hand, $K = 0$ is also a sufficient condition to describe geodesic trajectories.
Indeed, let us demonstrate this by starting from the geodesic equation of $x^\mu(v)$ labeled by an affine parameter along the curve v, and rewrite it more suggestively:

$$\frac{d^2x^\mu}{dv^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dv}\frac{dx^\beta}{dv} = 0 \tag{3.39}$$

$$\leftrightarrow \quad U^\alpha \nabla_\alpha U^\mu = 0. \tag{3.40}$$

$U^\mu \equiv \frac{dx^\mu}{dv}$ denotes the tangent vector along the path $\mathcal{C}$. The second identity follows from the chain rule applied to the covariant derivative. The normal vector n^μ is required to be orthogonal to the tangent vector along the entire curve:

$$n_\alpha(v)U^\alpha(v) \equiv 0. \tag{3.41}$$

Applying the generalized Leibnitz rule for covariant derivatives on this definition readily yields a relation between the variation of the tangent vector and the variation of the normal vector:

$$n_\alpha \nabla_\mu U^\alpha = -U^\alpha \nabla_\mu n_\alpha. \tag{3.42}$$

The geodesic equation follows from the variational solution of the free particle action along the curve. Since the variation in any direction can be decomposed into its tangential and normal direction, we may characterize the general variation entirely along $\delta x^\mu = n^\mu$:

$$\delta I \sim \int dv\ \delta x_\mu (U^\alpha \nabla_\alpha U^\mu) \sim \int dv\ n_\mu (U^\alpha \nabla_\alpha U^\mu) = -\int dv\ U^\mu U^\alpha \nabla_\alpha n_\mu.$$

The last equality follows from the orthogonality of the normal vector with U^μ. Here, we recognize the definition of the pulled back extrinsic curvature along the curve Eq 1.86;

$$K = U^\mu U^\alpha \nabla_\alpha n_\mu. \tag{3.43}$$

The variation of the action is therefore completely specified by the value of the extrinsic curvature:

$$\delta I \sim \int dv\ K. \tag{3.44}$$

Hence, on every curve for which $K \equiv 0$, the variation in the normal direction vanishes, constraining it to solutions of the geodesic equation.

Inserted in the JT path integral, we schematically have:

$$\int_{\text{paths}\sim\mathcal{C}} \mathcal{D}x\ e^{-\int_{\mathcal{C}} dv\sqrt{g_{vv}}(\mu-\Phi K)} \xrightarrow{\text{Integrate over }\Phi} \int_{\text{geodesics}\sim\mathcal{C}} \mathcal{D}x\ e^{-\mu\int_{\mathcal{C}} dv\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}. \tag{3.45}$$

To evaluate the path integral over geodesics, we use a method of steepest descent to localize along geodesics. The path integral effectively localizes on those classical solutions, including its one-loop determinant, in the limit where $\mu \gg 1$. Note that this does not impose any additional constraints on the value of μ. If we were to reintroduce an effective scale $\hbar$ into the theory, the localization requirement would simply require $\mu \gg \hbar$. We can therefore take μ to be very large by simply tuning the relative scales in the theory.
Anyway, to make contact with the previous section, we know that the free particle path integral is equivalent to a Wilson operator insertion in the BF theory, whose mass tunes the representation label according to Eq 3.36. In the limit where we localize along geodesics, we may neglect the linear term and identify:

$$\boxed{\mu^2 \approx j^2}. \tag{3.46}$$

3.3.3 Geometric interpretation of the hyperbolic group parameter

To make contact with the calculation in the boundary particle formalism, we need a structural link between the group theoretical variables (such as the parameters of the Gauss parametrization Eq 2.148) and geometric variables (such as the geodesic length). Since we are working in the mixed parabolic basis that diagonalizes $e^{\gamma_- iJ_-}$ and $e^{\gamma_+ iJ_+}$ with eigenvalues determined by the coset constraints, the only relevant variable is the hyperbolic parameter ϕ (c.f. Eq. 2.254, and the discussion thereafter):

$$R_{k,1_-1_+}(g) = \langle k, 1_- \mid g \mid k, 1_+\rangle = e^{\gamma_- - \gamma_+}\langle k, 1_- \mid e^{2i\phi J_0} \mid k, 1_+\rangle\,. \tag{3.47}$$

In the context of hyperbolic defects giving rise to the trumpet partition function, we have already established the identification Eq 2.313:

$$2\phi = -b. \tag{3.48}$$

This follows from calculating the metric associated to the monodromy M in the presence of a defect insertion $z = e^{-2\pi\lambda(iJ_0)}$ along the Cartan element iJ_0, and comparing with the generic Gauss parametrization of the hyperbolic group element $z = e^{2i\phi J_0}$. For the metric at hand (Eq 2.309), the geodesics at the neck of the trumpet have length $b = 2\pi\lambda$, which immediately yields the former identification.

Geodesics in Euclidean hyperbolic geometry

We are interested whether this identification also holds for Wilson lines anchored to the asymptotic boundary. Thereto, let us calculate general geodesic lengths within the Poincaré upper half plane ($Z > 0$, $-\infty < T < +\infty$) of a Euclidean AdS_2-patch:

$$ds^2 = \frac{dT^2 + dZ^2}{Z^2}. \tag{3.49}$$

Geodesic trajectories $\mathcal{C}$ parameterized by the affine coordinate v are given by maps $(Z(v), T(v))$ satisfying the geodesic equation Eq 3.39.
The non-vanishing Christoffel symbols are given by $\Gamma^Z_{ZZ} = \Gamma^T_{ZT} = \Gamma^T_{TZ} = -\frac{1}{Z}$, $\Gamma^Z_{TT} = \frac{1}{Z}$, yielding the geodesic equations for Z, respectively T:

$$Z'' + \frac{1}{Z}T'T' - \frac{1}{Z}Z'Z' = 0, \qquad T'' - \frac{2}{Z}T'Z' = 0. \tag{3.50}$$

Multiplying the second equation by Z'

$$2Z'^2 - ZZ'\frac{T''}{T'} = 0$$

we may use the identity for Z'^2 in the first equation to obtain:

$$Z'^2 + T'^2 + ZZ'' - ZZ'\frac{T''}{T'} = 0.$$

After dividing by T', we may write the geodesic equations in terms of a total derivative

$$\frac{Z'Z'}{T'} + \frac{ZZ''}{T'} - \frac{ZZ'}{T'^2}T'' + T' = \left(\frac{ZZ'}{T'} + T\right)' = 0. \tag{3.51}$$

This is easily integrated in terms of some constant C:

$$ZZ' + (T - C)T' = 0 \tag{3.52}$$

, which itself can be integrated to the locus of a circles of radius R, shifted along the timelike axis across $(C, 0)$

$$Z^2 + (T - C)^2 = R^2. \tag{3.53}$$

Therefore, the collection of geodesics in the Poincaré upper half plane are the set of half-arcs in the region $Z > 0$. This is the same conclusion reached in [99]. The general solution is given by the circular trajectories $(T = C + R\cos\theta, Z = R\sin\theta)$ for $\theta \in [0, 2\pi[$. These circles can likewise be visualized on the Euclidean disk by making T periodic from $-\infty$ to $+\infty$. The geodesic trajectories then divide the hyperbolic disk into congruent triangles, as shown in figure 1. This has been known for a long time (see e.g. the famous drawings by *M.C. Escher* in hyperbolic geometries). By parameterizing the geodesics in terms of the angular parameter θ, the general line element ds deduced from the hyperbolic metric is simply $ds = \frac{d\theta}{\sin\theta}$, allowing us to integrate the total length from a point $P = (T = C + R\cos\theta_P, Z = R\sin\theta_P)$ to a point $Q = (T' = C + R\cos\theta_Q, Z' = R\sin\theta_Q)$ with $(\theta_Q > \theta_P)$ along a geodesic trajectory labeled by the same parameters R, C [99]:

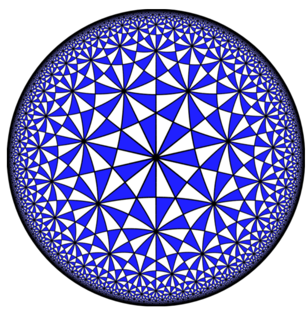


Figure 3.1: Geodesics divide the Euclidean hyperbolic disk into congruent triangles, where each of the indicated triangles has an angle of respectively $\pi/2$, $\pi/3$, $\pi/7$. Figure taken from [98].

$$b(P, Q) = \int_{\mathcal{C}} ds = \int_{\theta_P}^{\theta_Q} \frac{d\theta}{\sin\theta} = \log\left(\frac{1 + \cos\theta_P}{\sin\theta_P}\frac{1 - \cos\theta_Q}{\sin\theta_Q}\right). \tag{3.54}$$

This identity is obtained using the standard integral identity $\int d\theta \frac{d\theta}{\sin\theta} = \log\left(\sin\frac{\theta}{2}\right) - \log\left(\cos\frac{\theta}{2}\right) = \log\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$, together with $1+\cos\theta_P = 2\cos^2\frac{\theta_P}{2}$ and $1-\cos\theta_Q = 2\sin^2\frac{\theta_Q}{2}$, yielding:

$$\log\left(\frac{1+\cos\theta_P}{\sin\theta_P}\frac{1-\cos\theta_Q}{\sin\theta_Q}\right) = \log\left(\frac{2\cos^2\frac{\theta_P}{2}}{\sin\theta_P}\frac{2\sin^2\frac{\theta_Q}{2}}{\sin\theta_Q}\right) = \log\left(\frac{\cos\frac{\theta_P}{2}}{\sin\frac{\theta_P}{2}}\frac{\sin\frac{\theta_Q}{2}}{\cos\frac{\theta_Q}{2}}\right).$$

An insight from [99] was to write this final expression in terms of an isometric invariant $\delta(P,Q)$ of AdS_2:

$$b(P,Q) = 2\text{Arcsinh}(\delta(P,Q)), \qquad \delta(P,Q) = \sqrt{\frac{(Z-Z')^2+(T-T')^2}{4ZZ'}}. \tag{3.55}$$

Indeed, using the definition $\text{Arcsinh}(x) = \log(x+\sqrt{x^2+1})$, and expressing the Poincaré coordinates in terms of the angular variable θ, we may write:

$$\begin{aligned}
&2\text{Arcsinh}(\delta(P,Q))\\
&= 2\log\left[\sqrt{\frac{(\sin\theta_P-\sin\theta_Q)^2+(\cos\theta_P-\cos\theta_Q)^2}{4\sin\theta_P\sin\theta_Q}} + \sqrt{\frac{(\sin\theta_P+\sin\theta_Q)^2+(\cos\theta_P-\cos\theta_Q)^2}{4\sin\theta_P\sin\theta_Q}}\right]\\
&= 2\log\left[\sqrt{\frac{1-\sin\theta_P\sin\theta_Q-\cos\theta_P\cos\theta_Q}{2\sin\theta_P\sin\theta_Q}} + \sqrt{\frac{1+\sin\theta_P\sin\theta_Q-\cos\theta_P\cos\theta_Q}{2\sin\theta_P\sin\theta_Q}}\right]\\
&= 2\log\left[\sqrt{\frac{1-\cos(\theta_P-\theta_Q)}{2\sin\theta_P\sin\theta_Q}} + \sqrt{\frac{1-\cos(\theta_P+\theta_Q)}{2\sin\theta_P\sin\theta_Q}}\right]\\
&= 2\log\left[\frac{-\sin((\theta_P-\theta_Q)/2)+\sin((\theta_P+\theta_Q)/2)}{\sqrt{\sin\theta_P\sin\theta_Q}}\right] = 2\log\left[\frac{2\cos(\theta_P/2)\sin(\theta_Q/2)}{\sqrt{\sin\theta_P\sin\theta_Q}}\right]\\
&= \log\left[\frac{(1+\cos\theta_P)(1-\cos\theta_Q)}{\sin\theta_Q\sin\theta_P}\right].
\end{aligned}$$

This coincides with the goniometric expression in Eq 3.54. Close to the asymptotic boundary at $Z\to 0$, we may approximate the Arcsinh by a logarithm:

$$b(P,Q) \approx 2\log(\delta(P,Q)). \tag{3.56}$$

Coupled to gravity, we turn on the wiggly reparametrization modes of the asymptotic boundary curve ($T = F(\tau), Z = \epsilon F'(\tau)$) Eq 1.111 in terms of a proper time variable τ. The geodesic length between two end points P, Q along the boundary curve can therefore be written as:

$$\begin{aligned}
b(P(\tau),Q(\tau)) &\approx 2\log(\delta(P(\tau),Q(\tau))) = \log\left(\frac{(F_P-F_Q)^2}{4\epsilon^2 F'_P F'_Q}\right)\\
&= \log\left(\frac{(F_P-F_Q)^2}{F'_P F'_Q}\right) - \log(4\epsilon^2).
\end{aligned} \tag{3.57}$$

Hyperbolic group parameter

Matrix elements for Wilson line operators $\mathcal{W}_{\ell,00}(\tau_1,\tau_2)$ have been evaluated in section 2.8.6, both in gravity and in holography. In particular, it was argued that the gravitational coset constraints only require the lowest-weight matrix element due to current conservation. In the mixed parabolic basis, the matrix element of the lowest-weight discrete series representation is given by Eq 2.279:

$$R_{\ell,00} = e^{2\ell\phi}. \tag{3.58}$$

In terms of holography, the Wilson line operator in the lowest-weight discrete series is equivalent to the Schwarzian bilocal operator $\mathcal{O}^{\ell}(\tau_1,\tau_2)$ due to Hill's equations, Eq 2.275. Concretely, we have the identification:

$$\mathcal{O}^{\ell}(\tau_1,\tau_2) = \frac{F'(\tau_1)^{\ell}F'(\tau_2)^{\ell}}{(F(\tau_1)-F(\tau_2))^{2\ell}} \equiv R_{\ell,00} = e^{2\ell\phi} \tag{3.59}$$

The expression of the bilocal operator matches, up to a regularization factor, with the geodesic length Eq 3.57. Identifying both expressions relates the hyperbolic group parameter ϕ to the geodesic length b:

$$\boxed{b = -2\phi}. \tag{3.60}$$

Unsurprisingly, this is the same identification for geodesics at the neck of a trumpet Eq 3.48. This gives a physical interpretation for the hyperbolic group parameter of a Wilson line connecting two points on the boundary, in terms of the shortest distance between these two points. Since EOW branes effectively localize onto geodesics, this is exactly the distance of the path described by the Wilson line in this case. Of course, this path is not unique in the path integral since the boundary conditions are unspecified on the wiggly boundary curve. This quantum feature requires an integration over all possible geodesic lengths connecting two boundary points.

3.4 Gravitational amplitudes involving EOW branes

3.4.1 Half-moon gravitational amplitudes

We start by calculating the gravitational amplitude associated to an EOW brane attached to the asymptotic boundary curve of the Euclidean disk, which I will denote *half-moon* amplitudes from here on due to their suggestive shape. These are essentially the same amplitudes discussed in the introduction of this chapter that purify the TFD state in Lorentzian signature. The explicit amplitudes have been used before to model pure states during black hole evaporation in [34], using the boundary-particle formalism. Here, we work entirely in group theory, extending the discussion of the previous chapter.

We start again from the total action involving an EOW brane Eq 3.1, and insert it into the gravitational path integral. Due to the operator identity Eq 3.31, we may formally write:

$$\int \mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A}\, \mathcal{W}_{\ell,00}(\mathcal{C}_{\tau_1\tau_2})e^{-I_{BF}[\mathbf{B},\mathbf{A}]} = \int \mathcal{D}g\mathcal{D}\Phi \int_{\text{geodesics}\sim\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}x\; e^{-\mu\int_{\mathcal{C}_{\tau_1\tau_2}} dv\, \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}}\; e^{-I_{JT}[g,\Phi]} \tag{3.61}$$

, where $\mathcal{C}_{\tau_1\tau_2}$ is the path along the EOW brane. We have integrated out the dilaton, yielding the identification $\mu^2 \approx \ell^2$:

$$\mu \sim \ell, \qquad \mu \gg 1. \tag{3.62}$$

As will be elaborated further, we are interested in lowest-weight discrete series modules for which $\ell > 0$. From the positivity of μ, we are required to take the plus sign in the square root of this relation.
Since the Wilson line on the disk is sandwished between two asymptotic states on the coset boundary, current conservation restricts the evaluation of the Wilson line between two mixed parabolic lowest-weight states:

$$R_{\ell,0_-,0_+}(\phi) = \langle \ell, 0_- \mid \mathcal{P}\exp\left(-\int_{\mathcal{C}_{\tau_1\tau_2}} \mathbf{A}\right) \mid \ell, 0_+\rangle. \tag{3.63}$$

The diagonal matrix element is given in general by the modified Bessel functions of the first kind Eq 2.278:

$$R_{\ell,\nu_-,\nu_+}(\phi) = e^{\phi} I_{2\ell-1}(\nu e^{\phi}). \tag{3.64}$$

Evaluated in the lowest-weight state, the limit $\ell \to 0$ yields an exponential behaviour in the group label ϕ Eq 2.279:

$$R_{\ell,0_-0_+}(\phi) \sim e^{2\ell\phi}. \tag{3.65}$$

On the other hand, it is interesting to note that any other (diagonal) mixed parabolic state yields this result (up to a ϕ-independent prefactor) in the geodesic approximation. Indeed, using the geodesic approximation Eq 3.62, together with the asymptotics of the Bessel function for $\ell \to \infty$ [100]:

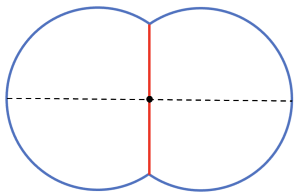
$$I_{2\ell-1}(z) \sim \frac{1}{\sqrt{2\pi(2\ell-1)}}\left(\frac{ez}{2(2\ell-1)}\right)^{2\ell-1} \tag{3.66}$$

, we find that in the geodesic approximation, all diagonal states evaluate to $R_{\ell,\nu_-\nu_+}(\phi) \sim e^{2\ell\phi}$ up to a ϕ-independent prefactor.
Anyway, written in terms of geodesic length $b = -2\phi$ and mass tension μ, this implies that the geodesic approximation of the free particle path integral is one-loop exact (inside a path integral over b):

$$R_{\mu,0_-0_+}(b) = \int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}x\; e^{-\mu\int ds} \simeq e^{-\mu b} \tag{3.67}$$

, where the RHS is obtained from both the above large ℓ-limit of Wilson line operators, or by simply taking the on-shell (geodesic) approximation of the full path integral, while approximating the length element by the geodesic length $\int ds \approx b$.
Using the open channel formalism reviewed in section 2.5.1, we have found the precise amplitude of a two-

sided Euclidean black hole with a Wilson line insertion Eq 2.276, which we pictorially denote as:

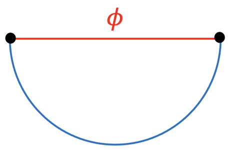


$$= \langle \mathbf{1} \mid e^{-\beta H} \mathcal{W}_{\ell,00} \mid \mathbf{1} \rangle = \int_{-\infty}^{\infty} d\phi \, e^{-2\phi} \langle \mathbf{1} \mid e^{-\beta_2 H} \mid \phi \rangle R_{\ell,00}(\phi) \langle \phi \mid e^{-\beta_1 H} \mid \mathbf{1} \rangle . \tag{3.68}$$

$dg = e^{-2\phi} d\phi$ is the usual Haar measure. The result is essentially equivalent to inserting a complete set of group elements in the disk partition function, diagonalizing the Wilson line to its lowest-weight discrete series matrix element:

$$\mathcal{W}_{\ell,00} = \int_{-\infty}^{\infty} d\phi \, e^{-2\phi} R_{00}(\phi) \, |\phi\rangle \langle \phi| \, . \tag{3.69}$$

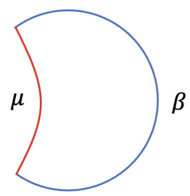
$\langle \phi \mid e^{-\beta_1 H} \mid \mathbf{1} \rangle$ is the Hartle-Hawking state preparing the vacuum Eq 2.260, where the physical boundary segment is characterized by a trivial group element $\mathbf{1}$:



$$= \langle \phi \mid e^{-\beta_1 H} \mid \mathbf{1} \rangle = \int_0^{\infty} dk \; k \sinh 2\pi k \; e^{\phi} K_{2ik}(e^{\phi}) e^{-\beta_1 k^2} . \tag{3.70}$$

Performing a $\mathbb{Z}_2$-quotient along the geodesic EOW brane fixed points essentially removes one Hartle-Hawking state, yielding:

$$Z_{EOW}(\beta) =$$



$$= \int_{-\infty}^{\infty} d\phi \, e^{-2\phi} R_{\ell,00}(\phi) \; \langle \phi \mid e^{-\beta H} \mid \mathbf{1} \rangle . \tag{3.71}$$

Note that it is important to constrain the EOW brane trajectory to the geodesic fixed points of the doubled Euclidean geometry in order to perform this $\mathbb{Z}_2$-quotient. The remaining integral over the group element ϕ makes sense since it is related to the geodesic length along the EOW brane. Since the group label along the EOW is undetermined by the gravitational wiggly boundary conditions along the UV boundary, we ought to perform an integral over all possible lengths. Inserting the explicit value of the Hartle-Hawking state finally yields:

$$Z_{EOW}(\beta) = \int_0^{\infty} dk \; k \sinh 2\pi k \; e^{-\beta k^2} \int_{-\infty}^{\infty} d\phi \; e^{-\phi} e^{2\ell \phi} K_{2ik}(e^{\phi}). \tag{3.72}$$

Written in terms of the gravitational geodesic length $2\phi = -b$, and the tension $\ell = \mu$, we have up to an overall factor:

$$Z_{EOW}(\beta) = \int_0^{\infty} dk \; k \sinh 2\pi k \; e^{-\beta k^2} \int_{-\infty}^{\infty} db \; e^{\left(\frac{1}{2} - \mu\right) b} K_{2ik}(e^{-\frac{b}{2}}). \tag{3.73}$$

The integral over b has been evaluated in [34] in the free particle context, and describes the overlap between the matrix elements of the continuous, and discrete series representations (interpreted in a different context in that reference):

$$\int_{-\infty}^{\infty} db\, e^{\left(\frac{1}{2}-\mu\right)b} K_{2ik}(e^{-\frac{b}{2}}) = \frac{2^{2\mu}}{4}|\Gamma(\mu - \frac{1}{2} + ik)|^2. \tag{3.74}$$

The result coincides with Eq 3.2, obtained in [34]:

$$\boxed{Z_{EOW}(\beta) = \int_0^{\infty} dk\; k \sinh 2\pi k\; e^{-\beta k^2} 2^{2\mu-2} |\Gamma(\mu - \frac{1}{2} + ik)|^2}. \tag{3.75}$$

3.4.2 Trumpet gravitational amplitudes

As mentioned in Eq 3.30, an EOW brane along a closed loop $\mathcal{C}$ in the interior is not constrained by the gravitational coset boundary conditions. The path integral over the closed contour generates a Wilson loop $\mathcal{W}_\ell(\mathcal{C})$. This evaluates to a trace of the holonomy of $\mathbf{A}$ around this contour, evaluated in a discrete series representation $-j = \ell \approx \mu$.
Since the trumpet amplitude is constructed by inserting a hyperbolic defect, the holonomy associated to an EOW loop at the neck of the trumpet will be evaluated by a hyperbolic class function labeled by the Gauss parameter ϕ. The trace of the holonomy will therefore be evaluated by a hyperbolic character of the discrete series representation.

Discrete series representation: revisited

We construct the discrete series representation in the momomial realization, where we choose a basis that diagonalizes iJ_0, and denote its eigenvalue as the *weight* under iJ_0. In the Borel-Weil realization of the algebra Eq A.11, the generator associated to iJ_-, iJ_+ is associated to a raising and lowering operator respectively, in the sense that (c.f. A.5)

$$[iJ_0, iJ_\pm] = \mp iJ_\pm. \tag{3.76}$$

Note that one should be aware of this perhaps misleading notation. To construct a highest-weight module, we start from a highest-weight state and apply successive lowering operators iJ_+ on the highest-weight state.
Highest-weight states themselves are annihilated by the raising operator $iJ_- = \partial_x$.

$$\psi_{HW}(x) = 1. \tag{3.77}$$

The associated weight is simply:

$$iJ_0\psi_{HW}(x) = (-x\partial_x + j)\psi_{HW}(x) = j.$$

A *highest-weight* module is generated by acting with iJ_+. E.g., the first excited state is

$$iJ_+\psi_{HW}(x) = (-x^2\partial_x + 2jx)\psi_{HW}(x) = 2jx$$

, for which we indeed find that the weight is lowered by one: $iJ_0(2jx) = (-x\partial_x + j)(2jx) = (j-1)(2jx)$.

Lowest-weight modules are defined by applying successive raising operators iJ_- to some lowest-weight state

$$\psi_{LW}(x) = x^{2j} \tag{3.78}$$

, whose weight under iJ_0 is

$$iJ_0\psi_{LW}(x) = -j\psi_{LW}(x).$$

This state is annihilated by the lowering operator iJ_+ as one can check explicitly.
Finite-dimensional representations are constructed by taking $2j \in \mathbb{N}$ in the lowest-weight state, where it is readily seen that successive application of iJ_- on $\psi_{LW}(x)$ eventually vanishes after $2j$ operations. On the other hand, infinite dimensional lowest-weight modules are constructed by taking[2] $j \equiv -\ell < 0$. This defines the lowest-weight module, defined in the context of Wilson lines Eq 2.270:

$$\boxed{\psi_{LW}(x) = \frac{1}{x^{2\ell}}}. \tag{3.79}$$

There is a further restriction to $2\ell \in \mathbb{N}$ for $\mathrm{SL}(2,\mathbb{R})$ that I neglect in the current treatment. Generalizations to $2\ell \notin \mathbb{N}$ are a priori harder to make sense of in genuine $\mathrm{SL}(2,\mathbb{R})$ (or $\mathrm{SL}^+(2,\mathbb{R})$). This restriction is lifted in the universal covering group $\tilde{SL}(2,\mathbb{R})$ [40]. Since the tension along the EOW brane is a priori any real number, we will resort our attention to the these lowest-weight modules, and imagine that we analytically continue the results of discrete ℓ to their universal covering value. This lowest-weight is indeed annihilated by the lowering operator iJ_+ for $j = -\ell$:

$$iJ_+\psi_{LW}(x) = (-x^2\partial_x + 2jx)\frac{1}{x^{2\ell}} = 2\ell x^2\frac{1}{x^{2\ell+1}} + 2j\frac{1}{x^{2\ell-1}} \equiv 0.$$

Furthermore, its weight under iJ_0 is:

$$iJ_0\,\psi_{LW}(x) = (-x\partial_x - \ell)\frac{1}{x^{2\ell}} = 2\ell x\frac{1}{x^{2\ell+1}} - \ell\frac{1}{x^{2\ell}} = \ell\,\frac{1}{x^{2\ell}}. \tag{3.80}$$

Using the $\mathfrak{sl}(2,\mathbb{R})$ algebra $[iJ_0, iJ_-] = iJ_-$, we know that the next consecutive state will have a weight of $(\ell+1)$ (which can also be checked explicitly by application of the differential operator).

The hyperbolic character is defined in analogy with the continuous series representation in section A.1.2. However, this representation was realized on functions of $L^2(\mathbb{R})$, whose eigenvalues under iJ_a span a continuous range. In the present context, the eigenvalues under iJ_0 increase in discrete steps, and are ultimately annihilated by some $iJ_\pm$ at one side of the infinite module.
The evaluation of a discrete series character therefore proceeds differently from the calculation in the principal series representation in Eq 2.323. The latter involves taking the (continuous) trace of the representation matrices. Characters in a lowest-weight discrete series module labeled by j on the other hand, are defined as

[2] We could also start from $j = \mathbb{R}^+/\mathbb{N}$. However, we want to include the natural numbers since these represent the conformal dimension of Wilson line operators.

functions on the weight space [77]:

$$\chi_j(\mu) = \sum_{\lambda} \text{mult}_j(\lambda) e^{(\lambda,\mu)} \tag{3.81}$$

, where λ denotes a weight vector in the lowest-weight discrete series module. $(\lambda, \mu) = \lambda_i G^{ij} \mu_j$ represents the inner product between the two weights with respect to the (inverse) of the restriction of the Cartan-Killing metric to the Cartan subalgebra $G_{ij} = \kappa(iH_i, iH_j)$. $\text{mult}_j(\lambda)$ denotes the dimension of the vector space $V_j(\lambda)$ of each weight λ in the representation j, defined by:

$$V_\lambda = \{v_\lambda \in V \mid R_j(iH_i)v_\lambda = \lambda_i v_\lambda\}. \tag{3.82}$$

iH_i in general denote the elements in the Cartan subalgebra, and $\lambda \equiv (\lambda_i)$ denote the weight vector with respect to the Cartan subalgebra.

Since the multiplicity of each weight of the $\mathfrak{sl}(2,\mathbb{R})$ algebra is simply one, and the weight vectors contain only a single component with respect to the Cartan subalgebra $iH = iJ_0$, the inner product on the weight space simply amounts to taking a product $(\mu, \lambda) = \mu\lambda$. Calculating the character of a Gauss parameterized hyperbolic group element $g = e^{2\phi i J_0}$ (c.f. Eq A.23) in a lowest-weight module now involves taking a sum over (an a priori natural) ℓ (and analytically continuing the result towards continuous ℓ):

$$\chi_\ell(\phi) = \text{Tr}(e^{2\phi i J_0}) \equiv \sum_{n=\ell}^{\infty} e^{2n\phi} \tag{3.83}$$

, where we have used that the eigenvalue under iJ_0 is $\ell + \mathbb{N}$. The sum may be worked out as a geometric series:

$$\chi_l(\phi) = \frac{e^{2\ell\phi}}{1 - e^{2\phi}} = \frac{e^{2\ell\phi - \phi}}{2\sinh(-\phi)}. \tag{3.84}$$

Written in terms of the geodesic length $2\phi = -b$, and using the geodesic approximation $\mu \approx \ell \gg 1$, we may neglect the additional term in the exponent:

$$\boxed{\chi_\mu(b) = \frac{e^{-\mu b}}{2\sinh\frac{b}{2}}}. \tag{3.85}$$

This limit can also be interpreted as the on-shell approximation of the free-particle path integral (c.f. Eq 3.67) with an additional one-loop determinant that is a priori hard to determine from a gravitational perspective. Note that in order for the geometric series Eq 3.83 to converge, we must constrain the regime of integration to positive lengths $b > 0$ only.

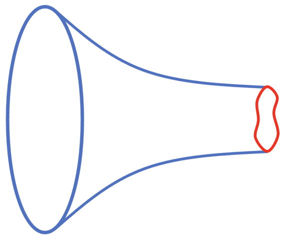
Gluing along the trumpet partition function

The amplitude of an EOW brane attached to the neck of a single trumpet can now be readily deduced from the usual cutting and gluing axioms. As a first step, one should introduce a hyperbolic defect in the bulk, creating a non-trivial monodromy along the thermal boundary circle. The procedure was explained in section 2.9 and shall not be repeated. The essential takeaway is to introduce a (normalized) character into the constrained disk partition function:

$$D_k(\phi) = \frac{\cos(-2k\phi)}{k \sinh 2\pi k} = \frac{\cos(kb)}{k \sinh 2\pi k}. \tag{3.86}$$

In a closed channel approach, the interior itself is unconstrained by the gravitational coset conditions, and the Wilson loop evaluates to the discrete series hyperbolic character derived above Eq 3.85. Gluing along positive b finally yields the same partition function Eq 3.3 derived by [36] in the boundary particle formalism;

$$Z_{EOW}(\beta) = \quad \beta$$



$$\mu \quad = \int_0^\infty db\, Z_{\text{trumpet}}(\beta, b) \frac{e^{-\mu b}}{2\sinh(b/2)}. \tag{3.87}$$

Here, we have derived it completely from first principles in group theory. This opens up a way to extrapolate the notion of EOW branes to more exotic theories of JT supergravity that have not yet been considered before in the literature.

A subtlety is that we glue the hyperbolic character of the discrete series representation corresponding to the EOW brane on the trumpet with the flat integration measure over geodesic lengths, while group theory instructs us to use the Weyl integration measure corresponding to class elements instead (c.f. Eq 2.316). We argue that the correspondence between JT gravity and group theory is only formally applicable and that the single trumpet partition function is obtained from only the numerator of the hyperbolic continuous character Eq 2.323. After all, it is only the numerator that appears directly from the holographic perspective in Eq 2.317. On the other hand, we have proven that the free particle integral coincides *exactly* with the insertion of a discrete series character (c.f. Eq 3.30). Therefore, it is important to keep both the numerator and denominator in the full quantum amplitude.

3.5 Physical application: Black Hole evaporation process

Within the holographic duality, a powerful tool to calculate the entanglement entropy is the Ryu-Takayanagi formula [33]. Here, the entropy of a boundary region is calculated from the generalized Bekenstein-Hawking entropy over an extremal surface in the bulk. Recently, this rule was modified in the context of black hole evaporation [32] [31] by adding the contribution of some island region I in the bulk. Explicitly, the rule states that the actual fine-grained (von Neumann) entropy of the Hawking radiation $S(R)$ is given by [34]:

$$S(R) = \min\left[\text{ext}_I\left(\frac{\text{Area}(\partial I)}{4G_N} + S_{\text{bulk}}(I \cup R)\right)\right] \tag{3.88}$$

, where $S_{\text{bulk}}(I \cup R)$ calculates the semiclassical entropy of the island and the Hawking radiation. This rule involves searching for all extremal surfaces of the island I that minimize the entropy in the spatial direction but maximize it in the time direction. Finally, we take the minimal value among all the extremal contributions. The resulting plot of the fine-grained entropy over time yields a unitary Page curve of the Hawking radiation in the black hole evaporation process. In particular, since we know that the black hole starts out in a unitary state, the central dogma of quantum gravity requires the final state to remain pure after evaporation. This would imply that in the early stages of evaporation, the entropy of the radiation should rise as a consequence of the entanglement between the interior black hole microstates. However, as more and more radiation comes out, there should be a unitary Page transition where the entropy decreases proportional to the area of the black hole horizon. A lot more is to be said about this Page curve. However, for brevity and to avoid shifting the main focus (EOW brane calculations), I redirect the reader to the extensive literature on this subject (in particular the excellent recent review notes [101] [25]).

The crux of the matter is thus to derive the island rule directly from gravitational path integral calculations, without resorting to the boundary holographic considerations of [32] [31]. Since the classical Ryu-Takayanagi formula can be derived from gravitational path integral calculations involving replicas, one suspects that the island prescription describing the generalized entropy should also be derivable by considering multiple copies of the evaporating black hole.

Clearly, one needs to go beyond the semiclassical calculation of the gravitational path integral and consider non-perturbative corrections to the latter. A series of seminal papers [35] and [34] demonstrated how the Page transition is achieved by including Euclidean wormholes connecting the different replica contributions. [34] especially will be the most relevant in the subject of this thesis. They considered a simple toy model of an evaporating black hole in JT gravity, where the Hawking radiation in some auxiliary reference system R is entangled with the black hole microstates. These interior partners are modelled precisely by EOW branes behind the BH horizon. The relevant gravitational amplitudes are precisely the half-moon disk amplitudes that we have calculated from an alternative perspective in group theory Eq 3.75.

One first calculates the Rényi entropy $S_R = \text{Tr}(\rho^n)$ by replicating the quantum system into n independent copies, and considering its total gravitational amplitude. Analytically continuing this result in n and taking the smooth limit $n \to 1$ recovers the standard von Neumann entanglement entropy $\rho_{vN} = -\text{Tr}(\rho \log \rho)$. We will see how the exact gravitational path integral can be obtained by performing a resummation of the exact planar quantum amplitudes obtained before, consistent with the prescribed boundary conditions. To recover the analogue of the Page curve, we will see that before the Page time, the dominant contribution consists of a disconnected topology with n separate replicas. After the Page time however, the dominant topology transitions to an n-boundary connected Euclidean wormhole solution. This will turn out to be consistent only if we reinterpret the path integral as an ensemble average of quantum theories, reminiscent of the analysis of random matrices in [30].

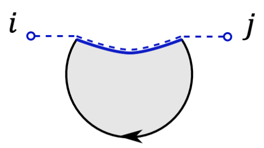
The starting point is to consider a black hole in JT gravity with an EOW brane anchored behind the BH horizon, for which the total action is given in Eq 3.1. Reintroducing the purely topological Einstein-Hilbert term and the proper prefactors, the precise action (in Euclidean signature) is given by (c.f. Eq 1.49):

$$I = -\frac{S_0}{4\pi}\left[\int_{\mathcal{M}} \sqrt{g}\, R + \int_{AdS} \sqrt{g_{uu}} K\right] - \frac{1}{16\pi G_N}\left[\int \Phi\sqrt{g}(R+2) + \int_{AdS} du\ \Phi\sqrt{g_{uu}}(K-1)\right] + \int_{EOW} dv\sqrt{g_{vv}}(\mu - \Phi K). \tag{3.89}$$

S_0 denotes the extremal entropy contribution of the parent nearly extremal black hole Eq 1.45 $S_0 = \Phi_0/(4G_N)$ in terms of the extremal value of the dilaton field. The EOW branes are located behind the BH horizon in Lorentzian signature if we restrict the range of the tension parameter to $\mu \geqslant 0$ [36]. Performing a $\mathbb{Z}_2$-quotient along the EOW brane fixed points purifies the doubled solution in Lorentzian signature.

We will henceforth be interested in the regime where the EOW brane describes a very large number k of possible internal states. Thus, we model the pure black hole internal states in terms of the possible microstates of the EOW brane, which we will denote by $|\psi_i\rangle_B$. The subscript B denotes the pure states behind the BH horizon. The overlaps $\langle\psi_i \mid \psi_j\rangle$ compute the relevant gravitational amplitudes, which we represent by [34]:

$$Z_1(\beta) = \langle\psi_i \mid \psi_j\rangle =$$



. (3.90)

The black boundary represents the asymptotic AdS_2 boundary, while the blue line represents the EOW brane connecting the pure states labeled by i and j. As always, the time direction flows from the ket $|\psi_j\rangle$ to the bra $\langle\psi_i|$, indicated by the direction of the arrow. These are precisely the half-moon amplitudes calculated earlier. For now, we will assume that these are simply orthogonal such that $\langle\psi_i \mid \psi_j\rangle \approx \delta_{ij}$.
Now, we imagine that we maximally entangle these pure states with an auxiliary quantum system R describing the early radiation states $|i\rangle_R$:

$$|\Psi\rangle = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} |\psi_i\rangle_B\, |i\rangle_R\,. \tag{3.91}$$

Here, $|\Psi\rangle$ represents the whole purified quantum state of the BH and the radiation.

A crude application of the island formula Eq 3.88 demonstrates the validity of this procedure. If we first take the island to be the empty set, then the area term vanishes and the result simply evaluates to the entanglement entropy $S_{vN} = -\mathrm{Tr}(\rho\log\rho)$ between the early radiation and the BH interior. The density matrix ρ is constructed by summing over the internal BH states $\rho = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$, which results in:

$$\rho_R = \frac{1}{k}\sum_{ij}^{k} |j\rangle\langle i|_R \langle\psi_i \mid \psi_j\rangle_B \tag{3.92}$$

$$\approx \frac{1}{k}\sum_{i}^{k} |i\rangle\langle i|_R\,. \tag{3.93}$$

Simple application of the von Neumann entropy formula yields $S_{vN} = -\text{Tr}(\rho \log \rho) = \log k$. On the other hand, taking the island region over the whole BH interior, the semiclassical entanglement entropy term $S_{vN}(R \cup I)$ vanishes (since the radiation and BH states together form a pure state). From the total action Eq 3.89, we infer the remaining area term of the island:

$$\frac{\text{Area}(\partial I)}{4G} \to S_0 + \frac{\Phi}{4G}. \tag{3.94}$$

This simply evaluates to the coarse-grained entropy of the black hole S_{BH} from the Bekenstein-Hawking prescription. The extremization in the island conjecture amounts to putting the boundary of the island at an extremal point of the dilaton field, which in this case lies at the bifurcation point of the Euclidean black hole. Thus, during evaporation, the island conjecture predicts that the entropy of radiation will exhibit a continuous transition between these two extremal cases:

$$S(R) = \min(S_{BH}, \log(k)). \tag{3.95}$$

[34] obtained this result directly from a gravitational calculation using the replica trick. One starts from the standard replica prescription to compute the von Neumann entropy of the radiation S_R from the Rényi entropy in the limit $n \to 1$:

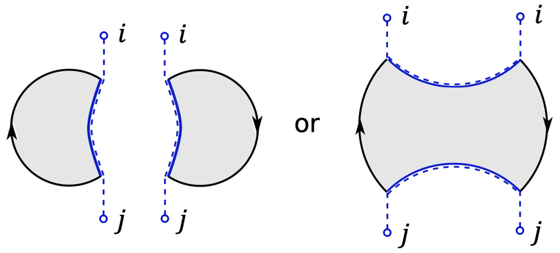
$$S_R = -\text{Tr}(\rho_R \log \rho_R) = -\lim_{n \to 1} \frac{1}{n-1} \log \text{Tr}(\rho_R^n) \tag{3.96}$$

This identity is readily verified by the rule of 'l Hôpital. We compute $\text{Tr}(\rho^n)$ by summing over all possible geometries with an n-boundary configuration. As an example, consider the related purity $\text{Tr}(\rho_R^2)$. Leaving $\langle \psi_i \mid \psi_j \rangle_B$ explicit in Eq 3.92, we calculate

$$\text{Tr}(\rho_R^2) = \frac{1}{k^2} \sum_{ij}^{k} \langle \psi_i \mid \psi_j \rangle_R \langle \psi_j \mid \psi_i \rangle_R = \frac{1}{k^2} \sum_{ij}^{k} |\langle \psi_i \mid \psi_j \rangle_R|^2. \tag{3.97}$$

We compute the gravitational amplitude by summing over all possible topologies compatible with a two-boundary configuration. To simplify the calculations, we assume that only planar geometries contribute in this discussion. The relevant contributions are a completely disconnected and a completely connected geometry [34]:

$$|\langle \psi_i \mid \psi_j \rangle_R|^2 =$$



. (3.98)

The connected geometry is achieved by a Euclidean wormhole linking the two asymptotic boundaries. We denote this geometry by $Z_2(\beta)$. More general connected n-boundary replicated geometries with asymptotic lengths β are denoted $Z_n(\beta)$. Using this notation, the purity is readily represented by:

$$\mathrm{Tr}(\rho_R^2) = \frac{1}{Z_1^2}\left(\frac{Z_1^2}{k} + Z_2\right) = \frac{1}{k} + \frac{Z_2}{Z_1^2} \tag{3.99}$$

, where one normalizes the solution by an overall Z_1^2. The first term represents the first geometry, where the EOW branes connect the two different labels i and j. In the sum Eq 3.97, this yields a factor k which is compensated by the k^2 in the overall normalization. On the other hand, both EOW branes in the Euclidean wormhole geometry connect to the same labels i and j. Summing yields a factor k^2, which is absorbed completely in the normalization of the denominator.

Retaining only dependence on the topological Einstein-Hilbert term in the total action Eq 3.89, the gravitational amplitudes scale only with the Euler characteristic $e^{\chi S_0}$. This is a topological invariant that is specified completely by the genus g and number of boundaries n: $\chi = 2 - 2g - n$. For disk-shaped amplitudes, both Z_1 and Z_2 are proportional to e^{S_0} with $\chi = 1$. Thus schematically;

$$\mathrm{Tr}(\rho_R^2) = \frac{1}{k} + \frac{1}{e^{S_0}}. \tag{3.100}$$

Loosely interpreting k as a time variable, we see that for small k, the disconnected geometry dominates in the purity with $\sim 1/k$. When k becomes of the order of the black hole entropy e^{S_0}, there appears an interchange in dominance, and the connected replicated geometry starts to dominate. This will purify the purity by a constant e^{-S_0}. This interchange in dominance is the main mechanism behind the Page transition.

Considering the more general Rényi entropy $S_R = \mathrm{Tr}(\rho_R^n)$, we again fill up the boundary conditions with all possible (planar) geometries consistent with the prescribed n-boundary boundary conditions. Normalizing the solution by Z_1^n, the completely disconnected phase has only a single k-loop, yielding [34]:

$$\mathrm{Tr}(\rho_R^n) \supset \frac{kZ_1^n}{k^n Z_1^n} = \frac{1}{k^{n-1}}. \tag{3.101}$$

Using the general prescription to calculate the von Neumann entropy Eq 3.96, the entanglement entropy in this phase of matter is characterized by:

$$S_R = \log(k). \tag{3.102}$$

On the other hand, when $k \gg e^{S_{BH}}$, the completely connected geometry dominates. In this geometry, the n EOW branes always connect to the same index, yielding

$$\mathrm{Tr}(\rho_R^n) \supset \frac{k^n Z_n}{k^n Z_1^n} = \frac{Z_n}{Z_1^n}. \tag{3.103}$$

Performing a $\mathbb{Z}_n$-quotient and continuing $n \to 1$, this simply becomes the unreplicated geometry for which the von Neumann entropy is given by the coarse-grained black hole entropy Eq 3.94

$$S_R = S_0 + \frac{\Phi_h}{4G} \tag{3.104}$$

, where the value of the dilaton at the horizon Φ_h represents the fixed point of the $\mathbb{Z}_n$ quotient. In this phase, the answer yields the thermodynamical entropy of the black hole expected at late times in a unitary evaporation process.
Thus, transitioning between geometries, we obtain a discontinuous version of the result predicted by the Island formula Eq 3.95.

Of course, this result is not consistent in the strict quantum mechanical interpretation, since on the one hand we assume

$$\langle \psi_i \mid \psi_j \rangle \approx \delta_{ij} \tag{3.105}$$

, while on the other hand

$$|\langle \psi_i \mid \psi_j \rangle|^2 = \delta_{ij} + \frac{Z_2}{Z_1^2}. \tag{3.106}$$

However if we assume that the true amplitude is in fact a random variable $\langle \psi_i \mid \psi_j \rangle = \delta_{ij} + e^{-S_0/2} R_{ij}$, where R_{ij} has mean zero, then one should interpret the gravitational path integral as calculating a coarse-grained average instead [34] :

$$\overline{\langle \psi_i \mid \psi_j \rangle} = \delta_{ij}, \qquad \overline{|\langle \psi_i \mid \psi_j \rangle|^2} = \delta_{ij} + \frac{Z_2}{Z_1^2}. \tag{3.107}$$

In this case, the Euclidean wormhole calculates the variance of this random variable. In fact, a whole branch of recent investigation in JT gravity studies the relation between Euclidean wormholes and random matrices, see e.g. [30] [29].

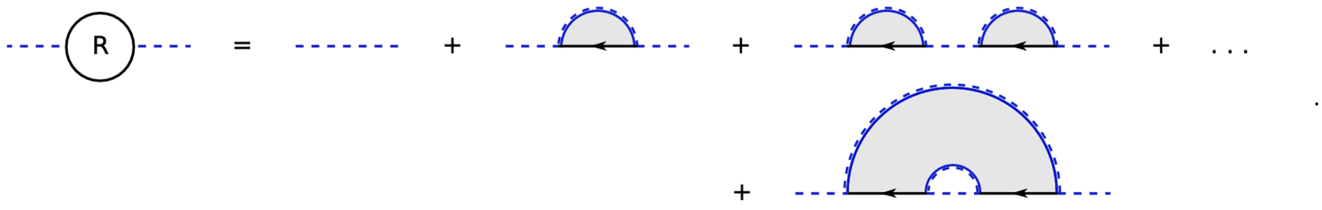
To obtain a smooth transition between these two regimes in the Page curve, one needs to be less schematic and study the exact correlators in JT gravity and sum over all possible planar geometries connecting the different boundary regions. The former task was already achieved in section 3.4. To investigate the latter, [34] used the technique of planar resummation. First of all, note that we can indeed constrain to planar geometries by assuming e^{S_0} and k to be large. In this regime, higher topological corrections will be suppressed in the gravitational amplitudes by e^{-2S_0} for adding handles ($\chi = 2 - 2g - n$) or by $1/k^2$ for introducing crossings.
A property familiar from random matrix theories involves the resolvent matrix $R_{ij}(\lambda)$ of ρ_R:

$$R_{ij}(\lambda) = \left(\frac{1}{\lambda \mathbf{1} - \rho_R} \right)_{ij}. \tag{3.108}$$

This matrix contains the eigenvalues of ρ_R at the location of its poles in λ. Using a geometric series, we expand this quantity in a perturbation series in $1/\lambda$:

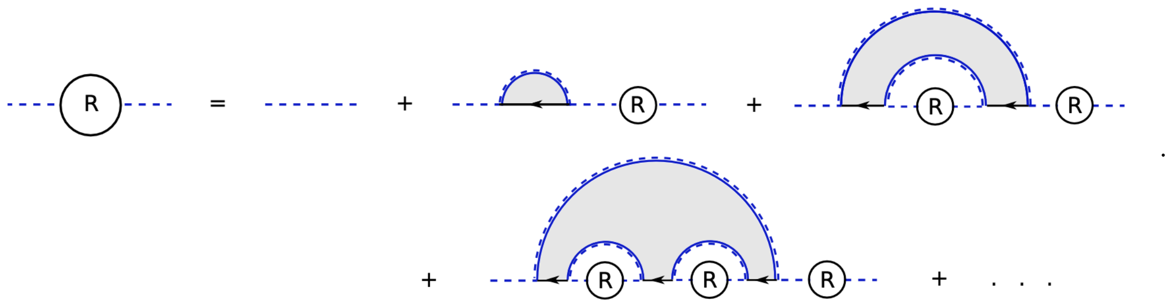
$$R_{ij}(\lambda) = \frac{1}{\lambda} \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} (\rho_R^n)_{ij}. \tag{3.109}$$

[34] denotes this pictorially as:



(3.110)

Here, the dashed index lines denote the free indices i, j on the left and right side respectively. We interpret these lines as bare propagators that come with a factor $1/\lambda$. The arrows on the other hand come with factors $1/(kZ_1)$ in the normalization of the density matrix. The shaded regions that fill up the boundary conditions with all possible planar diagrams, denote the proper gravitational amplitudes. An efficient way to fill up the boundary conditions with all possible geometries, is to use an iterative perturbation in the resolvent matrix itself [34]:



(3.111)

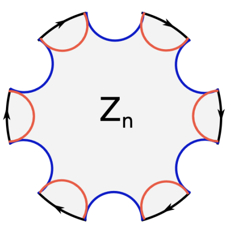
Using the diagrammatic rules, the explicit amplitude is:

$$R_{ij}(\lambda) = \frac{1}{\lambda}\delta_{ij} + \frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{Z_n}{(kZ_1)^n}R(\lambda)^{n-1}R_{ij}(\lambda) \tag{3.112}$$

, where $R(\lambda)$ represents the trace of the resolvent matrix: $R(\lambda) = \sum_i^k R_{ii}(\lambda)$. Taking the trace yields a relatively simple expression [34]:

$$\lambda R(\lambda) = k + \sum_{n=1}^{\infty} Z_n \frac{R(\lambda)^n}{k^n Z_1^n}. \tag{3.113}$$

To proceed, we need the explicit gravitational amplitude of an n-boundary Euclidean wormhole geometry connected by a collection of n EOW branes, represented by a pinwheel geometry [34]:



(3.114)

Here, red lines denote geodesic boundaries, while blue lines denote EOW branes as usual. We may relatively straightforwardly generalize the half-moon amplitudes in section 3.4 to the pinwheel geometry by gluing n Hartle-Hawking states Eq 3.70 and n states associated to the EOW brane (Wilson line) Eq 3.65 to a cap wavefunction with $2n$ geodesic boundaries. The latter is specified by an overlap $\langle \phi_1, \dots, \phi_{2n} \mid \mathbf{1} \rangle$ in the closed radial slicing with coset boundary conditions (c.f. 2.193 in the limit of vanishing β):

$$I_{2n}(\phi_1, \dots, \phi_{2n}) = \int_0^\infty dk\; k \sinh(2\pi k)\; e^{\phi_1} K_{2ik}(e^{\phi_1}) \dots e^{\phi_{2n}} K_{2ik}(e^{\phi_{2n}}). \tag{3.115}$$

Gluing with the appropriate Haar measure factor $e^{-2\phi}$ along the hyperbolic group elements yields the generalization of the half-moon amplitude[3]:

$$Z_n = e^{S_0} \int_{-\infty}^{+\infty} d\phi_1 \dots d\phi_{2n}\; e^{-2\phi_1} \dots e^{-2\phi_{2n}} I_{2n}(\phi_1, \dots, \phi_{2n}) \langle \mathbf{1} \mid e^{-\beta H} \mid \phi_1 \rangle e^{2\phi_2 \ell} \dots \tag{3.116}$$
$$\times \langle \mathbf{1} \mid e^{-\beta H} \mid \phi_{2n-1} \rangle e^{2\phi_{2n} \ell}.$$

Specifying to the geodesic approximation $\mu \approx \ell$ and using the gravitational identification Eq 3.60 $b = -2\phi$, there are n orthogonality integrations involving the asymptotic Hartle-Hawking states Eq 2.261:

$$\int_{-\infty}^\infty db\; K_{2ik}(e^{-b/2}) K_{2ik'}(e^{-b/2}) = \frac{\delta(k-k')}{k' \sinh 2\pi k'} \tag{3.117}$$

This removes the remaining Plancherel measure in the asymptotic Hartle-Hawking states. The remaining n integrals over the EOW brane wavefunctions are evaluated using the integral identity Eq 3.74

$$\int_{-\infty}^\infty db\; e^{\left(\frac{1}{2}-\mu\right)b} K_{2ik}(e^{-\frac{b}{2}}) = \frac{2^{2\mu}}{4} |\Gamma(\mu - \frac{1}{2} + ik)|^2. \tag{3.118}$$

This finally yields the factorized result:

$$Z_n = e^{S_0} \int dk k \sinh 2\pi k\; y(k)^n, \qquad y(k) = e^{-\beta k^2} 2^{2\mu-2} |\Gamma(\mu - \frac{1}{2} + ik|^2. \tag{3.119}$$

I should note again that this amplitude was obtained from the boundary particle perspective instead in [34]. This factorization gravely simplifies the expression of the resolvent Eq 3.113 by performing a geometric series under the integral:

$$\begin{aligned} \lambda R(\lambda) &= k + \sum_{n=1}^\infty \frac{Z_n R(\lambda)^n}{k^n Z_1^n} = k + e^{S_0} \int dk\; \sinh 2\pi k \sum_{n=1}^\infty \left(\frac{R(\lambda) y(k)}{k Z_1} \right)^n \\ &= k + e^{S_0} \int dk\; \sinh 2\pi k\; \frac{R(\lambda) y(k)}{k Z_1 - R(\lambda) y(k)}. \end{aligned} \tag{3.120}$$

From this resolvent, one can find the density of eigenvalues of the density matrix $\rho_R(\lambda)$ by analytically continuing the definition Eq 3.108 into the complex plane and taking the discontinuity along the positive real

[3] The factor e^{S_0} resembles the extremal entropy contribution from the Einstein-Hilbert action, for which the disk geometry always has $\chi = 1$.

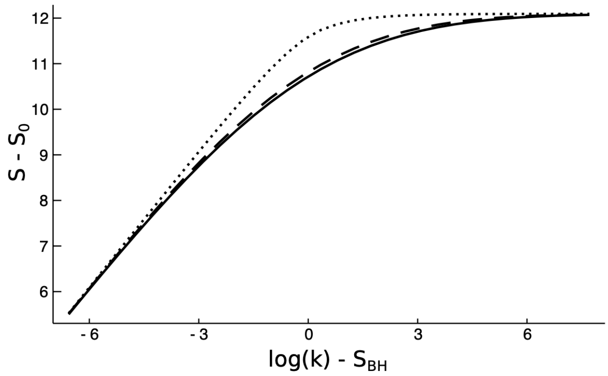


Figure 3.2: Exact von Neumann entropy calculated in the planar approximation using different numerical techniques (plotted for $\beta = 3$ and large μ). As a function of k, this exhibits a smoothed version of the Page transition around $\log(k) \sim S_{BH}$.

axis:

$$\frac{1}{2\pi i}\left(R(\lambda - i\epsilon) - R(\lambda + i\epsilon)\right) = \frac{1}{2\pi i}\sum_j \left(\frac{1}{(\lambda-\lambda_j) - i\epsilon} - \frac{1}{(\lambda-\lambda_j)+i\epsilon}\right)$$
$$= \frac{1}{2\pi i}\sum_j \frac{2i\epsilon}{(\lambda-\lambda_j)^2+\epsilon^2}$$

This last expression is precisely the Cauchy distribution $\frac{\epsilon}{(\lambda-\lambda_i)^2+\epsilon^2} = \pi\delta(\lambda-\lambda_i)$, yielding the spectral distribution of the density matrix:

$$\frac{1}{2\pi i}(R(\lambda - i\epsilon) - R(\lambda + i\epsilon)) = \rho_R(\lambda), \qquad \rho_R(\lambda) = \sum_j \delta(\lambda - \lambda_j). \tag{3.121}$$

The von Neumann entropy follows directly from the definition

$$S_R = -\int d\lambda \rho(\lambda)\lambda\log(\lambda). \tag{3.122}$$

Numerically calculating the distribution of $R(\lambda)$ from the exact planar expression Eq 3.120, [34] finds a smoothed out version of the qualitatively expected Page curve Eq 3.95 (figure 3.2). We conclude that the gravitational calculations involving EOW branes are able to shed a new light on the black hole evaporation process, and seem to produce explicitly a unitary Page transition.

Chapter 4

EOW branes in JT supergravity

"The most beautiful thing we can experience is the mysterious. It is the source of all true art and all science."

Einstein, Albert

4.1 Metric formulation of $\mathcal{N} = 1$ JT supergravity

Much of the current exploration in JT gravity investigates whether some of its generic lessons generalize to other related models of lower-dimensional dilaton gravity. By the time of writing this thesis, considerable amount of attention is devoted to understanding the supersymmetric extensions of the JT supergravity model. By introducing a boundary term analogous to [17], Forste et al. [39] showed that the dynamics of JT SUGRA in superspace is holographically described by a superanalogue of the Schwarzian boundary theory. Both the one-loop exact solution of the partition function, and the dual matrix ensemble have been generalized to JT supergravity, see [20] and [102] respectively. The boundary correlators have also been obtained for $\mathcal{N} = 1$ supergravity by exploiting its exact relation to 2d Liouville superconformal theory [21] [64]. More recently, its gauge theoretic description in terms of an OSp$(1|2, \mathbb{R})$-supergroup BF model was exploited to obtain a bulk interpretation of the bilocal operators in the super-Schwarzian theory in terms of super-Wilson line insertions [40], along the lines of the previous chapters based on [22] [24]. This study demonstrates that the constrained subsemigroup setup of the global supergroup persists in the supersymmetric extension.

A natural question to ask is whether we can generalize the group theoretic description of the EOW branes to the supergroup model. As a first step, we should start from the action of an EOW brane in superspace, analogous to the bosonic case in Eq 3.1, and reinvestigate its relation to Wilson line insertions in the supersymmetric path integral. Although the action of a free particle in 2d superspace has already been formulated in [46], there does not seem to be an appropriate action available in the current literature that captures the entire geodesic dynamics. We therefore define an appropriate supersymmetric analogue of the extrinsic curvature in superspace, and make an educated guess for the appearance of the boundary action. The most interesting results are again obtained for the quantum amplitudes of an EOW brane at the neck of a supersymmetric trumpet. Due to the global structure of the $\mathrm{OSp}(1|2,\mathbb{R})$-supergroup, we find two disconnected sectors depending on the periodicity of the fermionic coordinates around the thermal boundary circle. We find that the periodic Ramond sector does not yield the spurious UV divergence for $b \to 0$ of the bosonic case. This resembles how UV divergences in bosonic string theory are cured in superstring theory, see e.g. [103].

To arrive at these results, I will first review the supergroup structure of the $\mathcal{N}=1$ JT SUGRA model, along the lines of [40]. This implies formulating the first order action in superspace in terms of a topological $\mathfrak{osp}(1|2,\mathbb{R})$ BF theory. Using this identification, we obtain the exact quantum amplitudes in terms of the positive sub-semisupergroup formulation of $\mathrm{OSp}^+(1|2,\mathbb{R})$. This again restricts the integration space to the smooth hyperbolic component of the moduli space. The reader might enjoy the close resemblance to the development of the bosonic case in the previous chapters. To emphasize this resemblance, I use an isomorphism of the algebra compared to [40], to fit as much as possible with the conventions of the bosonic algebra of the previous chapters. As opposed to the structure of the thesis so far, I have deliberately stripped down the first parts of this review on the metric formulation and superconformal symmetry to its core essence. However, without going into too much of the calculational details, I have still tried to be as concrete as possible.

We start by formulating the action of $\mathcal{N}=1$ JT supergravity (including its natural boundary term as formulated in [39], up to a total factor of $-1/4\pi G$) in superspace as:

$$\boxed{I_{JT}^{\mathcal{N}=1} = \frac{1}{4}\left[\int_{\Sigma} d^2z d^2\theta \; E\Phi(R_{+-}+2) + 2\int_{\partial\Sigma} d\tau d\vartheta \; \Phi K\right]} \tag{4.1}$$

, where Φ denotes the supersymmetric dilaton field, R_{+-} the Ricci supercurvature, and E the frame field determinant in superspace. Note the striking resemblance between the action formulated in superspace and the bosonic JT gravity action in Eq 1.48.

The bulk term spans the $2|2$-dimensional supermanifold Σ, while the boundary curve $\partial\Sigma$ is infinitesimally thickened in its fermionic coordinate to co-dimension $1|1$.

In general, we locally parameterize the $\mathcal{N}=1$ superspace by a pair of bosonic and fermionic coordinates z^m, θ^μ ($m=0,1$, $\mu=0,1$) respectively. The latter satisfy the anti-commutative Grassmann algebra[1] $\{\theta^\mu,\theta^\nu\}=0$. Along the lines of [104], the $2|2$-dimensional supergeometry is easy to describe by a pair of holomorphic and anti-holomorphic coordinates

$$Z^M = (z^m, \theta^\mu) = (z, \bar{z}, \theta, \bar{\theta}). \tag{4.2}$$

Due to the anticommutative nature of the fermionic partners, care has to be taken when swapping two super-

[1] In [40], the authors also used the order preserving convention of complex conjugation of Grassmann numbers $(\theta^\mu\theta^\nu)^* = \theta^\mu\theta^\nu$.

space coordinates[2] (or its differentials) [40]

$$Z^M Z^N = (-)^{MN} Z^N Z^M, \quad dZ^M \wedge dZ^N = -(-)^{MN} dZ^N \wedge dZ^M, \quad dZ^M Z^N = (-)^{MN} Z^N dZ^M. \quad (4.3)$$

We imagine the numbers M, N in the exponent $(-)^{MN}$ to be $\mathbb{Z}_2$-valued (0,1), and are either even or odd if the respective coordinate is bosonic or fermionic.

Since any theory of supergravity is equipped with spinor fields whose transformation rule is defined in terms of the Lorentz group, we again introduce a set of local $U(1)$ Lorentz indices $A=(a,\alpha)$. $a=0,1$ are the bosonic frame indices and are raised and lowered by the flat Euclidean metric δ_{ab}. Upper and lower for the bosonic frame indices are therefore immaterial. In general D dimensions, the dimensionality of the Clifford matrices is $2^{[D/2]}$, where $[D/2]$ denotes the integer part of $D/2$ [59]. Therefore in $D=2$, the spinors have two free labels $\alpha=+,-$. The fermionic indices are raised and lowered with respect to the Levi-Civita tensor $\epsilon_{\alpha\beta}$ (with conventionally $\epsilon_{+-}=-1$). We collectively denote the local metric for both fermionic and bosonic entries by κ_{AB}, with:

$$\kappa_{ab} = \delta_{ab}, \qquad \kappa_{\alpha\beta} = \epsilon_{\alpha\beta}, \qquad \kappa_{a\alpha} = \kappa_{\alpha a} = 0. \quad (4.4)$$

A general feature is that the local metric is always block diagonal and is antisymmetric in the fermionic block. We denote this property as:

$$\kappa_{AB} = (-)^{AB}\kappa_{BA} = (-)^A \kappa_{BA} = (-)^B \kappa_{BA}. \quad (4.5)$$

The last identities hold since the Cartan-Killing metric is block-diagonal and the pair A, B always shares the same parity. Since the fermionic entries anticommute, we have in general

$$V_A W^A = (-)^A W^A V_A \neq W^A V_A. \quad (4.6)$$

Note that local Lorentz indices are collectively denoted by capital letters at the beginning of the alphabet, while superspace coordinates are labeled by capital letters in the middle of the alphabet.
To transition between the two frames, we again introduce a superframefield $E^A = dZ^M E_M{}^A$ and its inverse $E_A{}^M$, satisfying:

$$E_B{}^M E_M{}^A = \delta_B^A, \qquad E_N{}^A E_A{}^M = \delta_N^M. \quad (4.7)$$

Lorentz vectors and covectors are defined by their transformation under local Lorentz transformations

$$\delta V^A = L^A{}_B V^B, \qquad \delta V_A = -V_B L^B{}_A \quad (4.8)$$

, where the local Lorentz group in 2d is determined in terms of a single bosonic number L:

$$L^A{}_B = L E^A{}_B. \quad (4.9)$$

[2] We can expand any number in superspace in a Grassmann expansion $\mathbf{z} = z_0 + z_1\theta + z_2\bar{\theta} + z_3\bar{\theta}\theta$, where due to the Grassmann algebra, any term is at most linear in the Grassmann variables $\theta,\ \bar{\theta}$. We call the purely bosonic piece z_0 the body of the supernumber. The remainder is dubbed the soul, and can have both odd and even parity. Positivity of a supernumber is defined as $\mathbf{z} = \mathbf{z}^*$. It can be proven [40] that any supergroup number is positive iff its body is positive: $\mathbf{z} > 0 \ \leftrightarrow\ z_0 > 0$. The intuition is that Grassmann combinations are expanded to linear order in the Taylor expansion. As such, their value can be thought of as infinitesimal compared to the bosonic body, and has no effect on the positivity.

${E^A}_B$ is the generalized Levi-Civita symbol[3], defined in terms of the Levi-Civita symbol ϵ_{ab}, and the γ_5-matrix of the 2d Clifford algebra [40]:

$$ {E^a}_b = {\epsilon^a}_b, \qquad {E^\alpha}_b = {E^a}_\beta = 0, \qquad {E^\alpha}_\beta = -\frac{1}{2}{(\gamma_5)^\alpha}_\beta. \tag{4.10} $$

The gamma matrices in $D = 2$ satisfy the Clifford algebra

$$ \{\gamma_a, \gamma_b\} = 2\delta_{ab}, \qquad \gamma_5 = -\gamma_0\gamma_1 \tag{4.11} $$

, leading to $\{\gamma_a, \gamma_5\} = 0$, $\gamma_5^2 = -1$. As opposed to the entries of the gamma matrices, we imagine the entries of the spinors terms ψ_m^α to be Grassmann valued. In $D = 2$, we are free to choose both γ_0 and γ_1 to be symmetric. From the Clifford algebra, this implies that γ_5 is antisymmetric. In any case, the generalized Levi-Civita symbol ${E^A}_B$ is antisymmetric in both the bosonic and fermionic blocks:

$$ {L^A}_B = -{L_B}^A. \tag{4.12} $$

Since the matrix of infinitesimal Lorentz transformations ${L^A}_B$ contains no Grassmann entries, this property ensures that Lorentz bilinears are invariant under local Lorentz transformations Eq 4.8 in both orderings:

$$ \begin{aligned} \delta(V_A W^A) &= \delta V_A W^A + V_A \delta W^A \\ &= -V_B {L^B}_A W^A + V_A {L^A}_B V^B = 0 \end{aligned} \tag{4.13} $$

$$ \begin{aligned} \delta(V^A W_A) &= \delta V^A W_A + V^A \delta W_A \\ &\equiv {L^A}_B V^B W_A - V^A W_B {L^B}_A \\ &= -V^B {L_B}^A W_A + V^A {L_A}^B W_B = 0. \end{aligned} \tag{4.14} $$

We lower the spinor index of the Majorana conjugate with respect to the metric in the NW-SE direction (*north-west - south-east*) [59]:

$$ \overline{\psi}_\alpha = \psi^\beta C_{\beta\alpha}, \quad \leftrightarrow \quad \overline{\psi} = \psi^T C \tag{4.15} $$

, where $C_{\alpha\beta} = \epsilon_{\alpha\beta}$ in 2d. Thus spinor contactions in the inner product $\psi^\alpha\epsilon_{\alpha\beta}\psi^\beta$ are defined SW-NE:

$$ \psi^\alpha \epsilon_{\alpha\beta}\psi^\beta = \bar{\psi}_\beta \psi^\beta = \bar{\psi}\psi. \tag{4.16} $$

For Majorana spinors, both the definition of the Majorana and Dirac conjugate $\overline{\psi} = \psi^\dagger\gamma_0$ are in fact equivalent [59]. Note that we need to fix these conventions beforehand since the metric is antisymmetric in its fermionic block:

$$ V_A W^A = V^B \kappa_{BA} W^A = (-)^B V^B W^A \kappa_{AB} = (-)^B V^B W_B \neq V^A W_A \tag{4.17} $$

In 2d, the definitions Eqs 4.8 imply the correct transformation rules for spinors and conjugate spinors under

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[3] Mind the difference between the generalized Levi-Civita symbol and the super frame fields.

infinitesimal local Lorentz transformations:

$$\delta\psi^{\alpha} = -\frac{1}{2}L\gamma_5^{\alpha}{}_{\beta}\psi^{\beta}, \qquad \delta\overline{\psi}_{\alpha} = \frac{1}{2}L\overline{\psi}_{\beta}(\gamma_5)^{\beta}{}_{\alpha}. \tag{4.18}$$

In any number of dimensions, the spinor bilinear $\overline{\psi}\chi$ is Lorentz-invariant using the infinitesimal form of the Lorentz transformation above. Its symmetry properties, however, vary for every dimension[4]. The simple form of C in 2d allow us to readily deduce the correct symmetry properties

$$\overline{\psi}\chi = \overline{\chi}\psi, \quad \overline{\psi}\gamma_a\chi = -\overline{\chi}\gamma_a\psi, \quad \overline{\psi}\gamma_5\chi = -\overline{\chi}\gamma_5\psi. \tag{4.19}$$

In parallel to the first-order formalism discussed in section 2.1, we define a spin connection $\Omega^A{}_B$ that transforms inhomogeneously under local Lorentz transformations, and acts as its local gauge field:

$$\delta\Omega^A{}_B = L^A{}_C\Omega^C{}_B - \Omega^A{}_C L^C{}_B - dL^A{}_B. \tag{4.20}$$

$d = dZ^M\partial_M$ acts as the exterior derivative in superspace[5]. This defines a Lorentz-covariant derivative:

$$DV^A = dV^A + \Omega^A{}_B \wedge V^B, \qquad DV_A = dV_A - \Omega^B{}_A \wedge V_B \tag{4.21}$$

, which does transform covariantly under local Lorentz transformations in superspace:

$$\begin{aligned}
\delta(DV^A) &= d(\delta V^A) + \delta\Omega^A{}_B V^B + \Omega^A{}_B\delta V^B \\
&= \cancel{dL^A{}_B V^B} + L^A{}_B dV^B + L^A{}_C\Omega^C{}_B V^B - \cancel{\Omega^A{}_C L^C{}_B V^B} - \cancel{dL^A{}_B V^B} + \cancel{\Omega^A{}_B L^B{}_C V^C} \\
&= L^A{}_B\left(dV^B + \Omega^B{}_C V^C\right).
\end{aligned}$$

Due to the simplification in 2d (Eq 4.9), the spin connection can also be written in terms of the generalized Levi-Civita symbol:

$$\Omega^A{}_B = \Omega E^A{}_B \tag{4.22}$$

, where $\Omega = dZ^M\Omega_M$ is a single one-form in superspace. We likewise denote the covariant exterior derivative as $D = dZ^M D_M$. This e.g. defines a bosonic Lorentz-covariant derivative on the bottom component ψ_m^{α} of $E^{\alpha}{}_m$;

$$D_m\psi_n' = \partial_m\psi_n' - \frac{1}{2}\omega_m\gamma_5\psi_n', \quad \text{with } \omega_m \text{ the bottom component of } \Omega_m. \tag{4.23}$$

The definitions of torsion and curvature two-forms in terms of the first and second Cartan structure equations Eqs 2.5, 2.12 naturally generalize to the definitions of the supertorsion and supercurvature respectively:

$$T^A = DE^A = dE^A + \Omega^A{}_B \wedge E^B = \frac{1}{2}T^A{}_{BC}E^B \wedge E^C, \tag{4.24}$$

$$R^A{}_B = d\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B = \frac{1}{2}R^A{}_{BCD}E^C \wedge E^D. \tag{4.25}$$

[4]See [59] chapter 3 for a detailed discussion

[5]We define the Grassmann derivative acting from the left.

These are readily checked to satisfy the generalized Bianchi-identities

$$DT^A = R^A{}_B \wedge E^B, \qquad DR^A{}_B = 0. \tag{4.26}$$

In 2d, the definition of the supercurvature simplifies considerably since $\Omega \wedge \Omega = 0$, leading to:

$$R^A{}_B = FE^A{}_B, \quad \text{where} \quad F \equiv d\Omega. \tag{4.27}$$

Returning to the first order JT supergravity action in superspace Eq 4.1, we can unpack the dilaton superfield ϕ in supercoordinates in terms of a real dilaton field ϕ, the dilatino λ, and a scalar auxiliary field F;

$$\Phi = \phi + \overline{\theta}\lambda + \overline{\theta}\theta F. \tag{4.28}$$

[40] formulates the supergeometry in the Wess-Zumino gauge. This gauge fixes the components of the supertorsion, leaving only a non-zero $T^a_{\beta\gamma}$.
Within this gauge, all geometrical quantities can be expressed in terms of the spin-1 frame field e^a_m, the spin-3/2 gravitino ψ^α_m, and an auxiliary scalar field A that serves as the bottom component of R_{+-}. Within the superspace expansion, e^a_m and ψ^α_m correspond to the bottom components (lowest order in the Grassmann expansion) of $E^a{}_m$, and $E^\alpha{}_m$ respectively. The integral over the dilaton superfield localizes the integration contour of the path integral to surfaces of negative supercurvature $R_{+-} = -2$.

Denoting $E = \text{sdet}(E^A{}_M)$ as the superdeterminant, one can show that within this gauge, the top component of $E\Phi(R_{+-} + 2)$ is given by [40] [105]:

$$e\overline{\theta}\theta \left(F(A+2) + \frac{1}{2}\phi \left(R - A - \frac{1}{2}\epsilon^{mn}\overline{\psi}_m\gamma_5\psi_n \right) + \overline{\lambda}\epsilon^{mn}D_m\psi_n + \frac{1}{2}\overline{\lambda}\epsilon^{mn}\gamma_m\psi_n \right) \tag{4.29}$$

, where $R = 2\epsilon^{mn}\partial_m\omega_n$ is the scalar curvature corresponding to the bottom component ω_m of the superspinconnection Ω_m. A and e denote the bottom components of R_{+-} and E respectively. In the current gauge, this is in turn constrained by the torsion constraints to [40]:

$$\omega_m = -\epsilon^{n\ell}e_{am}\partial_n e^a_\ell + \frac{1}{2}\overline{\psi}_m\gamma^n\gamma_5\psi_n. \tag{4.30}$$

The constraint Eq 4.30 descends from the component-wise torsion constraints [40]:

$$T^a{}_{mn}| = \partial_{[m}e^a_{n]} + \epsilon^a{}_b\omega_{[m}e^b_{n]} - \frac{1}{4}\overline{\psi}_{[m}\gamma^a\psi_{n]} \equiv 0. \tag{4.31}$$

The bar indicates the bottom component, and the brackets denote antisymmetrization with weight one.
Integrating over the dilaton superfield imposes

$$A = -2, \qquad R = \frac{1}{2}\epsilon^{mn}\overline{\psi}_m\gamma_5\psi_n - 2. \tag{4.32}$$

Using the normalized integral over Grassmann fields $\int d^2\theta\ \overline{\theta}\theta = 2$, and the constraint for $A = -2$, we may plug the top component Eq 4.29 in the superspace action and obtain the JT supergravity action at the level of the components:

$$\begin{aligned} I_{JT}^{\mathcal{N}=1} = \frac{1}{2}\int_\Sigma d^2ze \Bigg[\frac{1}{2}\phi\left(R+2-\frac{1}{2}\epsilon^{mn}\overline{\psi}_m\gamma_5\psi_n\right)+\overline{\lambda}\epsilon^{mn}D_m\psi_n+\frac{1}{2}\overline{\lambda}\epsilon^{mn}\gamma_m\psi_n \\ -\phi_a\left(\epsilon^{mn}\partial_m e_n^a+\epsilon^a{}_b\epsilon^{mn}\omega_m e_n^b-\frac{1}{4}\epsilon^{mn}\overline{\psi}_m\gamma^a\psi_n\right)\Bigg]. \end{aligned} \tag{4.33}$$

The last term introduces a set of auxiliary Lagrange multipliers ϕ_a, enforcing the torsion constraints Eq 4.31 in the current gauge. Using the general form identities [40] $\xi \wedge \xi' = \xi_m\xi'_n e_a^m e_n^b \epsilon^{ab} e d^2z = d^2ze\ \epsilon^{mn}\xi_m\xi'_n = e^0 \wedge e^1\epsilon^{mn}\xi_m\xi'_n$, and $d\xi = d^2ze\ \epsilon^{mn}\partial_m\xi_n = e^0 \wedge e^1\epsilon^{mn}\partial_m\xi_n$, together with the definition of the scalar curvature $R = 2\epsilon^{mn}\partial_m\omega_n$, leads to $d\omega = \frac{1}{2}e^0 \wedge e^1 R = \frac{1}{2}d^2ze\ R$. Inserted in the action yields the equivalent first order form of the action in components [40]:

$$\boxed{I_{JT}^{\mathcal{N}=1} = \frac{1}{2}\int_\Sigma \left[\phi\left(d\omega+e^0e^1-\frac{1}{4}\overline{\psi}\gamma_5\psi\right)+\overline{\lambda}D\psi+\frac{1}{2}\overline{\lambda}e^a\gamma_a\psi-\phi_a\left(de^a+\epsilon^a{}_b\omega e^b-\frac{1}{4}\overline{\psi}\gamma^a\psi\right)\right]}. \tag{4.34}$$

Following [40], I omitted writing wedge products $\wedge$ explicitly, to avoid an awkward double notation with gamma matrices in between. One can check that the $\mathcal{N}=1$ local supersymmetry transformations [40] [105]

$$\begin{aligned} &\delta_\xi e^a = \frac{1}{2}\overline{\xi}\gamma^a\psi, \qquad \delta_\xi\omega = \frac{1}{2}\overline{\xi}\gamma_5\psi, \qquad \delta_\xi\psi^\alpha = D\xi^\alpha+\frac{1}{2}e^a(\gamma_a)^\alpha{}_\beta\xi^\beta \\ &\delta_\xi\phi^a = -\frac{1}{2}\overline{\xi}\gamma^a\lambda, \qquad \delta_\xi\phi = -\frac{1}{2}\overline{\xi}\gamma_5\lambda, \qquad \delta_\xi\lambda^\alpha = -\frac{1}{2}\phi^a(\gamma_a)^\alpha{}_\beta\xi^\beta+\frac{1}{2}\phi(\gamma_5)^\alpha{}_\beta\xi^\beta. \end{aligned} \tag{4.35}$$

leave this action invariant: $\delta_\xi I_{JT}^{\mathcal{N}=1} = 0$, using an appropriate Fierz rearrangement. $D\xi^\alpha$ denotes the bottom component of the Lorentz covariant derivative Eq 4.21; $D\xi^\alpha = d\xi^\alpha - \frac{1}{2}\omega \wedge (\gamma_5\xi)^\alpha$.
Although this first order component formulation is less transparent than its superspace counterpart Eq 4.1, it will allow us to transition to the $\mathfrak{osp}(1|2,\mathbb{R})$ BF formulation later.

4.2 Superconformal symmetry breaking and the Super-Schwarzian

The previous section has been quite abstract, where generic supergravity calculations at the level of the components necessarily require a lot of non-trivial algebra that I omitted for clarity in this discussion. In this section, we return to the superspace formulation of JT SUGRA (Eq 4.1), and focus on the boundary term. As well known by now, the dynamics of pure JT gravity is equivalently described by an effective Schwarzian boundary action that weights the different boundary reparametrization modes. Chapter 1 reviewed among others the well-known mechanism of the conformal symmetry breaking [17], in which the dilaton equation of motion imposes the metric to describe patches of pure AdS_2 in the bulk, while the remaining degrees of freedom in the conformal gauge can be translated to the reparametrization modes of the boundary cut-off in Poincaré coordinates. The bulk action vanishes on-shell, while the extrinsic curvature along the boundary curve is described in terms of the Schwarzian derivative.
Forste et al. [39] generalize this setup to JT supergravity, and come to the conclusion that $\mathcal{N}=1$ JT SUGRA

is holographically described by the super-Schwarzian action along a $1|1$-dimensional wiggly boundary curve. Let me rephrase their argument in the current setup.

4.2.1 Asymptotic superconformal reparametrization symmetries

In the superconformal gauge, one fixes the metric to

$$ds^2 = dz^M g_{MN} dz^N \equiv e^{2\Sigma} d\mathbf{z} \otimes d\bar{\mathbf{z}} \tag{4.36}$$

, for some bosonic superfield Σ. $d\mathbf{z}$ is the length element in superspace, defined as $d\mathbf{z} = dz + \theta d\theta$. This gauge fixes the value of the scalar curvature to [40]

$$R_{+-} = 2e^{-\Sigma} D\overline{D}\Sigma \tag{4.37}$$

, where $D \equiv \partial_\theta + \theta\partial_z$ and $\overline{D} = \partial_{\overline{\theta}} + \overline{\theta}\partial_{\overline{z}}$ are the 2d holomorphic and anti-hololorphic superderivatives respectively. Intuitively, the superderivative is the square root of the partial derivative ∂_z, in the sense that $D^2 = \partial_z$. The dilaton equations of motion set $R_{+-} \equiv -2$, leading to the super-Liouville equation in superspace

$$D\overline{D}\Sigma + e^{\Sigma} = 0. \tag{4.38}$$

Its solutions are well-known [104] [40]:

$$e^{\Sigma} = \frac{D\theta' \bar{D}\bar{\theta}'}{(z' - \overline{z}' - \theta'\overline{\theta}')} \tag{4.39}$$

for some parametrized holomorphic and anti-holomorphic bosonic $z'(z,\theta)$, $\overline{z}'(\overline{z},\overline{\theta})$, and fermionic $\theta'(z,\theta)$, $\overline{\theta}'(\overline{z},\overline{\theta})$ superfields. These are additionally constrained by the superconformal constraints:

$$Dz' = \theta' D\theta', \quad \overline{D}\overline{z}' = \overline{\theta}'\bar{D}\bar{\theta}'. \tag{4.40}$$

We should think of z' and $\overline{z}'$ as the superanalogues of the respective lightcone coordinates $U(u)$ and $V(v)$ defined in chapter 1. The constraint $Dz' = \theta' D\theta'$ and its complex conjugate imply that (z', θ') and $(\overline{z}', \bar{\theta}')$ are (anti)-holomorphic superconformal transformations. These can be solved explicitly in terms of a bosonic $F(z)$ and fermionic $\eta(z)$ reparametrization mode:

$$z'(z,\theta) \equiv A(z,\theta) = F(z + \theta\eta(z)), \tag{4.41}$$

$$\theta'(z,\theta) \equiv \alpha(z,\theta) = \sqrt{\dot{F}(z)}\left(\theta + \eta(z) + \frac{1}{2}\theta\eta(z)\dot{\eta}(z)\right) \tag{4.42}$$

, where the dot denotes partial integration with respect to z. We can check this explicitly by expanding the bosonic field F to linear order in the Grassmann variables θ and η: $z'(z,\theta) = F(z) + \dot{F}(z)\theta\eta$, leading to:

$$Dz'(z,\theta) = \dot{F}(z)\eta + \theta\dot{F}(z).$$

Similarly, applying the superderivative to θ' leads to:

$$D\theta' = \left(\sqrt{\dot{F}} + \frac{\sqrt{\dot{F}}}{2}\eta\dot{\eta}\right) + \theta\left(\frac{\ddot{F}}{2\sqrt{\dot{F}}}\eta + \sqrt{\dot{F}}\dot{\eta}\right)$$
$$\theta' D\theta' = \sqrt{\dot{F}}\theta\left(\sqrt{\dot{F}} + \frac{\sqrt{\dot{F}}}{2}\eta\dot{\eta}\right) + \dot{F}\eta + \dot{F}\eta\theta\dot{\eta} + \frac{1}{2}\dot{F}\theta\eta\dot{\eta} = \dot{F}\theta + \dot{F}\eta = Dz'.$$

Of course, we can likewise define an antiholomorphic bosonic $\bar{F}(\bar{z})$ and fermionic $\bar{\theta}(\bar{z})$ reparametrization mode, solving the anti-holomorphic constraint $\overline{D}\overline{z}' = \overline{\theta}'\bar{D}\bar{\theta}'$.

The isometry group of the metric Eq 4.39 consists of the group of super-Möbius OSp$(1|2,\mathbb{R})$-transformations, acting projectively on the coordinates as

$$z' \to \frac{az' - c - \beta\theta'}{-bz' + d + \delta\theta'}, \qquad \theta' \to \frac{\alpha z' - \gamma + e\theta'}{-bz' + d + \delta z'} \tag{4.43}$$

, and analogously for the anti-holomorphic parts. The entries are chosen from the OSp$(1|2,\mathbb{R})$ matrices defined in Eq B.1:

$$g = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \gamma \\ \hline \beta & \delta & e \end{array}\right) \tag{4.44}$$

, where the Greek letters denote Grassmann valued entries and roman letters denote bosonic entries. These entries satisfy the OSp$(1|2,\mathbb{R})$-constraints [40]

$$\alpha = \pm(a\delta - b\beta), \qquad \gamma = \pm(c\delta - d\beta), \qquad e = \pm(1+\beta\delta), \qquad ad - bc = 1 + \delta\beta \tag{4.45}$$

, for either sign $\pm$. More on the definition of the OSp$(1|2,\mathbb{R})$ supergroup can be found in appendix B and later on in the main text.

The general length element is denoted as $d\mathbf{z} \equiv dz + \theta d\theta$. Using the constraint $Dz' = \theta' D\theta'$, we readily deduce that $(D\theta')^2 = D^2 z' + \theta' D^2\theta' = \dot{z}' + \theta'\dot{\theta}'$. Under general superconformal transformations, we see that the length element transforms as $d\mathbf{z}' = (D\theta')^2 d\mathbf{z}$ [40]. The most general line element is therefore a superconformal transformation of the Poincaré super-upper half-plane (SUHP) metric:

$$ds^2 = e^{2\Sigma} d\mathbf{z} \otimes d\bar{\mathbf{z}} = \frac{(D\theta')^2(\bar{D}\bar{\theta}')^2}{|z' - \bar{z}' - \theta'\bar{\theta}'|^2}|dz + \theta d\theta|^2 = \frac{|dz' + \theta' d\theta'|^2}{|z' - \bar{z}' - \theta'\bar{\theta}'|^2}. \tag{4.46}$$

In terms of the real bosonic fields τ', y', we write

$$z' = \tau' + iy', \qquad \overline{z}' = \tau' - iy' \tag{4.47}$$

, where we think of y and τ as the spatial and temporal coordinates respectively. This showcases the analogy between the lightcone coordinates $U = T + Z$ and $V = T - Z$ in bosonic JT gravity.

In super-Poincaré coordinates, the $1|1$-dimensional boundary lies at $z \equiv \overline{z}$, $\theta \equiv \overline{\theta}$. The former is equivalent to

$y = 0$. To regularize divergences, we again place the holographic boundary inwards:

$$y = \epsilon, \qquad \theta = \overline{\theta} \equiv \vartheta, \qquad \epsilon > 0. \tag{4.48}$$

τ and ϑ are the natural coordinates that parameterize the $1|1$-dimensional boundary curve, whose effective metric in Poincaré SUHP coordinates is given by:

$$ds^2 = \frac{|dz' + \theta d\theta'|^2}{|z' - \bar{z}' - \theta' \overline{\theta}'|^2} = \frac{d\tau^2 + 2\vartheta d\vartheta d\tau}{4\epsilon^2}. \tag{4.49}$$

Reflecting the bosonic case, the most general solution of the boundary curve is again given by a superconformal reparametrization of the solution in preferred coordinates Eq 4.48, that preserves the asymptotic length element Eq 4.49. This leads to the constraint equation:

$$\boxed{z' - \bar{z}' - \theta' \overline{\theta}' \equiv 2i\epsilon D\theta' \bar{D}\overline{\theta}' + \mathcal{O}(\epsilon^2)}. \tag{4.50}$$

The single bosonic constraint can readily be solved using the reparametrization modes Eqs 4.41, 4.42 (to leading order in ϵ) [40]:

$$\begin{aligned}
\tau' &= \frac{1}{2}(z' + \bar{z}') = \frac{1}{2}(A(\tau + i\epsilon, \vartheta) + A(\tau - i\epsilon, \vartheta)) \simeq F(\tau + \vartheta\eta(\tau)) \\
y' &= \frac{1}{2i}(z' - \bar{z}') = \frac{1}{2i}(A(\tau + i\epsilon, \vartheta) - A(\tau - i\epsilon, \vartheta)) \simeq \epsilon\partial_\tau A = \epsilon D^2 A = \epsilon D(\alpha D\alpha) = \epsilon((D\alpha)^2 - \alpha D^2\alpha) \\
\theta' &= \alpha(\tau + i\epsilon, \vartheta) = \alpha + i\epsilon\partial_\tau\alpha = \alpha + i\epsilon D^2\alpha \\
\overline{\theta}' &= \alpha(\tau - i\epsilon, \vartheta) = \alpha - i\epsilon\partial_\tau\alpha = \alpha - i\epsilon D^2\alpha
\end{aligned} \tag{4.51}$$

, where in the second line we have used the constraint $DA = \alpha D\alpha$. These boundary reparametrizations are of course the superspace analogues of the bosonic reparametrization behaviour $(F(\tau), \epsilon F'(\tau))$, and parameterize the asymptotic behaviour of the wiggly $1|1$-dimensional boundary curve. Note that the boundary curve is infinitesimally thickened in the fermionic ϑ-direction, as opposed to genuine $1|0$-dimensional curves. We may readily verify that this asymptotic behaviour indeed obeys the correct bosonic constraint Eq 4.50:

$$\begin{aligned}
z' - \bar{z}' - \theta'\overline{\theta}' = 2iy' - \theta'\overline{\theta}' &= 2i\epsilon((D\alpha)^2 - \alpha D^2\alpha) - (\alpha + i\epsilon D^2\alpha)(\alpha - i\epsilon D^2\alpha) \\
&= 2i\epsilon((D\alpha)^2 - \alpha D^2\alpha) - (-i\epsilon\alpha D^2\alpha - i\epsilon\alpha D^2\alpha) \\
&= 2i\epsilon(D\alpha)^2 = 2i\epsilon(D\theta')(\bar{D}\overline{\theta}')
\end{aligned}$$

, again up to first order in ϵ. Given the reparameterization modes $(F(\tau), \eta(\tau))$ of the boundary curve, we can choose a metric Eq 4.46 that smoothly extrapolates this behaviour into the entire bulk. The bottom component of this metric is the familiar bosonic Poincaré submetric, in terms of the bosonic reparametrization mode $F(z)$:

$$ds^2 = \frac{\partial F(z)\bar{\partial}F(\bar{z})}{(F(z) - F(\bar{z}))^2}\, dz d\bar{z}. \tag{4.52}$$

4.2.2 Super-Schwarzian boundary action

To deduce an effective boundary action weighting the different reparametrization modes, we rewrite the super-Poincaré metric in natural coordinates Eq 4.49 in terms of the first order superframe fields (c.f. Eq 4.280):

$$ds^2 = \frac{d\tau^2 + 2\vartheta d\vartheta d\tau}{4\epsilon^2} = dZ^M E_M^1 \, dZ^N E_N^{\bar{1}} \tag{4.53}$$

, where we choose the non-vanishing coordinates of the flat space metric as $\kappa_{1\bar{1}} = \kappa_{\bar{1}1} = 1/2$. We may readily infer the frame fields corresponding to the super-Poincaré metric in conformal gauge as

$$E_\theta^1 = -\theta' e^\Sigma, \qquad E_z^1 = e^\Sigma \tag{4.54}$$

, together with their complex conjugates. Indeed,

$$ds^2 = (dz' e^\Sigma - d\theta'\theta' e^\Sigma)(d\overline{z}' e^\Sigma - d\bar{\vartheta}'\bar{\theta} e^\Sigma) = e^{2\Sigma}|dz' + \theta' d\theta'|^2.$$

Using the constraint Eq 4.53 and the first order frame fields, we can verify that the remaining superconformal reparametrization degrees of freedom along the boundary curve should satisfy the superconformal constraint $Dz = \theta D\theta$ and $D\overline{z} = \overline{\theta} D\overline{\theta}$. Indeed, expanding the RHS, we calculate:

$$\begin{aligned}
|dz' + \theta' d\theta'|^2 &= \left(\theta' d\theta' + dz'\right)\left(\overline{\theta}' d\overline{\theta}' + d\overline{z}'\right) \\
&= \left(\theta'\left(d\tau\frac{\partial\theta'}{\partial\tau} + d\vartheta\frac{\partial\theta'}{\partial\vartheta}\right) + d\tau\frac{\partial z'}{\partial\tau} + d\vartheta\frac{\partial z'}{d\vartheta}\right)\left(\overline{\theta}'\left(d\tau\frac{\partial\overline{\theta}'}{\partial\tau} + d\vartheta\frac{\partial\overline{\theta}'}{\partial\vartheta}\right) + d\tau\frac{\partial\overline{z}'}{\partial\tau} + d\vartheta\frac{\partial\overline{z}'}{d\vartheta}\right) \\
&= \left(d\tau\left(\theta'\frac{\partial\theta'}{\partial\tau} + \frac{\partial z'}{\partial\tau}\right) + d\vartheta\left(-\theta'\frac{\partial\theta'}{\partial\vartheta} + \frac{\partial z'}{\partial\vartheta}\right)\right) \\
&\quad\times\left(d\tau\left(\overline{\theta}'\frac{\partial\overline{\theta}'}{\partial\tau} + \frac{\partial\overline{z}'}{\partial\tau}\right) + d\vartheta\left(-\overline{\theta}'\frac{\partial\overline{\theta}'}{\partial\vartheta} + \frac{\partial\overline{z}'}{\partial\vartheta}\right)\right) \\
&= \left|\theta'\frac{\partial\theta'}{\partial\tau} + \frac{\partial z'}{\partial\tau}\right|^2 d\tau^2 \\
&\quad + \left[\left(\theta'\frac{\partial\theta'}{\partial\tau} + \frac{\partial z'}{\partial\tau}\right)\left(\overline{\theta}'\frac{\partial\overline{\theta}'}{\partial\vartheta} - \frac{\partial\overline{z}'}{\partial\vartheta}\right) + \left(\theta'\frac{\partial\theta'}{d\vartheta} - \frac{\partial z'}{\partial\vartheta}\right)\left(\overline{\theta}'\frac{\partial\overline{\theta}'}{\partial\tau} + \frac{\partial\overline{z}'}{\partial\tau}\right)\right] d\tau d\vartheta.
\end{aligned}$$

Notice the ordering (NW-SE) in which we take the fermionic chain rule.

This expression can only be compatible with the metric in preferred coordinates Eq 4.53 if the ratio between the two components is compatible:

$$\frac{\overline{\theta}'\frac{\partial\overline{\theta}'}{\partial\vartheta} - \frac{\partial\overline{z}'}{\partial\vartheta}}{\overline{\theta}'\frac{\partial\overline{\theta}'}{\partial\tau} + \frac{\partial\overline{z}'}{\partial\tau}} + \frac{\theta'\frac{\partial\theta'}{d\vartheta} - \frac{\partial z'}{\partial\vartheta}}{\theta'\frac{\partial\theta'}{\partial\tau} + \frac{\partial z'}{\partial\tau}} \equiv 2\vartheta.$$

This constraint is trivially satisfied if both the holomorphic and anti-holomorphic coordinates are superconformal reparametrizations of the preferred coordinates with

$$\boxed{Dz' \equiv \theta' D\theta', \qquad D\overline{z}' \equiv \overline{\theta}' D\overline{\theta}'} \tag{4.55}$$

, where $D = \frac{\partial}{\partial\vartheta} + \vartheta\frac{\partial}{\partial\tau}$ denotes the superderivative along the boundary. Indeed, we can show explicitly that the first conformal restriction solves the second term:

$$\frac{\theta'\frac{\partial\theta'}{d\vartheta} - \frac{\partial z'}{\partial\vartheta}}{\theta'\frac{\partial\theta'}{\partial\tau} + \frac{\partial z'}{\partial\tau}} = \frac{\vartheta\frac{\partial z'}{\partial\tau} - \theta'\vartheta\frac{\partial\theta'}{\partial\tau}}{\frac{\partial z'}{\partial\tau} + \theta'\frac{\partial\theta'}{\partial\tau}} = \vartheta.$$

The other constraint follows from complex conjugation.

Path integrating over the dilaton superfield Φ in the total superaction Eq 4.1 restricts the bulk supermetric to patches of super-AdS_2 with $R_{+-} + 2 = 0$. Thereby, the bulk term vanishes and the only remaining degrees of freedom are the conformal reparametrization modes at the asymptotic boundary:

$$\boxed{I_{JT}^{\mathcal{N}=1} \simeq \frac{1}{2}\int_{\partial\Sigma} d\tau d\vartheta \Phi K}. \tag{4.56}$$

In [39], an appropriate definition for the first order extrinsic curvature along the $1|1$-dimensional boundary curve is given by:

$$K = \frac{T^A D_T n_A}{T^A T_A}. \tag{4.57}$$

$T^A = \frac{\partial Z^M}{\partial\tau} E_M^A$ denotes the tangent vector along the boundary curve (Z^M) in local Lorentz coordinates. n_A is the normal vector along the boundary, defined by the orthogonality relation:

$$n_A T^A = 0. \tag{4.58}$$

This definition of the extrinsic curvature is the natural generalization of the bosonic definition $T^\nu T^\mu \nabla_\mu n_\nu = T^\nu \nabla_T n_\nu$ in its first order form (using the identification $\nabla_\mu V^\nu \equiv e_a^\nu D_\mu V^a$). The covariant superderivative along the $1|1$-dimensional boundary is given in terms of the spin connections [39]:

$$D_T n_A = Dn_A + n_A \frac{\partial Z^M}{\partial\vartheta}\Omega_M + n_A\vartheta\frac{\partial Z^M}{\partial\tau}\Omega_M. \tag{4.59}$$

To begin, we calculate the holomorphic component of the tangent vector

$$T^1 = \frac{\partial Z'^M}{\partial\tau} E_M^1 = e^\Sigma\left(\frac{\partial z'}{\partial\tau} + \theta'\frac{\partial\theta'}{\partial\tau}\right) \tag{4.60}$$

$$= e^\Sigma (D\theta')^2. \tag{4.61}$$

In the last line, we again made use of the constraint equation $Dz' = \theta' D\theta'$, leading to $\frac{\partial z'}{\partial\tau} = D^2 z' = (D\theta')^2 - \theta' D^2\theta' = (D\theta')^2 - \theta'\frac{\partial\theta'}{\partial\tau}$. The conformal factor in the metric Eq 4.46, together with the constraint equation along the wiggly boundary Eq 4.50, is given by:

$$e^{2\Sigma} = \frac{1}{|z' - \bar{z}' - \theta'\bar{\theta}'|^2} = \frac{1}{4\epsilon^2 |D\theta' D\bar{\theta}'|^2} \tag{4.62}$$

, leading to:

$$T^1 = \frac{1}{2\epsilon} \frac{D\theta'}{D\bar{\theta}'}. \tag{4.63}$$

$T^{\bar{1}}$ is readily obtained by complex conjugation. The normal vector n_A is defined by its orthogonality Eq 4.58 and choice of normalization $n_1 n_{\bar{1}} = 1/4$ [39]:

$$n_1 = \frac{i}{2} \frac{D\bar{\theta}'}{D\theta'}. \tag{4.64}$$

The first contribution to the extrinsic curvature is therefore:

$$\frac{T^A D n_A}{T^1 T^{\bar{1}}} = 4\epsilon \frac{i}{4} \left(\frac{D\theta'}{D\bar{\theta}'} \frac{D^2\bar{\theta}'}{D\theta'} - \frac{D\theta'}{D\bar{\theta}'} \frac{D^2\theta' D\bar{\theta}'}{(D\theta')^2} - \frac{D\bar{\theta}'}{D\theta'} \frac{D^2\theta'}{D\bar{\theta}'} + \frac{D\bar{\theta}'}{D\theta'} \frac{D^2\bar{\theta}' D\theta'}{(D\bar{\theta}')^2} \right) = 4\epsilon \Im \left(\frac{D^2\theta'}{D\theta'} \right).$$

Note that in general the chain rule of fermionic derivatives works as an operator on the left. Here, the combination $D\theta$ is bosonic either way. In terms of the leading order reparametrization Eq 4.51, we find to leading order:

$$\frac{T^A D n_A}{T^1 T^{\bar{1}}} = 4\epsilon^2 \left(\frac{D^4\alpha}{D\alpha} - \frac{D^2\alpha D^3\alpha}{(D\alpha)^2} \right). \tag{4.65}$$

Calculating the terms proportional to the spin connection in the covariant superderivative requires explicit knowledge of the relation between the spin connections and local rotations. I quote the result of [39], which itself was based on [106]:

$$\frac{T^1 n_1}{|T^1|^2} \left(\frac{\partial Z^M}{\partial \vartheta} \Omega_M + \vartheta \frac{\partial Z^M}{\partial \tau} \Omega_M \right) = -4\epsilon^2 \frac{D^2\alpha D^3\alpha}{(D\alpha)^2} \tag{4.66}$$

Added in the covariant superderivative Eq 4.59, the result can be written in terms of the super-Schwarzian derivative

$$K = 4\epsilon^2 \{F, \alpha; \tau, \varphi\}, \qquad \text{with} \qquad \{F, \alpha; \tau, \varphi\} = \left(\frac{D^4\alpha}{D\alpha} - 2\frac{D^2\alpha D^3\alpha}{(D\alpha)^2} \right) \tag{4.67}$$

, subject to the constraint $DF = \alpha D\alpha$. The super-Schwarzian derivative is a purely fermionic superfield, and may be expanded to linear order in ϑ in terms of a fermionic $T_F(\tau)$ and bosonic $T_B(\tau)$ field, related to the boundary reparametrization modes $F(\tau)$, $\eta(\tau)$ (c.f. Eqs 4.41, 4.42) as [39] [40]:

$$\{F, \alpha; \tau, \varphi\} \equiv -T_F(\tau) - \vartheta T_B(\tau) \tag{4.68}$$

$$T_B(\tau) = \frac{1}{2} \left(\{F, \tau\} + \eta \partial_\tau^3 \eta + 3 \partial_\tau \eta \partial_\tau^2 \eta - \{F, \tau\} \eta \partial \eta \right) \tag{4.69}$$

$$T_F(\tau) = \partial_\tau^2 \eta + \frac{1}{2} \eta \partial_\tau \eta \partial_\tau^2 \eta + \frac{1}{2} \eta \{F, \tau\}. \tag{4.70}$$

This can be interpreted as a redefinition $(T_B, T_F) \to (F, \eta)$. Plugged into the action Eq 4.56, the holographic description of JT supergravity is described by the super-Schwarzian boundary action that weights explicitly the super-conformal reparametrization modes of the $1|1$-dimensional boundary curve

$$\boxed{I_{JT}^{\mathcal{N}=1} \simeq \int d\tau d\vartheta\ \Phi_r(\tau,\vartheta)\{F,\alpha;\tau,\varphi\}} \tag{4.71}$$

, in terms of the super-Schwarzian derivative and a renormalized dilaton superfield $\Phi_r(\tau,\vartheta)$. This action breaks explicitly the full conformal reparametrization symmetry, up to a super-Möbius ambiguity of the $\mathrm{OSp}(1|2,\mathbb{R})$ supergroup, acting as:

$$\tau' \to \frac{a\tau' - c - \beta\theta'}{-b\tau' + d + \delta\theta'}, \qquad \theta' \to \frac{\alpha\tau' - \gamma + e\theta'}{-b\tau' + d + \delta\theta'}. \tag{4.72}$$

Since this is the isometry group of the underlying Poincaré SUHP metric, these modes are interpreted as the zero modes of an underlying gauge symmetry. In particular, one should identify configurations differing only by such transformations. The entries of this projective action are taken from the general $g \in \mathrm{OSp}(1|2,\mathbb{R})$ group element Eq B.1, satisfying the $\mathrm{OSp}(1|2,\mathbb{R})$ constraints Eq B.3.
One restricts to one of the signs in these constraints in the projective subgroup $\mathrm{OSp}(1|2,\mathbb{R})/\mathbb{Z}_2$ of super-Möbius transformations.

In the following, we imagine the dilaton field to diverge as $\Phi \sim \frac{a}{\epsilon}$, in terms of a single bosonic constant a. Decomposing the super-Schwarzian derivative using Eq 4.68, and integrating out the fermionic superpartner ϑ by the standard Grassmann integral $\int d\vartheta\ \vartheta = 1$, $\int d\vartheta = 0$, yields the effective Schwarzian boundary action

$$\boxed{I_{Sch}^{\mathcal{N}=1} \simeq -\oint d\tau\ T_B(\tau)}. \tag{4.73}$$

By Eq 4.69, we see that this depends on the standard Schwarzian derivative of the bosonic reparametrization mode $F(\tau)$, supplemented by corrections proportional to the fermionic superpartner η.

4.3 BF formulation of $\mathcal{N}=1$ JT supergravity

4.3.1 On-shell equivalence

Using the component formulation of the $\mathcal{N}=1$ JT SUGRA action (Eq 4.34), I will follow the conventions of [40] leading to an equivalent gauge theoretic BF description. This section largely generalizes the discussion of section 2.2.
We recall the traceless 2×2 P_I-matrices defined in Eq 2.23, satisfying the $\mathfrak{sl}(2,\mathbb{R})$-algebra. Following [40], we denote them as (in the ordening $+,-$):

$$\Gamma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.74}$$

, and embed them in the $\mathfrak{osp}(1|2,\mathbb{R})$-valued P_I-matrices:

$$P_I = \left(\begin{array}{c|c} \frac{1}{2}\Gamma_I & \mathbf{0}_{2\times 1} \\ \hline \mathbf{0}_{1\times 2} & 0 \end{array}\right). \tag{4.75}$$

In the following, I will continue to denote the STr operation of a supermatrix $g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ as

$$\mathrm{STr}(g) = \mathrm{Tr}(A) - \mathrm{Tr}(D) \tag{4.76}$$

This has the convenient property that it is still symmetric under cyclic permutations of fermionic supermatrices. Using the explicit form of the P_I-matrices above, we recall that they satisfy the $\mathfrak{sl}(2,\mathbb{R})$-algebra and normalization defined in Eq 2.19:

$$[P_I, P_J] = \epsilon_{IJK}P^K, \qquad \mathrm{STr}(P_I P_J) = \frac{1}{2}\eta_{IJ}, \qquad \epsilon_{012} = -1, \qquad \eta_{IJ} = \mathrm{diag}(1,1,-1). \tag{4.77}$$

The $\mathfrak{osp}(1|2,\mathbb{R})$-algebra is spanned by two additional supermatrices[6]

$$Q_- = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline -1 & 0 & 0 \end{array}\right), \qquad Q_+ = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) \tag{4.78}$$

, satisfying the normalization

$$\mathrm{STr}(Q_\alpha Q_\beta) = \frac{1}{2}\epsilon_{\alpha\beta}, \qquad \epsilon_{+-} = -1. \tag{4.79}$$

Note that although the $Q_\pm$ matrices are the generators of the fermionic subgroup, they contain only bosonic numbers. Indices are raised and lowered with respect to the appropriate Cartan-Killing matrices η_{IJ} and $\epsilon_{\alpha\beta}$. One can check that the representation matrices above satisfy the complete $\mathfrak{osp}(1|2,\mathbb{R})$-algebra, here defined as[7]:

$$[P_I, P_J] = \epsilon_{IJK}P^K, \qquad [P_I, Q_\alpha] = \frac{1}{2}(\Gamma_I)^\beta{}_\alpha Q_\beta, \qquad \{Q_\alpha, Q_\beta\} = -\frac{1}{2}(C\Gamma^I)_{\alpha\beta}P_I \tag{4.80}$$

, with C the Majorana conjugate matrix $C_{\alpha\beta} = \epsilon_{\alpha\beta}$, defined in Eq 4.15 (in the ordening $\{+,-\}$):

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.81}$$

The algebra commutation relations may be formulated more transparently by defining different linear combinations [107] [40]:

$$P_0 = iH, \quad P_1 = \frac{1}{2}(iE_- + iE_+), \quad P_2 = \frac{1}{2}(iE_- - iE_+), \quad Q_- = -iF_-, \quad Q_+ = iF_+. \tag{4.82}$$

[6] As opposed to genuine supermatrices, the entries in the off-diagonal fermionic blocks are commutative numbers instead of Grassmann numbers. Within the algebra $\mathfrak{osp}(1|2,\mathbb{R})$, their expansion are necessarily Grassmann-valued.

[7] Note that in the last anticommutation relation, the bosonic index $C\Gamma^I$ is also raised with the metric η^{IJ}.

We collectively denote these new combinations as iJ_I, which are found to satisfy (an isomorphism of) the $\mathfrak{osp}(1|2,\mathbb{R})$-superalgebra

$$\begin{aligned}
&[H,E_\pm]=\pm iE_\pm, \qquad [E_+,E_-]=2iH, \qquad [H,F_\pm]=\pm\frac{1}{2}iF_\pm,\\
&[E_\pm,F_\mp]=iF_\pm, \qquad \{F_+,F_-\}=\frac{1}{2}iH, \qquad \{F_\pm,F_\pm\}=\mp\frac{1}{2}iE_\pm.
\end{aligned} \tag{4.83}$$

The restriction to $iH, iE^\pm$ satisfies the familiar bosonic $\mathfrak{sl}(2,\mathbb{R})$-algebra of Eq A.5. This set of generators is used in the general discussion on the $\mathfrak{osp}(1|2,\mathbb{R})$ algebra and subsequent representation theory in section B.1.2 of the appendix.

Returning to the original P_I-generators, we group the gauge and auxiliary fields into $\mathfrak{osp}(1|2,\mathbb{R})$-valued vectors with respect to the P_I-generators, whose restriction to $\mathfrak{sl}(2,\mathbb{R})$ agrees with Eq 2.18;

$$B_I=(-\phi_a,\phi), \qquad A^I=(e^a,\omega), \qquad \mathbf{B}=B^IP_I+\lambda^\alpha Q_\alpha, \qquad \mathbf{A}=A^IP_I+\psi^\alpha Q_\alpha. \tag{4.84}$$

Both the dilatino λ^α and the gaugino ψ^α entries are Grassmann-valued. In matrix notation, both the auxiliary- and gauge vector read:

$$\mathbf{B}=\frac{1}{2}\left(\begin{array}{cc|c} \phi_0 & -\phi_1+\phi & \lambda^+\\ -\phi_1-\phi & -\phi_0 & \lambda^-\\ \hline -\lambda^- & \lambda^+ & 0\end{array}\right), \qquad \mathbf{A}=\frac{1}{2}\left(\begin{array}{cc|c} -e^0 & e^1-\omega & \psi^+\\ e^1+\omega & e^0 & \psi^-\\ \hline -\psi^- & \psi^+ & 0\end{array}\right). \tag{4.85}$$

Note that we have raised the B_I indices using the Cartan-Killing metric η^{IJ}. The $\mathfrak{osp}(1|2,\mathbb{R})$-valued vectors comprise both diagonal bosonic and off-diagonal fermionic blocks.
Using the general definition $\mathbf{F}=d\mathbf{A}+\mathbf{A}\wedge\mathbf{A}$, we can explicitly derive the field strength:

$$\begin{aligned}
\mathbf{F}&=dA^IP_I+d\psi^\alpha Q_\alpha+\frac{1}{2}A^I\wedge A^J[P_I,P_J]+\frac{1}{2}\psi^\alpha\wedge A^I[Q_\alpha,P_I]+\frac{1}{2}A^I\wedge\psi^\alpha[P_I,Q_\alpha]+\frac{1}{2}\psi^\alpha\wedge\psi^\beta\{Q_\alpha,Q_\beta\}\\
&=\left(dA^I+\frac{1}{2}\epsilon^{IJK}A_J\wedge A_K-\frac{1}{4}\psi^\alpha\wedge(C\Gamma^I)_{\alpha\beta}\psi^\beta\right)P_I+d\psi^\alpha Q_\alpha+\frac{1}{2}A^I\wedge(\Gamma_I)^\beta{}_\alpha\psi^\alpha Q_\beta.
\end{aligned}$$

The anticommutator in the last term readily follows from the anticommutative nature of the spinors entries ψ^α and the wedge product between them. This ensures that the wedge product between spinor entries $\psi^\alpha\wedge\psi^\beta$ is symmetric (c.f. Eq 4.3), thus

$$\psi^\alpha Q_\alpha\wedge\psi^\beta Q_\beta=\frac{1}{2}\psi^\alpha\wedge\psi^\beta\{Q_\alpha,Q_\beta\}+\frac{1}{2}\psi^\alpha\wedge\psi^\beta[Q_\alpha,Q_\beta]=\frac{1}{2}\psi^\alpha\wedge\psi^\beta\{Q_\alpha,Q_\beta\}.$$

Also note that the Q_α-matrices defined in Eq 4.78 are bosonic, and commute with the Grassmann spinor entries above. Absorbing the antisymmetric matrix C into the definition of the Majorana conjugate, and introducing an $\mathfrak{so}(2,1)$-covariant derivative

$$\mathcal{D}\psi^\alpha\equiv d\psi^\alpha+\frac{1}{2}A^I\wedge(\Gamma_I)^\alpha{}_\beta\psi^\beta \tag{4.86}$$

, we may rewrite the field strength more conveniently as:

$$\mathbf{F}=\left(F^I-\frac{1}{4}\overline{\psi}_\alpha\wedge(\Gamma^I\psi)^\alpha\right)P_I+\mathcal{D}\psi^\alpha Q_\alpha. \tag{4.87}$$

F^I is the usual definition of the $\mathfrak{sl}(2,\mathbb{R})$-valued field strength, appearing in Eq 2.25;

$$F^I=dA^I+\frac{1}{2}\epsilon^{IJK}A_J\wedge A_K. \tag{4.88}$$

The Clifford algebra $\{\gamma_a,\gamma_b\}=2\delta_{ab}\mathbf{1}$ is explicitly realized in $D=2$ by the set of Pauli sigma matrices [59]

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \gamma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \gamma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}. \tag{4.89}$$

Since we need only two linearly independent (symmetrical) gamma matrices in two dimensions, we find that the restriction

$$\Gamma_I=(\gamma_a,\gamma_0\gamma_1=-\gamma_5),\qquad \gamma_5=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad (a=0,1) \tag{4.90}$$

satisfies the Clifford algebra. In this basis, the γ_a are symmetric, while γ_5 is antisymmetric.
Contracting $\mathbf{B}$ and $\mathbf{F}$ eventually yields:

$$\begin{aligned}\mathrm{STr}(\mathbf{BF})&=\frac{1}{2}B^IF^J\eta_{IJ}-\frac{1}{8}B^I\overline{\psi}_\alpha\wedge(\Gamma^J\psi)^\alpha\eta_{IJ}+\frac{1}{2}\lambda^\alpha\mathcal{D}\psi^\beta\epsilon_{\alpha\beta}\\&=\frac{1}{2}\left(dA^I+\frac{1}{2}\epsilon^{IJK}A_J\wedge A_K\right)B_I-\frac{1}{8}B^a\overline{\psi}_\alpha\wedge(\gamma_a\psi)^\alpha+\frac{1}{8}B^2\overline{\psi}_\alpha\wedge(\gamma_5\psi)^\alpha+\frac{1}{2}\overline{\lambda}_\alpha\mathcal{D}\psi^\alpha\end{aligned} \tag{4.91}$$

, since $\mathrm{STr}(J_IQ_\alpha)=\mathrm{STr}(Q_\alpha J_I)=0$ are orthogonal. Inserting the proper entries of $A^I=(e^a,\omega)$, $A_I=A^J\eta_{JI}=(e^a,-\omega)$ and $B_I=(-\phi_a,\phi)$, $B^I=B_J\eta^{IJ}=(-\phi_a,-\phi)$ yields:

$$\begin{aligned}\mathrm{STr}(\mathbf{BF})=&-\frac{1}{2}\phi_a\left(de^a+\epsilon^a{}_b\omega\wedge e^b-\frac{1}{4}\overline{\psi}_\alpha\wedge(\gamma^a\psi)^\alpha\right)+\frac{1}{2}\phi\left(d\omega+e^0\wedge e^1-\frac{1}{4}\overline{\psi}_\alpha\wedge(\gamma_5\psi)^\alpha\right)\\&+\frac{1}{2}\overline{\lambda}_\alpha D\psi^\alpha+\frac{1}{4}\overline{\lambda}_\alpha e^a(\gamma_a\psi)^\alpha\end{aligned} \tag{4.92}$$

, using that $\epsilon^{ab2}=-\epsilon_{ab2}=\epsilon_{ab}$ (from the convention $\epsilon_{012}=-1$). Furthermore, using the relation between the Γ_I and γ_a matrices Eq 4.90, we have used that the $\mathfrak{so}(2,1)$-covariant derivative expands into the Lorentz covariant derivative defined in Eq 4.21 plus a correction term;

$$\mathcal{D}\psi^\alpha=d\psi^\alpha-\frac{1}{2}\omega\wedge(\gamma_5\psi)^\alpha+\frac{1}{2}e^a(\gamma_a)^\alpha{}_\beta\psi^\beta=D\psi^\alpha+\frac{1}{2}e^a(\gamma_a)^\alpha{}_\beta\psi^\beta. \tag{4.93}$$

Specifying to the bottom component of the Lorentz covariant derivative 4.21, together with the spinor component of the generalized Levi-Civita symbol Eq 4.10 indeed yields the proper Lorentz covariant derivative on spinors:

$$D\psi^\alpha=d\psi^\alpha-\frac{1}{2}\omega(\gamma_5)^\alpha{}_\beta\psi^\beta.$$

The action obtained by integrating over the superspace volume Σ exactly coincides with the $\mathcal{N}=1$ super

JT action in the first order metric formulation (Eq 4.34); thereby identifying $\mathcal{N}=1$ JT supergravity with an $\mathfrak{osp}(1|2,\mathbb{R})$ gauge BF theory

$$\boxed{I_{JT}^{\mathcal{N}=1} = \int_{\Sigma} \mathrm{STr}(\mathbf{BF})}. \tag{4.94}$$

Infinitesimal gauge transformations with a fermionic supercurrent $\varepsilon = \xi^\alpha Q_\alpha$ are equivalent to the $\mathcal{N}=1$ local SUSY transformations Eq 4.35. Indeed, local gauge transformations act on the gauge fields $\mathbf{A}$ and on the fields in the adjoint representation $\mathbf{B}$ according to the conventions of section 2.2:

$$\delta\mathbf{A} = d\varepsilon + [\mathbf{A},\varepsilon], \qquad \delta\mathbf{B} = [\mathbf{B},\varepsilon]. \tag{4.95}$$

Using the supercurrent $\epsilon = \xi^\alpha Q_\alpha$, and the decomposition Eq 4.84, the transformations of the gauge field are:

$$\begin{aligned}\delta(A^I P_I + \psi^\alpha Q_\alpha) &= d\xi^\alpha Q_\alpha + [A^I P_I + \psi^\alpha Q_\alpha, \xi^\beta Q_\beta] = d\xi^\alpha Q_\alpha + A^I \xi^\alpha [P_I, Q_\alpha] + \psi^\alpha \xi^\beta \{Q_\alpha, Q_\beta\} \\ &= d\xi^\alpha Q_\alpha + \frac{1}{2} A^I (\Gamma_I)^\beta{}_\alpha \xi^\alpha Q_\beta + \frac{1}{2}\bar{\xi}\Gamma^I \psi P_I\end{aligned}$$

, leading to the local infinitesimal transformations

$$\delta_\xi A^I = \frac{1}{2}\bar{\xi}\Gamma^I\psi, \qquad \delta_\epsilon \psi^\alpha = \mathcal{D}\xi^\alpha. \tag{4.96}$$

Of course, the same transformations on the adjoint vector $\mathbf{B}$ lead to:

$$\delta_\xi B^I = \frac{1}{2}\bar{\xi}\Gamma^I\lambda, \qquad \delta_\xi \lambda^\alpha = \frac{1}{2} B^I (\Gamma_I)^\alpha{}_\beta \xi^\beta. \tag{4.97}$$

Inserting $B_I = (-\phi_a, \phi)$ and $A^I = (e^a, \omega)$ recovers the component-wise local SUSY transformations Eqs 4.35 that preserve the $\mathcal{N}=1$ JT SUGRA action, establishing complete on-shell equivalence between the metric and BF formulation of JT supergravity.

4.3.2 Recovering the Super-Schwarzian boundary action

We will now exploit the equivalence between the holographic particle-on-SL$(2,\mathbb{R})$ and Schwarzian theory of the bosonic case (c.f. sections 2.5.3, 2.6.1, 2.183) to consider the analogous boundary action and boundary condition to the otherwise completely topological $\mathfrak{osp}(1|2,\mathbb{R})$ BF theory (c.f. Eq 2.103), along the lines of [40]:

$$I_{JT}^{\mathcal{N}=1} = \int_{\mathcal{M}} \mathrm{STr}(\mathbf{BF}) - \frac{1}{2}\oint_{\partial\mathcal{M}} d\tau\, \mathrm{STr}(\mathbf{BA}_\tau), \qquad \mathbf{B}|_{\partial\mathcal{M}} = \mathbf{A}_\tau|_{\partial\mathcal{M}}. \tag{4.98}$$

τ is the affine bosonic coordinate along the boundary, tangential to $\partial\mathcal{M}$. This matches with Eq 2.103 up to a total sign factor since the total JT action itself was formulated up to a factor of $-1/4\pi G$ in 4.1 instead of $1/4\pi G$, following the conventions of [40]. The boundary term may again be motivated by dimensionally reducing the full 3d Chern-Simons theory, or by simply demanding a proper variational principle, along the lines of Eq 2.49.

One recovers the boundary-on-OSp$(1|2,\mathbb{R})$ theory by path integrating over $\mathbf{B}$, yielding a vanishing $\mathbf{F}=0$.

Inserting the mixed boundary condition yields:

$$I[g] = -\frac{1}{2}\oint_{\partial\mathcal{M}} d\tau\ \mathrm{STr}(\mathbf{A}_\tau^2). \tag{4.99}$$

The remaining configurational degrees of freedom are again the boundary group elements $g \in \mathrm{OSp}(1|2,\mathbb{R})$, which render $\mathbf{A}_\tau$ flat:

$$\mathbf{A}_\tau|_{\partial\mathcal{M}} = g\partial_\tau g^{-1} = -\partial_\tau g g^{-1}. \tag{4.100}$$

The Schwarzian boundary action cannot be recovered from pure particle-on-OSp$(1|2,\mathbb{R})$ theory. Instead, we have to constrain the dynamics by the coset boundary conditions, analogous to the bosonic case. A natural generalization of Eq 2.181 to the supersymmetric case are the Brown-Henneaux boundary conditions [108]:

$$\mathbf{A}_\tau = \left(\begin{array}{cc|c} 0 & T_B(\tau) & T_F(\tau) \\ 1 & 0 & 0 \\ \hline 0 & T_F(\tau) & 0 \end{array}\right) \tag{4.101}$$

, in terms of the bosonic $T_B(\tau)$ and fermionic $T_F(\tau)$ Schwarzian fields[8] Eqs 4.69, 4.70 with $T_B(\tau+\beta) = T_B(\tau)$ and $T_F(\tau+\beta) = \pm T_F(\tau)$. Computing the STr readily yields the super-Schwarzian boundary action encountered in Eq 4.73:

$$I_{Sch}^{\mathcal{N}=1} = -\oint_{\partial\mathcal{M}} d\tau\ T_B(\tau). \tag{4.102}$$

We may again use Eqs 4.69, 4.70 to express the super-Schwarzian action in terms of the boundary reparametrization mode $F(\tau)$ and its superpartner $\eta(\tau)$. The integration space is still over the loop group $L(\mathrm{OSp}(1|2,\mathbb{R}))/\mathrm{OSp}(1|2,\mathbb{R})$, but care has to be taken since the group OSp$(1|2,\mathbb{R})$ decomposes into two distinct sectors, depending on the periodicity of the fermionic superpartner $T_F(\tau+\beta) = \pm T_F(\tau)$ while traversing the thermal boundary circle. The periodic sector is called the *Ramond* (**R**) sector, while the anti-periodic sector is called the *Neveu-Schwarz* (**NS**) sector, resembling the nomenclature from superstring theory.
Using the constrained boundary behaviour of $\mathbf{A}_\tau$ in Eq 4.101, the periodicity of $T_F(\tau)$ is implemented on the gauge field by means of the sCasimir operator $(-)^F \equiv \mathrm{diag}(1,1,-1)$;

$$\begin{aligned} \textbf{(NS)}: &\qquad \mathbf{A}_\tau(\tau+\beta) = (-)^F\mathbf{A}_\tau(\tau)(-)^F \\ \textbf{(R)}: &\qquad \mathbf{A}_\tau(\tau+\beta) = \mathbf{A}_\tau(\tau). \end{aligned} \tag{4.103}$$

A general discussion on the different sectors of the OSp$(1|2,\mathbb{R})$ supergroup is differed to section B.1.1 in the appendix.

[8] Note that this analysis does not a priori require to define the bosonic T_B and fermionic field T_F as in Eqs 4.69, 4.70, since it is a general fact [40] that any single fermionic superfield $T_F(\tau) + \vartheta T_B(\tau)$ may be written in terms of a super-Schwarzian derivative of two new superfields $A(\tau,\vartheta)$ and $\alpha(\tau,\vartheta)$:

$$T_F(\tau) + \vartheta T_B(\tau) = -\frac{D^4\alpha}{D\alpha} + \frac{2D^3\alpha D^2\alpha}{(D\alpha)^2} = -\{A,\alpha,\tau,\vartheta\}.$$

A and α are bosonic, respectively fermionic fields satisfying the constraint $DA = \alpha D\alpha$. In this way, we recover a posteriori Eq 4.68.

4.4 Super-gravitational amplitudes

The previous section motivates the local description of $\mathcal{N}=1$ JT SUGRA in terms of an $\mathfrak{osp}(1|2,\mathbb{R})$ BF theory. Along the lines of the previous chapter, the full quantum dynamics are described by a suitable exponentiation of this algebra. To begin, we need to understand the representation theory governing the orthosymplectic group OSp$(1|2,\mathbb{R})$ in a similar fashion to the representation theory of SL$(2,\mathbb{R})$ described in appendix A. The purpose of the publication [40] was in fact two-folded. On the one hand, the main goal was to understand the group-theoretic structure governing the quantum amplitudes of $\mathcal{N}=1$ JT SUGRA. As opposed to the extensive mathematical literature existing on the representation theory of SL$(2,\mathbb{R})$, there seemed to be no comprehensive treatment for the case of OSp$(1|2,\mathbb{R})$ available. Therefore, the second goal was to develop the representation theory from first principles along the lines of the development of SL$(2,\mathbb{R})$ in [90].
Since I will need some of these results in the development of EOW brane amplitudes in superspace, I review the necessary ingredients in the appendix B. In particular, the definition and representation theory of the sub-semisupergroup OSp$^+(1|2,\mathbb{R})$ is reviewed in section B.2. Note that I will use slightly different conventions compared to [40] in order to match with the bosonic case for clarity. Besides summarizing the appendix of [40], I have worked out a lot of the calculational steps in detail for extra clarity. I summarize the conventions and results hereunder.

4.4.1 Overview of the OSp$(1|2,\mathbb{R})$ representation theory

The general linear supergroup GL$(1|2,\mathbb{R})$ consists of all invertible 3×3 matrices comprised of five bosonic variables a,b,c,d,e and four fermionic Grassmann variables $\alpha,\beta,\gamma,\delta$, separated into bosonic diagonal and fermionic off-diagonal blocks

$$g=\left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \gamma \\ \hline \beta & \delta & e \end{array}\right). \tag{4.104}$$

The subgroup OSp$(1|2,\mathbb{R})\subset$ GL$(1|2,\mathbb{R})$ that preserves the orthosymplectic form $g^{\text{st}}\Omega g=\Omega$ with

$$\Omega=\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{4.105}$$

defines the group OSp$(1|2,\mathbb{R})$, where the supertranspose is defined as

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{\text{st}}=\left(\begin{array}{c|c} A^T & -C^T \\ \hline B^T & D^T \end{array}\right). \tag{4.106}$$

This operation is not an involution, but is of order four: $\text{st}^4=\mathbf{1}$. The embedding of OSp$(1|2,\mathbb{R})$ in GL$(1|2,\mathbb{R})$ leads to the defining OSp$(1|2,\mathbb{R})$ constraints Eq B.3:

$$ad-bc-\delta\beta=1,\quad -a\gamma+c\alpha-\beta e=0,\quad e^2+2\gamma\alpha=1,\quad -\alpha d+b\gamma+e\delta=0. \tag{4.107}$$

These can be solved by imposing $ad - bc - \delta\beta = 1$ and parameterizing (c.f. Eq B.4):

$$\alpha = \pm(a\delta - b\beta), \qquad \gamma = \pm(c\delta - d\beta), \qquad e = \pm(1 + \beta\delta). \tag{4.108}$$

These relations lead to the natural inverse element of $g \in \mathrm{OSp}(1|2,\mathbb{R})$:

$$g^{-1} = \left(\begin{array}{cc|c} d & -b & -\delta \\ -c & a & \beta \\ \hline \gamma & -\alpha & e \end{array}\right). \tag{4.109}$$

The Berezinian or superdeterminant of an invertible $\mathrm{GL}(1|2,\mathbb{R})$ matrix $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is defined as [40]:

$$\mathrm{Ber}(M) = \det(A - BD^{-1}C)\det(D)^{-1}. \tag{4.110}$$

Taking into account the anticommutativity between the fermionic blocks B and C, the definition of the Berezinian is naturally invariant under the supertranspose st operation $\mathrm{Ber}(M^{\mathrm{st}}) = \mathrm{Ber}(M)$. For $\mathrm{OSp}(1|2,\mathbb{R})$, the Berezinian can take either of the two signs $\mathrm{Ber}(g) = \pm 1$, related to the sign in the relations Eq 4.108.
The group $\mathrm{OSp}(1|2,\mathbb{R})$ naturally decomposes into two disconnected sectors, depending on the sign of the Berezinian. Both sectors are related by applying the sCasimir operator $(-)^F = \mathrm{diag}(1, 1 \mid -1)$.
Just as for $\mathrm{SL}(2,\mathbb{R})$, the different conjugacy class elements are labeled according to the value of the supertrace $\mathrm{STr}(g)$, defined in Eq 4.76. Choosing either sign $\pm$ in Eq 4.108 leads to

$$\mathrm{STr}(g) = a + d - (\pm(1 + \beta\delta)). \tag{4.111}$$

Choosing the **NS**-sector $(-)$, group elements with $|\mathrm{STr}(g)| > 3$, $|\mathrm{STr}(g)| = 3$, $|\mathrm{STr}(g)| < 3$ are called hyperbolic, parabolic and elliptic respectively.
In the **R**-sector $(+)$, the conjugacy classes are instead: $|\mathrm{STr}(g)| > 1$, $|\mathrm{STr}(g)| = 1$, $|\mathrm{STr}(g)| < 1$, corresponding to hyperbolic, parabolic and elliptic respectively.

The generators iJ_I $(= iH, iE_\pm, iF_\pm)$ of the $\mathfrak{osp}(1|2,\mathbb{R})$ superalgebra are defined in Eq 4.82. Written out explicitly, the 3×3 supermatrices comprise the defining representation of $\mathfrak{osp}(1|2,\mathbb{R})$:

$$\begin{gathered} iH = \frac{1}{2}\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad iE_- = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad iE_+ = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \\ iF_- = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & -1 \\ \hline 1 & 0 & 0 \end{array}\right), \quad iF_+ = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right). \end{gathered} \tag{4.112}$$

Note that the fermionic generators contain only bosonic entries. These generators satisfy the known $\mathfrak{osp}(1|2,\mathbb{R})$-

algebra Eq 4.83:

$$\begin{aligned} &[H, E_\pm] = \pm i E_\pm, \qquad [E_+, E_-] = 2iH, \qquad [H, F_\pm] = \pm\frac{1}{2} i F_\pm, \\ &[E_\pm, F_\mp] = i F_\pm, \qquad \{F_+, F_-\} = \frac{1}{2} iH, \qquad \{F_\pm, F_\pm\} = \mp\frac{1}{2} i E_\pm. \end{aligned} \tag{4.113}$$

The Cartan-Killing metric is defined from the normalization of the generators with respect to the STr operation:

$$\mathrm{STr}((iJ_I)(iJ_J)) \equiv \frac{\kappa_{IJ}}{2}. \tag{4.114}$$

We again label the representations by the simultaneous eigenvalue of the quadratic Casimir, defined in terms of the inverse Cartan-Killing metric $\mathcal{C}_2 = -iJ_I\kappa^{IJ} iJ_J \equiv -j(j+1/2)$. This is given by Eq B.13:

$$\mathcal{C}_2 = -\kappa^{IJ} iJ_I iJ_J = H^2 + \frac{1}{2}(E_+E_- + E_-E_+) - (F_+F_- - F_-F_+) \equiv -j(j+1/2) \tag{4.115}$$

and is seen to commute with all generators of the algebra. The eigenvalue is again built from a spin label j. The defining generators Eq 4.112 constitute a spin-1/2 representation.

We can also consider operators that commute with all bosonic generators, while anticommuting with all fermionic generators. For $\mathrm{OSp}(1|2,\mathbb{R})$, this operation is given by the sCasimir $\mathcal{Q}$:

$$\mathcal{Q} = (iF_+)(iF_-) - (iF_-)(iF_+) + \frac{1}{8} = -F_+F_- + F_-F_+ + \frac{1}{8} = \left(\frac{j}{2} + \frac{1}{8}\right)(-)^F \tag{4.116}$$

, where $(-)^F = \mathrm{diag}(\mathbf{1}_{2j+1} \mid -\mathbf{1}_{2j})$ is the operator that commutes with all bosonic generators, while anticommuting with the fermionic generators. This operator is in turn related to the quadratic Casimir by its square:

$$\mathcal{Q}^2 - \frac{1}{64} = -\frac{1}{4}H^2 - \frac{1}{8}(E_+E_- + E_-E_+) + \frac{1}{4}(F_+F_- - F_-F_+) = -\frac{1}{4}\mathcal{C}_2 = \frac{j(j+1/2)}{4}. \tag{4.117}$$

To construct more general spin-j representations, we look at the action on the square integrable functions on the superline $\mathbb{R}^{1|1}$. The space of square integrable functions $L^2(\mathbb{R}^{1|1})$ on the superline is equipped with an inner product

$$\langle F \mid G\rangle = \int_{\mathbb{R}} dx \int d\vartheta\; F^*(x,\vartheta) g(x,\vartheta) \tag{4.118}$$

This defines a complete set of configurational sates $|x,\vartheta\rangle$

$$\int_{\mathbb{R}} dx \int d\vartheta\; |x,\vartheta\rangle\langle x,\vartheta| = \mathbf{1} \tag{4.119}$$

, whose overlap with vectors in $L^2(\mathbb{R}^{1|1})$ is given by: $\langle x,\vartheta \mid F\rangle = F(x,\vartheta)$.

The (spherical) projective action of $\mathrm{OSp}(1|2,\mathbb{R})$ on the space of square integrable functions is given by Eq

B.21 [40]:

$$\boxed{\langle x,\vartheta \mid g \mid f\rangle = (g\cdot f)(x,\vartheta) = \frac{|bx+d+\delta\vartheta|^{2j}}{\text{sgn}(e)^{1/2}\text{sgn}(bx+d+\delta\vartheta)^{1/2}}\, f\left(\frac{ax+c+\beta\vartheta}{bx+d+\delta\vartheta}, -\frac{\alpha x+\gamma-e\vartheta}{bx+d+\delta\vartheta}\right)}. \tag{4.120}$$

where the entries are taken from the general $g \in \text{OSp}(1|2,\mathbb{R})$ group element. This action composes naturally under group multiplication, and defines the *principal series representation* of $\text{OSp}(1|2,\mathbb{R})$.
Restricting the value of j leads to the unitary *continuous series representation* Eq B.23:

$$\boxed{j = \frac{ik}{2} - \frac{1}{4}} \qquad k \in \mathbb{R}. \tag{4.121}$$

Exponentiating the fundamental spin-1/2 generators leads to the corresponding group elements in $\text{OSp}(1|2,\mathbb{R})$:

$$\begin{gathered}
e^{2\phi iH} = \left(\begin{array}{cc|c} e^{-\phi} & 0 & 0 \\ 0 & e^{\phi} & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad e^{\gamma^- iE_-} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ \gamma^- & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad e^{\gamma^+ iE_+} = \left(\begin{array}{cc|c} 1 & \gamma^+ & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \\
e^{2\theta^- iF_-} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -\theta^- \\ \hline \theta^- & 0 & 1 \end{array}\right), \quad e^{2\theta^+ iF_+} = \left(\begin{array}{cc|c} 1 & 0 & \theta^+ \\ 0 & 1 & 0 \\ \hline 0 & \theta^+ & 1 \end{array}\right)
\end{gathered} \tag{4.122}$$

Linearizing their respective principal series action leads to the spin-j Borel Weil generators of $\mathfrak{osp}(1|2,\mathbb{R})$:

$$\begin{aligned}
iH = -x\partial_x - \frac{1}{2}\vartheta\partial_\vartheta + j, \quad iE_- = \partial_x, \quad iE_+ = -x^2\partial_x - x\vartheta\partial_\vartheta + 2jx \\
iF_- = \frac{1}{2}(\partial_\vartheta + \vartheta\partial_x), \quad iF_+ = -\frac{1}{2}x\partial_\vartheta - \frac{1}{2}x\vartheta\partial_x + j\vartheta.
\end{aligned} \tag{4.123}$$

We find that they satisfy the $\mathfrak{osp}(1|2,\mathbb{R})$ algebra up to a sign in the anticommutators Eq B.34:

$$\begin{aligned}
[H,E_\pm] = \pm iE_\pm, \qquad [E_+,E_-] = 2iH, \qquad [H,F_\pm] = \pm\frac{1}{2}iF_\pm, \\
[E_\pm,F_\mp] = iF_\pm, \qquad \{F_+,F_-\} = -\frac{1}{2}iH, \qquad \{F_\pm,F_\pm\} = \pm\frac{1}{2}iE_\pm.
\end{aligned} \tag{4.124}$$

This is consistent with the fact that the fermionic Borel-Weil generators are genuine Grassmann valued, while the fundamental generators contain only bosonic entries. This should be taken into account in the definition of the sCasimir Eq B.35:

$$\mathcal{Q} = (iF_-)(iF_+) - (iF_+)(iF_-) + \frac{1}{8} = \left(\frac{1}{8} + \frac{j}{2}\right)(1 - 2\vartheta\partial_\vartheta). \tag{4.125}$$

This is consistent with the earlier spin-j definition of the sCasimir Eq 4.116 upon recognizing that $(-)^F = (1-2\vartheta\partial_\vartheta)$ commutes with bosonic functions, while it anticommutes with purely fermionic functions.
The Gauss parametrization of the $\text{OSp}(1|2,\mathbb{R})$ supergroup manifold is characterized in terms of three bosonic

ϕ, γ^+, γ^- and two fermionic θ^+, θ^- parameters:

$$g(\phi, \gamma^-, \gamma^+ \mid \theta^-, \theta^+) = e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+}. \tag{4.126}$$

Using the explicit exponentiations Eq B.26 yields:

$$g(\phi, \gamma^-, \gamma^+ \mid \theta^-, \theta^+) = \left(\begin{array}{cc|c} e^{-\phi} & \gamma^+ e^{-\phi} & e^{-\phi}\theta^+ \\ \gamma^- e^{-\phi} & e^{\phi} + \gamma^-\gamma^+ e^{-\phi} - \theta^-\theta^+ & \gamma^- e^{-\phi}\theta^+ - \theta^- \\ \hline e^{-\phi}\theta^- & \gamma^+ e^{-\phi}\theta^- + \theta^+ & 1 + e^{-\phi}\theta^-\theta^+ \end{array} \right). \tag{4.127}$$

This covers only the Poincaré patch of the $\mathrm{OSp}(1|2,\mathbb{R})$ supergroup manifold, which should be taken into account in the derivation of the Plancherel measure [40].
One determines the Haar measure on this group manifold from the natural volume form of the metric on the particle-on-$\mathrm{OSp}(1|2,\mathbb{R})$ target space:

$$ds^2 = \frac{1}{2}\mathrm{STr}((g^{-1}dg)^2) \tag{4.128}$$

This leads to the Haar measure Eq B.43:

$$\boxed{dg = \frac{1}{2}e^{-\phi}[d\phi d\gamma^- d\gamma^+ \mid d\theta^- d\theta^+]}. \tag{4.129}$$

To deduce the Plancherel measure on $\mathrm{OSp}(1|2,\mathbb{R})$, one considers a generalization of the orthogonality theorem Eq A.30 to functions defined on the superline $\mathbb{R}^{1|1}$:

$$\int dg\ R^k_{x|\alpha,y|\beta}(g)^*\ R^{k'}_{x'|\alpha',y'|\beta'}(g) \equiv \frac{\delta(k-k')\delta(x-x')\delta(y-y')\delta(\alpha-\alpha')\delta(\beta-\beta')}{\rho(k)}. \tag{4.130}$$

The fermionic delta function evaluates to its argument as elaborated later in footnote 9.
For gravitational applications, we consider the mixed parabolic basis. The matrix expressions are worked out in detail in appendix B to which the reader is averted.
The main takeaway is the resulting Plancherel measure on $\mathrm{OSp}(1|2,\mathbb{R})$ Eq B.66:

$$\boxed{\rho(k) = \frac{1}{16\pi^2}\frac{\cosh(\pi k)}{1+\cosh(\pi k)}}. \tag{4.131}$$

4.4.2 Representation theory of $\mathrm{OSp}^+(1|2,\mathbb{R})$

The subsemisupergroup in the defining representation consists of all $\mathrm{OSp}(1|2,\mathbb{R})$-matrices (Eq 4.104) whose bosonic entries take only positive values $a, b, c, d > 0$, with no further constraints on the fermionic Grassmann entries since positivity of a supernumber is determined entirely by its body (see footnote 2).
This property is preserved under group multiplication $g_1 \cdot g_2$, for which the explicit composition law is written in Eq B.22. We see that each entry remains positive if the bosonic entries of both g_1 and g_2 are constrained to positive values. On the other hand, $\mathrm{OSp}^+(1|2,\mathbb{R})$ contains no natural inverse although it is well-defined from the parent $\mathrm{OSp}(1|2,\mathbb{R})$ group.

The entire (semi)supergroup manifold is covered by the single Poincaré patch in the Gauss decomposition

$$g(\phi,\gamma^-,\gamma^+ \mid \theta^-,\theta^+) = e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+} \tag{4.132}$$

, where we only need to constrain $\gamma^+,\ \gamma^- > 0$. The Haar measure in any case is the same as $\mathrm{OSp}(1|2,\mathbb{R})$.

The principal series representation of $\mathrm{OSp}^+(1|2,\mathbb{R})$ now acts projectively on the half superline $\mathbb{R}^{+1|1} = \{(x|\vartheta)\ :\ \ x > 0\}$ defined by the positive bosonic coordinate. The principal series action of a group element $g \in \mathrm{OSp}^+(1|2,\mathbb{R})$ on a square integrable function $f \in L^2(\mathbb{R}^{1|1})$ is defined as Eq B.21 but without the additional sign factors and absolute values:

$$\boxed{\langle x \mid g \mid f\rangle = (g\cdot f)(x,\vartheta) = (bx+d+\delta\vartheta)^{2j}\ f\left(\frac{ax+c+\beta\vartheta}{bx+d+\delta\vartheta}, -\frac{\alpha x+\gamma-e\vartheta}{bx+d+\delta\vartheta}\right)}. \tag{4.133}$$

The infinitesimal action defines a spin-j representation of the opposite $\mathfrak{osp}(1|2,\mathbb{R})$ superalgebra Eq B.32 in the anticommutators.

The principal continuous series representation on $\mathrm{OSp}^+(1|2,\mathbb{R})$ is both unitary and irreducible [40]. In particular, one again requires the spin label to be constrained to:

$$j = -\frac{1}{4} + \frac{ik}{2}, \qquad \text{with } k \in \mathbb{R}. \tag{4.134}$$

The explicit expressions for the parabolic eigenstates and the mixed parabolic matrix elements are derived in appendix B and shall not be repeated here. One eventually deduces the Plancherel measure on $\mathrm{OSp}^+(1|2,\mathbb{R})$:

$$\rho(k) = \cosh(\pi k) \tag{4.135}$$

4.4.3 $\mathrm{OSp}^+(1|2,\mathbb{R})$ structure of JT supergravity

To calculate the exact quantum gravitational amplitudes of $\mathcal{N} = 1$ JT SUGRA, we resort to its first-order $\mathfrak{osp}(1|2,\mathbb{R})$ BF formulation. Once we know the proper exponentiation of the algebra, all gravitational amplitudes are fixed by the relevant representation theory in terms of the constrained BF quantization of the previous chapters.

To arrive at the correct exponentiation, the Plancherel measure should match with the gravitational density of states. The correct density of states was first considered in [20] using the one-loop exact result of the super-Schwarzian theory. The derivation proceeds in parallel to the bosonic result. In particular, the classical solution turns off all the fermions. Therefore, the classical term in the partition function is the same as in the bosonic model: $I \simeq -\frac{\pi^2}{\beta}$ (c.f. Eq 1.106 with $C = 1/2$, $\beta = 1/T$). Since the total action is invariant under the class of super-conformal $\mathrm{OSp}(1|2,\mathbb{R})$ transformations, we need to divide by a number of bosonic and fermionic

zero-modes. Since the fermionic one-loop determinant is proportional to the Gaussian Grassmann path integral, whilst the scalar one-loop determinant is inversely proportional to a Gaussian path integral, one argues directly that we are in fact quotienting by $(\beta^{1/2})^{n_B-n_F}$, where n_F are the number of fermionic modes and n_B are the number of bosonic modes. In the case of $\mathrm{OSp}(1|2,\mathbb{R})$, we know that there are three bosonic generators $iH, iE_\pm$, and two fermionic $iF_\pm$ generators.
Since the super-Schwarzian emerges from a particular coadjoint orbit of the super-Virasoro algebra, the result is again one-loop exact, and we may write immediately:

$$Z(\beta) \propto \frac{1}{\beta^{1/2}}\, e^{\pi^2/\beta} = \int dE\ \rho(E) e^{-\beta E}. \tag{4.136}$$

The inverse Laplace transform readily yields [20]:

$$\boxed{\rho(E) \simeq \frac{1}{\sqrt{E}} \cosh(2\pi\sqrt{E})}. \tag{4.137}$$

This profile has the same characteristic exponential growth at large energies of the bosonic density of states $\rho(E) \xrightarrow{E>>1} e^{2\pi\sqrt{E}}$. This is as expected since the classical solution turns off the presence of the fermions in the path integral. On the other hand, the $\mathcal{N}=1$ density has a singular pole $\rho \to 1/\sqrt{E}$ towards low energies $E \to 0$. Translating this behaviour to a momentum variable $E = k^2$, the gravitational density of states $\rho(k)dk = \rho(E)dE$ readily reads:

$$\rho(k) \simeq \cosh(2\pi k). \tag{4.138}$$

This has a regular behaviour under $k \to 0$. This result matches with the Plancherel measure of the subsemisupergroup Eq B.82 under a shift of the continuous representation label $k \to 2k$. Noticeably, there is a complete mismatch with the Plancherel measure of the full group $\mathrm{OSp}(1|2,\mathbb{R})$ Eq B.66, whose classical limit has a power-law behaviour $\rho(E) \xrightarrow{E>>1} 1/\sqrt{E}$ instead of the expected exponential growth. This hints to again consider the subsemisupergroup BF description.

A technical argument in favour of the subsemisupergroup is that only the principal continuous series representation matrices turn up in the Plancherel decomposition of unitary irreducible representations, reminiscent of the bosonic case for $\mathrm{SL}^+(2,\mathbb{R})$ [40]:

$$\mathcal{L}^2(\mathrm{OSp}^+(1|2,\mathbb{R})) = \int_\oplus dk \cosh(\pi k)\ P_k \otimes P_k. \tag{4.139}$$

Therefore, regions inside the disk are labeled solely by the continuous k-label of the principal series representation.

A physical argument in favour of the subsemisupergroup is that this choice naturally restricts the entire moduli space of all flat connections on an arbitrary 2d super-Riemann surface to the subset of hyperbolic conjugacy class elements in super-Teichmüller space. Just like in the bosonic case, this restricts the metric on the Riemann surface to a smooth class of configurations only. The elliptic and parabolic holonomy classes necessarily lead to a singular (respectively conical, and cusp-like) behaviour. The different holonomies of a supergroup element

g are characterized by the value of its STr according to Eq B.8. Following [40], we may label the different sectors as $\epsilon = 0$ for **NS** and $\epsilon = 1$ for **R**, where using $a, b, c, d > 0$ in the subsemisupergroup perspective, and $ad - bc = 1 + \delta\beta$ as the natural $\text{OSp}(1|2,\mathbb{R})$ constraint Eq B.3, leads to:

$$|\text{STr}(g)| = |a + d + (-)^{\epsilon}(1 + \beta\delta)| \geqslant 2 + (-)^{\epsilon} \tag{4.140}$$

, by the fact that positivity of a supernumber is determined entirely by its body (footnote 2). Therefore, the restriction to the subsemisupergroup immediately leads to a restriction to the hyperbolic conjugacy class elements for both the **R** ($\text{STr}(g) > 1$) and **NS** ($\text{STr}(g) > 3$) sectors.

4.4.4 Thermal partition function

The thermal partition function can be re-derived purely from bulk considerations along the lines of section 2.8.5 of the bosonic theory. In particular, we cover the disk by boundary anchored Cauchy slices and propagate them from an initial asymptotic state labeled by $|\phi_i\rangle$, to a final state labeled by $|\phi_f\rangle$ using Hamiltonian evolution factor $e^{-\beta H}$:

$$Z_{\text{disk}}(\phi_i, \phi_f) = \langle \phi_f \mid e^{-\beta H} \mid \phi_i \rangle . \tag{4.141}$$

The Hamiltonian is proportional to the quadratic Casimir, which labels the representation. The latter is therefore diagonalized in the representation basis, which for $\text{OSp}^{(+)}(1|2,\mathbb{R})$ reads (Eq B.13):

$$\mathcal{C}_2 = -j(j + 1/2) = \frac{1}{16} + \frac{k^2}{4} \tag{4.142}$$

, where we inserted the unitarity constraint $j = -\frac{1}{4} + \frac{ik}{2}$ of the continuous principal series representation Eq B.23. To relate the continuous representation label k with the momentum label in gravity, we again rescale $k \to 2k$, and obtain up to a constant shift:

$$\mathcal{C}_2(k) \simeq k^2. \tag{4.143}$$

The Cauchy slices that diagonalize the Casimir operator are the representation matrices that reach the boundary. This open-channel Hilbert space is spanned by the representation eigenstates $|j, \nu_-, \epsilon_-; \nu_+, \epsilon_+\rangle$, whose overlap with the configurational group eigenstates $|g\rangle$ determine the complete set of wavefunctions:

$$\begin{aligned} \langle g \mid j, \nu_-, \epsilon_-; \nu_+, \epsilon_+ \rangle &= \sqrt{\rho(k)}\, \langle j, \nu_-, \epsilon_- \mid g \mid j, \nu_+, \epsilon_+ \rangle \\ &\equiv \sqrt{\rho(k)} R^k_{\nu_+,\epsilon_+;\nu_-,\epsilon_-}(\phi). \end{aligned} \tag{4.144}$$

The eigenfunctions of the Hilbert space are normalized with respect to the Plancherel measure of $\text{OSp}^+(1|2,\mathbb{R})$:

$$\rho(k) = \cosh(2\pi k) \tag{4.145}$$

, conform the normalization in the Peter-Weyl theorem Eq 2.111.

The principal series eigenstates $|j, \nu_-, \epsilon_-; \nu_+, \epsilon_+\rangle$ furnish a complete set of states on the Hilbert space due to the Peter-Weyl theorem in the restriction to the subsemisupergroup. Parameterizing the general group element $g \in \text{OSp}^+(1|2,\mathbb{R})$ in the Gauss decomposition Eq B.67, the matrix element diagonalizes up to the hyperbolic group element $e^{2i\phi H}$. This matrix element is precisely the Whittaker functions derived in Eq B.79:

$$\begin{aligned} R^k_{\nu_+,\epsilon_+;\nu_-,\epsilon_-}(\phi) =& \langle \nu_-, \epsilon_- \mid e^{2\phi i H} \mid \nu_+, \epsilon_+ \rangle \\ =& \frac{1}{\pi i} \frac{(\nu_+)^{\frac{1}{4}+\frac{ik}{2}}}{(\nu_-)^{-\frac{1}{4}+\frac{ik}{2}}} e^{\phi} \left(\epsilon_- K_{\frac{1}{2}+ik}(2e^{\phi}\sqrt{\nu_-\nu_+}) + \epsilon_+ K_{\frac{1}{2}-ik}(2e^{\phi}\sqrt{\nu_-\nu_+}) \right). \end{aligned} \tag{4.146}$$

The irrelevant prefactors corresponding to the eigenvalues of the parabolic group elements in the total matrix element can be absorbed in the choice of normalization.

Since this Hilbert space slicing reaches the asymptotic boundary, it is constrained by the coset boundary conditions that define JT supergravity. Just as in the bosonic case, this constrains the representation labels to $\nu_- = \nu_+ \equiv 1$. We indeed realize the Schwarzian theory with constrained entry $\mathcal{J}^- = 1$ in the expansion Eq 4.101: $\mathbf{A}_\tau = g\partial_\tau g^{-1} = \mathcal{J}^a i J_a$. Using similar reasoning to section 2.6.2 of the bosonic case (in particular the identification $i\mathcal{J}^- = J_+$), the eigenvalue under (here) E_+ is $E_+ = i$. For the right parabolic eigenstates Eq B.74, this corresponds to fixing $\nu_+ = 1$. This also fixes $\nu_- = 1$ according to the reasoning in section 2.8.4.
In the constrained setup, the representation eigenstates carry no additional labels and are simply specified by the representation label k. This coset setup leads to a weight one in the exponent of the Plancherel measure (c.f. Eq 2.204):

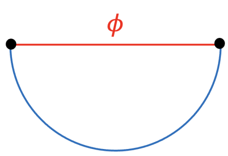
$$Z_{\text{disk}}(\phi_i, \phi_f) = \int_0^\infty dk \ \langle \phi_f \mid k \rangle \langle k \mid \phi_i \rangle e^{-\beta \mathcal{C}_2(k)} = \int_0^\infty dk \ \cosh(2\pi k) e^{-\beta k^2} R^k(\phi_i)^* R^k(\phi_f). \tag{4.147}$$

Physical boundaries are characterized by a trivial holonomy for which $\phi_i = \phi_f \to \infty$. Within this limit $R^k(\phi) \to 1$, and we readily obtain

$$\boxed{Z_{\text{disk}} = \int_0^{+\infty} dk \ \cosh(2\pi k) e^{-\beta k^2}}. \tag{4.148}$$

This coincides with the one-loop exact super-Schwarzian amplitude Eq 4.136.

Just as in the bosonic case, we may consider the Hartle-Hawking states preparing the vacuum, obtained by propagating the asymptotic state onto some boundary state labeled by $|\phi\rangle$:

$Z_{\text{Hartle}}(\phi, \beta') =$ 

$$= \langle \phi \mid e^{-\beta' H} \mid \mathbf{1} \rangle = \int_0^\infty dk \ \cosh(2\pi k) e^{-\beta' k^2} \langle \phi \mid k \rangle \langle k \mid \mathbf{1} \rangle. \tag{4.149}$$

Using the asymptotic state $\langle k \mid \mathbf{1} \rangle = R^k(\mathbf{1})^* \to 1$, and the explicit mixed parabolic matrix element Eq B.79 with coset boundary conditions $\nu_- = \nu_+ \equiv 1$ leads to:

$$Z_{\text{Hartle}}(\phi, \beta') = \frac{1}{\pi i} \int_0^\infty dk \ \cosh(2\pi k) e^{-\beta' k^2} e^{\phi} \left(K_{1/2+2ik}(2e^{\phi}) + \epsilon_- \epsilon_+ K_{1/2-2ik}(2e^{\phi}) \right). \tag{4.150}$$

Using the orthogonality of the continuous series representation Eq B.81, the thermal partition function with boundary length $\beta = \beta' + \beta''$ is readily obtained by gluing two asymptotic Hartle-Hawking states with boundary lengths β' and β'' along a common group label ϕ. After all, this is equivalent to inserting a complete set of states $\int \frac{e^{-\phi}}{2} |\phi\rangle\langle\phi|$ into the total disk amplitude;

$$\begin{aligned} Z(\beta) = \langle \mathbf{1} \mid e^{-\beta H} \mid \mathbf{1}\rangle &= \int_{-\infty}^{+\infty} \left(\frac{e^{-\phi}}{2}\right) \langle \mathbf{1} \mid e^{-\beta' H} \mid \phi\rangle\langle\phi \mid e^{-\beta'' H} \mid \mathbf{1}\rangle \\ &= \int_{-\infty}^{\infty} \left(\frac{e^{-\phi}}{2}\right) Z_{\text{Hartle}}(\phi,\beta')^* Z_{\text{Hartle}}(\phi,\beta''). \end{aligned}$$

The relevant Haar measure for the $\text{OSp}(1|2,\mathbb{R})$ supergroup is determined in Eq B.43 using similar techniques of section 2.6.1 for a particle-on-$\text{SL}(1|2,\mathbb{R})$:

$$dg = \frac{1}{2} e^{-\phi}\, d\phi \tag{4.151}$$

4.4.5 Super-Wilson line insertion

Holographic description

We identify boundary-anchored Wilson lines in holography to bilocal operators in the particle-on-a-group theory, in parallel to the discussion of section 2.8.6 of the bosonic case. In particular, a Wilson line operator in superspace is defined as:

$$\mathcal{W}_j(\tau_1,\tau_2) = \mathcal{P}\exp\left[-\int_{\tau_1}^{\tau_2} d\tau\, R_j(\mathbf{A})\right] \tag{4.152}$$

, which forms a $\dim(R) \times \dim(R)$-dimensional matrix.
On the other hand, JT supergravity is a constrained BF theory whose dual description leads to the super-Schwarzian theory. The specific constraint that reduces the full particle-on-a-group to the super-Schwarzian theory is the Brown-Henneaux boundary parametrization Eq 4.101, satisfying $\mathbf{A}_\tau = g\partial_\tau g^{-1}$.
To parameterize the boundary group element g^{-1}, one introduces two bosonic superfields $\psi_i = \psi_{i,\text{bot}} + \vartheta\psi_{i,\text{top}}$ $(i = 1, 2)$, and one fermionic superfield $\psi_3 = \psi_{3,\text{bot}} + \vartheta\psi_{3,\text{top}}$. These fields specify

$$g^{-1} = \left(\begin{array}{cc|c} \psi_{1,\text{bot}} & \psi'_{1,\text{bot}} & \psi_{1,\text{top}} \\ \psi_{2,\text{bot}} & \psi'_{2,\text{bot}} & \psi_{2,\text{top}} \\ \hline \psi_{3,\text{bot}} & \psi'_{3,\text{bot}} & \psi_{3,\text{top}} \end{array}\right). \tag{4.153}$$

The constraint equation $\mathbf{A}_\tau = g\partial_\tau g^{-1}$

$$\left(\begin{array}{cc|c} \psi_{1,\text{bot}} & \psi'_{1,\text{bot}} & \psi_{1,\text{top}} \\ \psi_{2,\text{bot}} & \psi'_{2,\text{bot}} & \psi_{2,\text{top}} \\ \hline \psi_{3,\text{bot}} & \psi'_{3,\text{bot}} & \psi_{3,\text{top}} \end{array}\right)\left(\begin{array}{cc|c} 0 & T_B(\tau) & T_F(\tau) \\ 1 & 0 & 0 \\ \hline 0 & T_F(\tau) & 0 \end{array}\right) = \left(\begin{array}{cc|c} \psi'_{1,\text{bot}} & \psi''_{1,\text{bot}} & \psi'_{1,\text{top}} \\ \psi'_{2,\text{bot}} & \psi''_{2,\text{bot}} & \psi'_{2,\text{top}} \\ \hline \psi'_{3,\text{bot}} & \psi''_{3,\text{bot}} & \psi'_{3,\text{top}} \end{array}\right) \tag{4.154}$$

leads to the three coupled differential equations:

$$\begin{aligned} \psi_{1,\text{bot}}\, T_B(\tau) + \psi_{1,\text{top}}\, T_F(\tau) &= \psi''_{1,\text{bot}}, & \psi_{1,\text{bot}}\, T_F(\tau) &= \psi'_{1,\text{top}}, \\ \psi_{2,\text{bot}}\, T_B(\tau) + \psi_{2,\text{top}}\, T_F(\tau) &= \psi''_{2,\text{bot}}, & \psi_{2,\text{bot}}\, T_F(\tau) &= \psi'_{2,\text{top}}, \\ \psi_{3,\text{bot}}\, T_B(\tau) + \psi_{3,\text{top}}\, T_F(\tau) &= \psi''_{3,\text{bot}}, & \psi_{3,\text{bot}}\, T_F(\tau) &= \psi'_{3,\text{top}}. \end{aligned} \tag{4.155}$$

This set of equations coincides with the supersymmetric Hill's equation [40]

$$(D^3 - \mathcal{V})\psi = 0 \tag{4.156}$$

, where $D \equiv \partial_\vartheta + \vartheta\partial_\tau$ is the 1d superderivative, and $\mathcal{V} = T_F(\tau) + \vartheta T_B(\tau)$ is a single fermionic superfield that captures both the bosonic and fermionic super-Schwarzian fields T_B, resp. T_F. Using $D^2 = \partial_\tau$ and $D^3 = \partial_\tau\partial_\vartheta + \vartheta\partial_\tau^2$, we can solve for both the top and bottom components of the Hill's equation

$$\begin{aligned} &\left(\partial_\tau\partial_\vartheta + \vartheta\partial_\tau^2 - T_F - \vartheta T_B\right)(\psi_b + \vartheta\psi_t) = 0 \\ \leftrightarrow \quad & \psi'_t - T_F\psi_b + \vartheta\left(\psi''_b - \psi_t T_F - T_B\psi_b\right) = 0 \end{aligned}$$

, which indeed coincide with the above set of gravitational equations for each variable ψ_i. Writing the superfield $\mathcal{V}$ as a general super-Schwarzian derivative

$$\mathcal{V}(\tau,\vartheta) = -\frac{D^4\alpha}{D\alpha} + \frac{2D^3\alpha D^2\alpha}{(D\alpha)^2} = -\{A,\alpha,\tau,\vartheta\} \tag{4.157}$$

, the solutions of the Hill's equation are captured entirely by the super-conformal reparametrization modes $\tau'(\tau,\vartheta)$, $\theta'(\tau,\vartheta)$ in terms of the two bosonic superfields ψ_i $(i = 1,2)$, and one fermionic superfield ψ_3 [40]:

$$\psi_1 = (D\theta')^{-1}, \qquad \psi_2 = \tau'(D\theta')^{-1}, \qquad \psi_3 = -\theta'(D\theta')^{-1}. \tag{4.158}$$

These fields are again constrained by the super-conformal constraint $D\tau' = \theta' D\theta'$. One can furthermore prove that the fields above are well-defined on $\mathrm{OSp}(1|2,\mathbb{R})$, and that they satisfy the defining constraints Eq B.3. In terms of the Gauss parametrization ($\gamma_\pm > 0$) Eq B.67

$$g^{-1} = e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{2\theta^+ iF_+} = \left(\begin{array}{cc|c} e^{-\phi} & \gamma^+ e^{-\phi} & e^{-\phi}\theta^+ \\ \gamma^- e^{-\phi} & e^{\phi} + \gamma^-\gamma^+ e^{-\phi} - \theta^-\theta^+ & \gamma^- e^{-\phi}\theta^+ - \theta^- \\ \hline e^{-\phi}\theta^- & \gamma^+ e^{-\phi}\theta^- + \theta^+ & 1 + e^{-\phi}\theta^-\theta^+ \end{array}\right)$$

, we may readily identify:

$$e^{-\phi} = \psi_{1,\text{bot}}, \quad \gamma^+ = \frac{\psi'_{1,\text{bot}}}{\psi_{1,\text{bot}}}, \quad \gamma^- = \frac{\psi_{2,\text{bot}}}{\psi_{1,\text{bot}}}, \quad \theta^- = \frac{\psi_{3,\text{bot}}}{\psi_{1,\text{bot}}}, \quad \theta^+ = \frac{\psi_{1,\text{top}}}{\psi_{1,\text{bot}}}. \tag{4.159}$$

Analogous to the bosonic case, the super-bilocal operator is represented by a matrix element in the lowest-weight discrete series representation on the superline $\mathbb{R}^{1|1}$, whose representation label is $j = -\ell$. We will see momentarily that ℓ coincides with the conformal weight of the super-bilocal operator (for $2\ell \in \mathbb{N}$). The lowest-weight state, and its adjoint are given by:

$$\langle x,\vartheta \mid j=-\ell,0\rangle = x^{2j} = \frac{1}{x^{2\ell}}, \qquad \langle -j=\ell,0 \mid x,\vartheta\rangle = \delta(x,\vartheta). \tag{4.160}$$

Using the general identification between Wilson lines and bilocal operators in particle-on-a-group Eq 2.138, the super-Wilson line is represented by:

$$\langle -j=\ell,0 \mid g(\tau_2)g^{-1}(\tau_1) \mid j=-\ell,0\rangle = \int dx d\vartheta\; \delta(x,\vartheta)g(\tau_2)g^{-1}(\tau_1)x^{-2\ell}. \tag{4.161}$$

Using the action of the principal series representation Eq B.68 under $g^{-1}(\tau_1)$ specified in Eq 4.153, we readily have:

$$x^{-2\ell} \xrightarrow{g^{-1}(\tau_1)} \left(\psi_{1,\text{bot}}(\tau_1)x+\psi_{2,\text{bot}}(\tau_1)+\psi_{3,\text{bot}}(\tau_1)\vartheta\right)^{-2\ell}. \tag{4.162}$$

The action under $g(\tau_2)$ is specified by its inverse (c.f. Eq B.6)

$$g(\tau_2) = \left(\begin{array}{cc|c} \psi'_{2,\text{bot}} & -\psi'_{1,\text{bot}} & -\psi'_{3,\text{bot}} \\ -\psi_{2,\text{bot}} & \psi_{1,\text{bot}} & \psi_{3,\text{bot}} \\ \hline \psi_{2,\text{top}} & -\psi_{1,\text{top}} & \psi_{3,\text{top}} \end{array}\right) \tag{4.163}$$

, leading to:

$$\begin{aligned}
&\xrightarrow{g(\tau_2)} \Big(\psi_{1,\text{bot}}(\tau_1)(\psi_{2,\text{bot}}(\tau_2)'x-\psi_{2,\text{bot}}(\tau_2)+\psi_{2,\text{top}}(\tau_2)\vartheta)+\psi_{2,\text{bot}}(\tau_1)(-\psi_{1,\text{bot}}(\tau_2)'x+\psi_{1,\text{bot}}(\tau_2)-\psi_{1,\text{top}}(\tau_2)\vartheta)\\
&\qquad +\psi_{3,\text{bot}}(\tau_1)(\psi_{3,\text{bot}}(\tau_2)'x-\psi_{3,\text{bot}}(\tau_2)+\psi_{3,\text{top}}(\tau_2)\vartheta)\Big)^{-2\ell}\\
&= \Big((\psi_{1,\text{bot}}(\tau_1)\psi_{2,\text{bot}}(\tau_2)'-\psi_{2,\text{bot}}(\tau_1)\psi_{1,\text{bot}}(\tau_2)'+\psi_{3,\text{bot}}(\tau_1)\psi_{3,\text{bot}}(\tau_2)')x\\
&\qquad +(\psi_{1,\text{bot}}(\tau_1)\psi_{2,\text{top}}(\tau_2)-\psi_{2,\text{bot}}(\tau_1)\psi_{1,\text{top}}(\tau_2)+\psi_{3,\text{bot}}(\tau_1)\psi_{3,\text{top}}(\tau_2))\vartheta\\
&\qquad -\psi_{1,\text{bot}}(\tau_1)\psi_{2,\text{bot}}(\tau_2)+\psi_{2,\text{bot}}(\tau_1)\psi_{1,\text{bot}}(\tau_2)-\psi_{3,\text{bot}}(\tau_1)\psi_{3,\text{bot}}(\tau_2)\Big)^{-2\ell}.
\end{aligned}$$

Taking the delta function in Eq 4.161 amounts to setting $x=\vartheta=0$. Within this limit, we may write directly

$$\langle -j=\ell,0 \mid g(\tau_2)g^{-1}(\tau_1) \mid j=-\ell,0\rangle = \left(-\psi_1(\tau_1)\psi_2(\tau_2)+\psi_2(\tau_1)\psi_1(\tau_2)-\psi_3(\tau_1)\psi_3(\tau_2)\right)^{-2\ell}|_{\text{bot}} \tag{4.164}$$

, where the vertical bar indicates that we consider the bottom component. Relating the superfields ψ_i to the solutions of the Hill's equation Eq 4.158 directly yields:

$$\boxed{\mathcal{W}_{\ell,00}(\tau_1,\tau_2) = \langle -j=\ell,0 \mid g(\tau_2)g^{-1}(\tau_1) \mid j=-\ell,0\rangle = \left(\frac{D_1\theta'_1\; D_2\theta'_2}{\tau'_1-\tau'_2-\theta'_1\theta'_2}\right)^{2\ell}\Bigg|_{\text{bot}}} \tag{4.165}$$

This can be identified directly to the $\text{OSp}(1|2,\mathbb{R})$-invariant super-bilocal operator in the super-Schwarzian theory. In the holographic bulk, this is to be identified with a lowest-weight Wilson operator insertion, to which we turn next.

Operator insertion

Within the super-gravitational amplitudes, a Wilson line is represented by a matrix element in the lowest-weight discrete series representation with $j = -\ell$. Up to some normalization, these are determined as solutions to the Casimir eigenvalue equation in terms of the Bessel functions of the first kind [40]:

$$R_{j,\nu_-\nu_+}(\phi) = e^{\phi} J_{2j+1}\left(2\sqrt{-\nu_-\nu_+ e^{\phi}}\right), \qquad e^{\phi} J_{2j}\left(2\sqrt{-\nu_-\nu_+ e^{\phi}}\right). \tag{4.166}$$

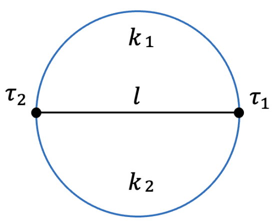
These states label the bottom and top components respectively. For now, we specialize to the bottom component. Within the super-gravitational amplitudes, these serve as operator insertions anchored to the holographic boundary. Just as in the bosonic case, their weight is constrained by the coset boundary conditions as a consequence of the current conservation property of the Clebsch-Gordan coefficients. In particular, sandwiched between two asymptotic states with $\nu_- = \nu_+ = 1$, the weight of the operator insertion is fixed to the lowest-weight state $\nu_- = \nu_+ = 0$. Taking the lowest-weight module with $j = -\ell$, and using the asymptotics of the Bessel functions of the first kind $J_\alpha(x) \sim x^{|\alpha|}$ for $x \to 0$, the bottom component asymptotes to (up to some ϕ-independent normalization):

$$R_{j=-\ell,00}(\phi) \;\sim\; e^{\phi}\, e^{|-2\ell+1|\phi}. \tag{4.167}$$

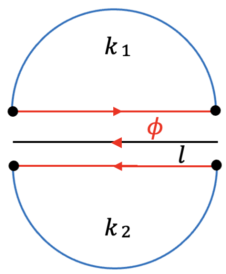
In the discretized theory, ℓ is constrained to $2\ell \in \mathbb{N}$. Taking $\ell > 1/2$, we identify

$$\boxed{R_{j=-\ell,00}(\phi) \;\sim\; e^{2\ell\phi}}. \tag{4.168}$$

The calculation proceeds along the lines of section 2.5.2. In particular, we obtain the super-gravitational amplitude of a Wilson line insertion by gluing two asymptotic Hartle-Hawking states along the Wilson line with common group label ϕ:



$$= \int d\phi$$



$$= \int_{-\infty}^{\infty} d\phi \left(\frac{1}{2}e^{-\phi}\right) Z_{\text{Hartle}}(\phi)^* \left(R_{-\ell,00}(f)\right) Z_{\text{Hartle}}(\phi)$$

The arrows indicate the orientation of the group element with respect to the Wilson line. Using the specified form of the Hartle-Hawking states in Eq 4.150, the integral over the group label ϕ leads to the $3j$-symbol of $\mathcal{N} = 1$ JT supergravity [21] [40]:

$$\begin{aligned}&\int_{-\infty}^{\infty} d\phi\, \frac{e^{\phi}}{2\pi^2}\left(K_{1/2-2ik_1}(2e^{\phi}) + \epsilon_-\epsilon_+ K_{1/2+2ik_1}(2e^{\phi})\right) e^{2\ell\phi}\left(K_{1/2+2ik_2}(2e^{\phi}) + \epsilon_-\epsilon_+ K_{1/2-2ik_2}(2e^{\phi})\right)\\ &= \frac{\Gamma(\frac{1}{2}+\ell \pm i(k_1+k_2))\Gamma(\ell \pm i(k_1-k_2)) + \Gamma(\frac{1}{2}+\ell \pm i(k_1-k_2))\Gamma(\ell \pm i(k_1+k_2))}{\pi^2\,\Gamma(2\ell)}\end{aligned} \tag{4.169}$$

, for any sign of $\epsilon_-\epsilon_+$. This leads to the final bottom component of the disk partition function with a single boundary anchored Wilson line insertion, up to some normalization (compare to Eq 2.282);

$$\begin{aligned}\langle \mathcal{W}_\ell(\tau_1,\tau_2)\rangle = &\int_0^\infty dk_1 \cosh(2\pi k_1) e^{-\beta_1 k_1^2} \int_0^\infty dk_2 \cosh(2\pi k_2) e^{-\beta_2 k_2^2} \\ &\times \frac{\Gamma(\frac{1}{2}+\ell \pm i(k_1+k_2))\Gamma(\ell \pm i(k_1-k_2)) + \Gamma(\frac{1}{2}+\ell \pm i(k_1-k_2))\Gamma(\ell \pm i(k_1+k_2))}{\Gamma(2\ell)}\end{aligned} \tag{4.170}$$

, with $\beta_1+\beta_2=\beta$.

4.5 Defects in JT supergravity

Defects in JT supergravity are again classified according to solutions of the supersymmetric Hill's equation. In particular, depending on T_B and T_F, the solutions can have non-trivial monodromies around the thermal boundary circle. Demanding the correct periodicity of the asymptotic gauge field $\mathbf{A}_\tau$ for both sectors Eq 4.103, the most general solutions to $g(\tau)\partial_\tau g^{-1}(\tau) = \mathbf{A}_\tau$ are classified according to:

$$\begin{aligned}(\textbf{NS}): \qquad & g(\tau+\beta) = (-)^F g(\tau) M \\ (\textbf{R}): \qquad & g(\tau+\beta) = g(\tau) M.\end{aligned} \tag{4.171}$$

$M \in \text{OSp}(1|2,\mathbb{R})$ labels the monodromy matrix. Since the solutions to the Hill's equation $g\partial_\tau g^{-1} = \mathbf{A}_\tau$ are redundant under the equivalence relation $g \sim gS$, for some constant $S \in \text{OSp}(1|2,\mathbb{R})$, the monodromies are parameterized by the different conjugacy class elements of $\text{OSp}(1|2,\mathbb{R})$:

$$M \sim SMS^{-1}. \tag{4.172}$$

This leads to the existence of stabilizer subgroups that preserve the monodromy matrices within each conjugacy class:

$$H = \{S \in \text{OSp}(1|2,\mathbb{R}) \mid MS = SM\}. \tag{4.173}$$

The different monodromy matrices and their stabilizers within each conjugacy class and spin sector have been thoroughly studied in [40], which was based on earlier accounts in [102]. Just as in the bosonic case [37], the classification of the different monodromies and their stabilizer is closely related to the classification of the coadjoint orbits of the super-Virasoro group. Within each coadjoint orbit, the super-Schwarzian pair $(T_B(\tau), T_F(\tau))$ is contained by acting with the super-Virasoro group on some constant representative element (T_B, T_F). The stabilizer subgroups of the different monodromies consequently preserve the value of the super-Schwarzian derivative within each orbit. This hence classifies the different super-Virasoro orbits directly according to the solutions of the super-Hill's equations.

Perhaps more interestingly in the current discussion are the gravitational implications of each non-trivial monodromy. Thereto, one should relate the structure of the monodromy matrices to the super-reparametrization fields (F,η). According to the general solutions of the Hill's equation Eq 4.158, these are related to the projective action of M on $\tau'(\tau,\vartheta)=\psi_2/\psi_1$, and $\theta'(\tau,\vartheta)=-\psi_3/\psi_1$. Using the parametrization of the inverse group element g^{-1} Eq 4.153, and the inverse monodromy matrix M^{-1}, one writes the monodromy equivalence relation as

$$g^{-1}(\tau+\beta)=M^{-1}g^{-1}(\tau)=\left(\begin{array}{cc|c} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ \hline 0 & 0 & M_{33} \end{array}\right)\left(\begin{array}{cc|c} \psi_{1,\text{bot}} & \psi'_{1,\text{bot}} & \psi_{1,\text{top}} \\ \psi_{2,\text{bot}} & \psi'_{2,\text{bot}} & \psi_{2,\text{top}} \\ \hline \psi_{3,\text{bot}} & \psi'_{3,\text{bot}} & \psi_{3,\text{top}} \end{array}\right). \tag{4.174}$$

The monodromy matrix is bosonic block diagonal, and $M_{33}=\pm 1$ [40]. Splitting the reparametrized superfields τ', θ' into their bottom and top components $\tau'(\tau,\vartheta)=\tau'_B(\tau)+\vartheta\tau'_F(\tau)$, $\theta'(\tau,\vartheta)=\theta'_F(\tau)+\vartheta\theta'_B(\tau)$, one can show that the action of the monodromy is projectively realized as [40]:

$$\tau'_B(\tau+\beta)=\frac{M_{21}+M_{22}\tau'_B(\tau)}{M_{11}+M_{12}\tau'_B(\tau)}, \qquad \theta'_B(\tau+\beta)=\frac{M_{33}\theta'_B(\tau)}{M_{11}+M_{12}\tau'_B(\tau)} \tag{4.175}$$

$$\tau'_F(\tau+\beta)=\frac{\tau'_F(\tau)}{(M_{11}+M_{12}\tau'_B(\tau))^2}, \qquad \theta'_F(\tau+\beta)=\frac{M_{33}\theta'_F(\tau)}{M_{11}+M_{12}\tau'_B(\tau)}. \tag{4.176}$$

Its constrained solutions are again given in terms of the reparametrization modes $(F(\tau),\eta(\tau))$ Eqs 4.41, 4.42:

$$\tau'(\tau,\vartheta)=F(\tau+\vartheta\eta(\tau)), \qquad \theta(\tau,\vartheta)=\sqrt{\dot F(\tau)}\left(\vartheta+\eta(\tau)+\frac{1}{2}\vartheta\eta(\tau)\partial_\tau\eta(\tau)\right). \tag{4.177}$$

The non-trivial monodromy is implemented as [40]:

$$F(\tau)=\tan\frac{\pi}{\beta}\Theta f(\tau), \qquad F(\tau)=\tanh\frac{\pi}{\beta}\Lambda f(\tau) \tag{4.178}$$

, for the elliptic and hyperbolic conjugacy classes respectively. f represents the reparametrization mode of the thermal boundary circle satisfying $f(\tau+\beta)=f(\tau)+\beta$, while $\eta(\tau+\beta)=\pm\eta(\tau)$ specifies the spin structure of either **R** $(+)$ or **NS** $(-)$. Parabolic defects on the other hand are specified by $F(\tau+\beta)=F(\tau)+\beta$, with the same (anti)periodicity of η as before.
Using the bosonic submetric Eq 4.52, we may readily extrapolate these monodromy relations into the bulk using similar reasoning of section 2.9, and conclude that elliptic monodromies with parameter Θ correspond to conical singularities with conical deficit angle $2\pi\Theta$. Integer values of n correspond to replicated geometries, although there is a subtlety between even and odd values of n since the stabilizer subgroup is different in both cases [40]. Parabolic monodromies again correspond to singular geometries with a cusp near the horizon.
The only regular non-trivial geometries are the hyperbolic monodromy classes. These correspond to wormhole geometries, ending on a geodesic boundary of length $2\pi\Lambda$.

4.5.1 Defect insertion and hyperbolic characters

To implement a non-trivial monodromy in the BF perspective, we insert a suitably normalized principal series character in the disk, associated to a defect with holonomy U. The technical details are explained in section 2.9 (recall in particular Eq 2.289 for a coset scenario). Since we will be interested in EOW branes ending at the neck of a trumpet, we consider the insertion of a hyperbolic character.

Hyperbolic character in the principal series representation

Characters of hyperbolic conjugacy class elements evaluated in the principal series representation, can be calculated along the lines of section 2.9.1 for $\mathrm{SL}(2,\mathbb{R})$. The specific representation theory was considered in [40], which itself was based on a generalization of [90] for $\mathrm{SL}(2,\mathbb{R})$. To start, wavefunctions corresponding to the state vectors $|f\rangle$ in configuration space $f(x,\vartheta) = \langle x,\vartheta \mid f\rangle$ transform as:

$$f(x,\vartheta) = \int dx'd\vartheta'\ K(x,\vartheta \mid x',\vartheta')f(x',\vartheta'). \tag{4.179}$$

$K(x,\vartheta \mid x',\vartheta')$ denotes the kernel of the representation. In the principal series representation Eq B.21, the latter is specified as[9]:

$$K(x,\vartheta \mid x',\vartheta') = \frac{|bx+d+\delta\vartheta|^{2j}}{\mathrm{sgn}(e)^{1/2}\mathrm{sgn}(bx+d+\delta\vartheta)^{1/2}}\delta\left(\frac{ax+c+\beta\vartheta}{bx+d+\delta\vartheta}-x'\right)\delta\left(\frac{-\alpha x-\gamma+e\vartheta}{bx+d+\delta\vartheta}-\vartheta'\right) \tag{4.181}$$

, where the fermionic delta function is defined in footnote 9. The principal series character of a group element g in representation j is consequently computed by the continuous trace of the kernel function:

$$\begin{aligned}\chi_j(g) &= \int dxd\vartheta\ K(x,\vartheta \mid x,\vartheta)\\ &= \int dxd\vartheta\frac{|bx+d+\delta\vartheta|^{2j}}{\mathrm{sgn}(e)^{1/2}\mathrm{sgn}(bx+d+\delta\vartheta)^{1/2}}\delta\left(\frac{ax+c+\beta\vartheta}{bx+d+\delta\vartheta}-x\right)\delta\left(\frac{-\alpha x-\gamma+e\vartheta}{bx+d+\delta\vartheta}-\vartheta\right).\end{aligned} \tag{4.182}$$

The delta function is evaluated by expanding in the zeros on the superline $\mathbb{R}^{1|1}$. Since the character is a class function, the calculation simplifies considerably by parameterizing the most general hyperbolic group element in the Gauss parametrization:

$$g(\phi) = e^{2\phi iH} \simeq \left(\begin{array}{cc|c} e^{-\phi} & \epsilon & 0\\ 0 & e^{\phi} & 0\\ \hline 0 & 0 & \pm 1\end{array}\right) \tag{4.183}$$

, for the different sectors **R** $(+)$ and **NS** $(-)$. The infinitesimal factor ϵ again regularizes the behaviour near $x=0$. Due to $e^{-\phi}+e^{\phi}\geqslant 2$, the STr of the different sectors is guaranteed to correspond to the hyperbolic conjugacy class for both sectors (up to the inclusion of parabolic class elements with measure zero)

$$\begin{aligned}(\mathbf{R}):&\qquad \mathrm{STr}(g)\geqslant 1\\ (\mathbf{NS}):&\qquad \mathrm{STr}(g)\geqslant 3.\end{aligned} \tag{4.184}$$

In both cases, the parameter ϵ serves as an infinitesimal regulator to fix the number of fixed points at $\epsilon=0$. In appendix E of [40], the character was derived explicitly for the **R** sector. To complement this calculation, I

[9] The definition of the delta function on Grassmann variables is subtle. In fact, any delta function imposing $\vartheta=\vartheta'$ is simply expressed as the argument inside the delta function: $\delta(\vartheta-\vartheta')=\vartheta-\vartheta'$. This is a consequence of the linear Taylor expansion of any function in ϑ: $(f(\vartheta)=a+\vartheta b)$, and the defining integral identities $\int d\vartheta=0$, $\int d\vartheta\vartheta=1$, where now indeed

$$\int d\vartheta\ \delta(\vartheta-\vartheta')f(\vartheta)=\int d\vartheta\ \delta(\vartheta-\vartheta')(a+\vartheta b)\equiv\int d\vartheta\ (\vartheta-\vartheta')(a+\vartheta b)=a+\vartheta' b. \tag{4.180}$$

work out the specific case for the **NS** sector hereunder.
First of all, specifying to $(-)$ in $g(\phi)$ and negative $\phi < 0$, the Borel-Weil action of $g(\phi)$ is translated to the kernel

$$\chi_j^{\textbf{NS}}(\phi) = \int dx d\vartheta \, \frac{|\epsilon x + e^{\phi}|^{2j}}{i} \delta\left(\frac{e^{-\phi}x}{\epsilon x + e^{\phi}} - x\right) \delta\left(\frac{-\vartheta}{\epsilon x + e^{\phi}} - \vartheta\right)$$

, where we choose the upper branch in $\mathrm{sgn}(-1)^{1/2} = i$. Due to the general identity $\delta(g(x)) = \sum_i \frac{g(x_i)}{|g'(x_i)|}$, the bosonic delta function decomposes into:

$$\delta\left(\frac{e^{-\phi}x}{\epsilon x + e^{\phi}} - x\right) = \frac{\delta(x)}{e^{-2\phi} - 1} + \frac{\delta\left(x - \frac{e^{-\phi} - e^{\phi}}{\epsilon}\right)}{1 - e^{2\phi}}.$$

The fermionic delta function on the other hand is just the argument of that function (see footnote 9):

$$\delta\left(\frac{-\vartheta}{\epsilon x + e^{\phi}} - \vartheta\right) = -\vartheta\frac{1 + \epsilon x + e^{\phi}}{\epsilon x + e^{\phi}}.$$

Performing the Grassmann integral over ϑ, and consequently integrating over x leads to:

$$\begin{aligned}
\chi_j^{\textbf{NS}}(\phi) &= i\frac{e^{2j\phi}}{e^{-2\phi} - 1}\frac{1 + e^{\phi}}{e^{\phi}} + i\frac{e^{-2j\phi}}{1 - e^{2\phi}}\frac{1 + e^{-\phi}}{e^{-\phi}} \\
&= i\frac{(e^{-\phi/2} + e^{\phi/2})e^{2j\phi + \phi/2}}{e^{-\phi} - e^{\phi}} + i\frac{(e^{\phi/2} + e^{-\phi/2})e^{-2j\phi - \phi/2}}{e^{-\phi} - e^{\phi}} \\
&= -2i\frac{\cosh(\phi/2)}{\sinh\phi}\cosh\left(2j\phi + \frac{\phi}{2}\right) \\
&= -i\frac{\cosh(ik\phi)}{\sinh(\phi/2)}
\end{aligned} \tag{4.185}$$

, where $\sinh\phi = 2\sinh(\phi/2)\cosh(\phi/2)$, and the unitarity constraint $j = -1/4 + ik/2$ were used in the last line. We finally obtain:

$$\boxed{\chi_k^{\textbf{NS}}(\phi) = i\frac{\cos(-k\phi)}{\sinh(-\phi/2)}}. \tag{4.186}$$

Similar reasoning for the **R** sector leads to [40]:

$$\boxed{\chi_k^{\textbf{R}}(\phi) = i\frac{\sin(-k\phi)}{\cosh(-\phi/2)}}. \tag{4.187}$$

It is interesting to note that the **NS** character can be decomposed as:

$$\frac{\cosh(-(4j+1)\phi/2)}{\sinh(-\phi/2)} = \frac{\cosh(-(2j+1)\phi)}{\sinh(-\phi)} + \frac{\cosh(-2j\phi)}{\sinh(-\phi)} \tag{4.188}$$

, which is the sum of two hyperbolic characters in the principal series representation of $\mathrm{SL}(2,\mathbb{R})$ (Eq 2.323) with spin j and $j - 1/2$ respectively. This is consistent with the direct sum decomposition Eq B.33, since it is a general fact that the character of the direct sum of irreducible representations equals the sum of the characters

evaluated in each representation. While it is true that both representations of $\mathrm{SL}(2,\mathbb{R})$ are irreducible, they are however not unitary since that would require $j = -\frac{1}{2} + ik$, whereas we have $\Re(j) = -\frac{1}{4}$.
Within each sector, the characters should be orthogonal with respect to the proper measure on the space of conjugacy class elements. The latter can be derived from the Weyl integration formula, where the bosonic result Eq 2.315 is generalized to the supergroup for each spin structure to [40]:

$$\Delta_{\mathbf{R}}(t) = \frac{\prod_{\alpha\in\Delta_B} |e^{\alpha(t)} - 1|}{\prod_{\alpha\in\Delta_F} (e^{\alpha(t)} - 1)}, \qquad \Delta_{\mathbf{NS}}(t) = \frac{\prod_{\alpha\in\Delta_B} |e^{\alpha(t)} - 1|}{\prod_{\alpha\in\Delta_F} (e^{\alpha(t)} + 1)}. \tag{4.189}$$

Δ_B and Δ_F denote the collection of bosonic and fermionic roots respectively. The roots are determined as the eigenvectors of the generators with respect to the commutator with the Cartan element iH:

$$[iH, iE_{\pm}] = \mp iE_{\pm}, \qquad [iH, iF^{\pm}] = \mp\frac{1}{2} iF^{\pm}. \tag{4.190}$$

This determines the root system of the algebra. Exponentiating the algebra to $\mathrm{Ad}(t^{-1})X^{\alpha} = t^{-1}X^{\alpha}t = e^{-\alpha(t)}X^{\alpha}$ defines the root vectors $\alpha(t)$. Within the hyperbolic conjugacy class, the relevant group elements are $X^{\alpha} \equiv e^{2iH\phi}$, and we find two bosonic and two fermionic roots, corresponding to the generators $iE_{\pm}$ and $iF_{\pm}$:

$$e^{\alpha_B(iE_{\pm})} = e^{\mp 2\phi}, \qquad e^{\alpha_F(iF_{\pm})} = e^{\mp\phi}. \tag{4.191}$$

Focusing on $\phi < 0$, it is straightforward to deduce the correct measure on each hyperbolic conjugacy class:

$$\Delta_{\mathbf{NS}}(t) = \frac{(e^{-2\phi} - 1)(1 - e^{2\phi})}{(e^{-\phi} + 1)(e^{\phi} + 1)} = 4\sinh^2(-\phi/2), \tag{4.192}$$

$$\Delta_{\mathbf{R}}(t) = \frac{(e^{-2\phi} - 1)(1 - e^{2\phi})}{(e^{-\phi} - 1)(e^{\phi} - 1)} = 4\cosh^2(-\phi/2). \tag{4.193}$$

The measures are fine-tuned to give precisely the anticipated character orthogonality:

$$\frac{1}{2}\int d\phi\; 4\sinh^2(-\phi/2)\chi_k^{\mathbf{NS}}(\phi)\chi_{k'}^{\mathbf{NS}}(\phi)^* = 2\int d\phi\; \cos(-k\phi)\cos(-k'\phi) = 2\pi\delta(k - k'), \tag{4.194}$$

$$\frac{1}{2}\int d\phi\; 4\cosh^2(-\phi/2)\chi_k^{\mathbf{R}}(\phi)\chi_{k'}^{\mathbf{R}}(\phi)^* = 2\int d\phi\; \sin(-k\phi)\sin(-k'\phi) = 2\pi\delta(k - k'). \tag{4.195}$$

A common practice is to strip off the Weyl-denominator of the characters immediately, and glue along a flat conjugacy class measure $d\phi$ on the supergroup. A more structural motivation why only the numerators of the characters are the essential objects to consider in the context of defect insertions, is the Schwarzian limit of super-Virasoro modular S-matrices. For example, one can relate the defect insertions in the bulk as topologically deformed Verlinde loop operators in the Schwarzian theory [37]. The latter are obtained in a $b \to 0$ limit of the super-Virasoro modular S-matrix S_s^P with $s = -\phi/2\pi b$ and $P = bk$. In particular, for the **NS** sector, its limit reproduces the continuous series character only up to the Weyl denominator [40]

$$\lim_{b\to 0} S_s^P = \lim_{b\to 0} \cos(4\pi s P) = \cos(-2\phi k). \tag{4.196}$$

This suggests that we should indeed constrain our attention to the numerator of the character when we consider the gravitational defect insertions as the fundamental objects. Note that, just as for $\mathrm{SL}(2,\mathbb{R})$, the elliptic character vanishes since there are no fixed points on the superline $\mathbb{R}^{1|1}$. However, in the interest of classifying defects, one analytically continues the character in the hyperbolic conjugacy class to the elliptic conjugacy class by simply replacing $\phi \to i\phi$. This leads to the character insertion $\cosh(-\phi k)$ for elliptic defects in the **NS** sector, and $\sinh(-\phi k)$ in the **R** sector.

Hyperbolic defect insertions

To relate the group theoretic character to gravity, we should again shift the continuous series label to the momentum label $k \to 2k$. The single trumpet amplitudes within each sector are readily obtained by inserting the respective suitably normalized characters Eqs 4.186, 4.187 within the disk partition function, and stripping off the Plancherel measure due to the coset boundary constraints. The result is:

$$Z^{\textbf{NS}}_{\text{trumpet}}(\beta,\phi) = \int_0^\infty dk\ \cos(-2\phi k)e^{-\beta k^2} = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}e^{-\phi^2/\beta}, \tag{4.197}$$

$$Z^{\textbf{R}}_{\text{trumpet}}(\beta,\phi) = \int_0^\infty dk\ \sin(-2\phi k)e^{-\beta k^2} = -\frac{1}{\sqrt{\beta}}D_+(\phi/\sqrt{\beta}). \tag{4.198}$$

The last function denotes the Dawson integral, defined as the one-sided Fourier-Laplace transform of the sine function with respect to the Gaussian kernel:

$$D_+(x) = e^{-x^2}\int_0^x dt\ e^{t^2} = \frac{1}{2}\int_0^\infty dt\ e^{-t^2/4}\sin(xt). \tag{4.199}$$

The integral $\int_0^x e^{t^2}dt$ is proportional to the analytically continued error function $\text{erfi}(x) = -i\text{erf}(ix)$, defined as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{4.200}$$

, normalized as $\lim_{x\to\infty}\text{erf}(x) = 1$. The **R** trumpet is therefore related to the **NS** trumpet up to a correction by the error function:

$$Z^{\textbf{R}}_{\text{trumpet}}(\beta,\phi) \simeq Z^{\textbf{NS}}_{\text{trumpet}}(\beta,\phi) \times \text{erfi}\left(-\frac{\phi}{\sqrt{\beta}}\right). \tag{4.201}$$

The origin of this error function is less clear from a one-loop exact super-Schwarzian perspective. In particular, there is a mismatch between the analysis of the **R** sector trumpet calculated directly in this perspective in [102]. However, this calculation was done entirely from one-loop argumentation in the super-Schwarzian perspective. The amplitude of the Ramond trumpet suggests that the result is not one-loop exact, and this argumentation is not applicable anyway. In many ways, the perspective provided from topological gauge theory should be exact to all orders in perturbation theory. We may therefore use the Taylor-Maclaurin expansion of the error function in the argument $z = \phi/\sqrt{\beta}$:

$$\text{erfi}(z) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}}\left(z + \frac{z^3}{3} + \frac{z^5}{10} + \frac{z^7}{42} + \frac{z^9}{216} + \dots\right) \tag{4.202}$$

, and treat the Ramond trumpet amplitude as the full perturbative answer to the gravitational path integral in increasing orders of the coupling $1/\beta$;

$$Z^{\mathbf{R}}_{\text{trumpet}}(\beta,\phi) = -\left(\frac{\phi}{\beta} + \frac{\phi^3}{3\beta^2} + \frac{\phi^5}{10\beta^3} + \frac{\phi^7}{42\beta^4} + \dots\right) e^{-\phi^2/\beta}. \tag{4.203}$$

Further investigation should reveal how this is related to the answer of [102].

4.6 Geodesic description of EOW branes in superspace

Our ambition is now to formulate an equivalent boundary action, in the likes of Eq 3.1, that captures the geodesic dynamics of EOW branes in superspace, and thereby extend the discussion of the previous chapter to applications of JT supergravity. First of all, we recapitulate the JT supergravity action in superspace (Eq 4.1) with the appropriate boundary term of [39];

$$I^{\mathcal{N}=1}_{JT} = \frac{1}{4}\left[\int_\Sigma d^2z d^2\theta \; E\Phi(R_{+-}+2) + 2\int_{\partial\Sigma} d\tau d\vartheta \; \Phi K\right]. \tag{4.204}$$

The extrinsic curvature along the UV boundary curve is defined in the first order form in Eq 4.57;

$$K = \frac{T^A D_T n_A}{T^A T_A} \tag{4.205}$$

, in terms of a covariant derivative that acts as a superderivative in superspace, equipped with the first order spin connections (c.f. Eq 4.59). An important realization is that the boundary curves are in fact $1|1$-dimensional sheets that are infinitesimally thickened in the fermionic ϑ-direction. I.e., the boundary curve is parameterized in terms of a bosonic τ- and fermionic ϑ-affine coordinate. In Poincaré SUHP coordinates discussed in section 4.2, the boundary curve covers the $1|1$-dimensional sheet in the parametrization

$$\tau'(\tau,\vartheta), \qquad y'(\tau,\vartheta), \qquad \theta'(\tau,\vartheta), \qquad \bar{\theta}'(\tau,\vartheta) \tag{4.206}$$

, with $z' \equiv \tau' + iy'$ and $\bar{z}' = \tau' - iy'$. However, we aim to describe EOW branes as geodesic curves in superspace. These are genuine $1|0$-dimensional curves in the $2|2$-dimensional superspace, describing the trajectory

$$z'(s), \qquad \bar{z}'(s), \qquad \theta'(s), \qquad \bar{\theta}'(s) \tag{4.207}$$

in terms of a single bosonic affine parameter s, which we may take to be the proper length along the curve. A nice visual representation was given in [40] (figure 4.1), where a $1|0$-dimensional geodesic is anchored to the $1|1$-dimensional wiggly boundary in Poincaré SUHP coordinates.

Surprisingly, the relevant differential geometry on superspace does not seem to be available in the existing literature. We therefore construct it from first principles, following the textbook development of bosonic GR.

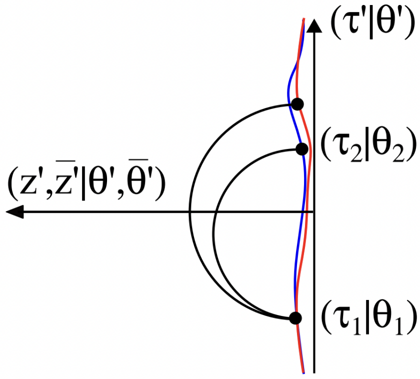


Figure 4.1: $1|0$-dimensional geodesic is anchored to the $1|1$-dimensional wiggly boundary in Poincaré SUHP coordinates $(z', \overline{z}', \theta', \overline{\theta}')$. Figure taken from [40].

4.6.1 Free particle action in superspace

A natural first step is to add to the JT supergravity action a term containing the free-particle action in superspace, labeled in terms of this bosonic affine parameter s:

$$I = -\mu \int_{EOW} ds \, \left(\dot{Z}^M g_{MN} \dot{Z}^N\right)^{1/2}. \tag{4.208}$$

$Z^M(s) = \big(z(s), \overline{z}(s), \theta(s), \overline{\theta}(s)\big)$ labels the trajectory in superspace, and the dot indicates differentiation with respect to the affine parameter. g_{MN} denotes the metric in superspace. This has a bosonic block for $(M+N) \mod 2 = 0$, and a fermionic block for $(M+N) \mod 2 = 1$, with M, N $\mathbb{Z}_2$-valued indices labeling the bosonic (0) or fermionic (1) parity of the superfield. We define an inverse metric tensor g^{MN}, satisfying

$$g^{NC} g_{CM} = g_{MC} g^{CN} = \delta_M^N. \tag{4.209}$$

This definition is non-trivial since in general one picks up additional sign factors when commuting two objects in superspace, depending on the bosonic or fermionic parity

$$g_{NC} g^{CM} = (-)^{C(1+M)+N(M+C)} g^{CM} g_{NC}. \tag{4.210}$$

For consistency, we take the conventions

$$\boxed{g^{MC} = (-)^{MC} \, g^{CN}, \qquad \text{and} \qquad g_{CN} = (-)^{CN+C+N} g_{NC}}. \tag{4.211}$$

Concretely, this means that both the fermionic ($MN+M+N = 1$) and doubly fermionic ($MN+M+N = 3$) block of the metric tensor g_{CN} are antisymmetric with respect to interchanges in the indices. In contrast, only the doubly fermionic block of the inverse metric g^{MC} is antisymmetric. Within these conventions, the natural definition $g^{MC} g_{CN} = \delta_N^M$ implies directly:

$$g_{NC} g^{CM} = (-)^{C(1+M)+N(M+C)} g^{CM} g_{NC} = (-)^{N(M+1)} g^{MC} g_{CN} = (-)^{N(M+1)} \delta_M^N \simeq \delta_M^N.$$

The last identity holds since if the delta function imposes $N \equiv M$, then the phase factor is always one: $(-)^{N(M+1)} \simeq (-)^{N+N} = 1$. Thus, the definition of the inverse metric is compatible in both orderings Eq 4.209. This should not conflict earlier definitions of the metric in this chapter. In particular, it will turn out to be compatible with the definition of the supermetric in terms of the superframe fields in Eq 4.53.
Since the line element $ds^2 = dZ^M g_{MN} dZ^N$ is by definition coordinate invariant, we define a covariant vector according to:

$$\boxed{\dot{Z}_M \equiv g_{MN}\dot{Z}^N}. \tag{4.212}$$

Contractions are thereby defined in the NW-SE (north-west - south-east) convention:

$$\dot{Z}^N g_{NM} \dot{Z}^M = \dot{Z}^M \dot{Z}_M. \tag{4.213}$$

Considering the fermionic derivative as an operator, we define a coordinate transformation from the left

$$\boxed{\dot{Z}'_M \equiv \frac{\partial Z^N}{\partial Z^{M\prime}} \dot{Z}_N} \tag{4.214}$$

, where coordinate invariant contractions again appear in the NW-SE ordening. The transformation rule on covectors Eq 4.214 is consistent with the coordinate transformation of a gradient in superspace ∂_M, where the (fermionic) chain rule acts also from the left (c.f. Eq 4.218):

$$\partial_M \;\rightarrow\; \partial'_M = \frac{\partial Z^N}{\partial Z^{M\prime}} \partial_N. \tag{4.215}$$

This property is not immediately obvious. Consider a function of one fermionic variable $f(\theta)$. This can always be expanded in a Taylor series to first order in θ: $f(\theta) = \alpha + \theta\beta = f(0) + \theta\frac{\partial}{\partial\theta} f(0)$. Note that we need to expand with θ on the left if $\frac{\partial}{\partial\theta}$ is a Grassmann derivative acting on the left. The definition of the latter being:

$$\frac{\partial}{\partial\theta} f(\theta) = \lim_{\Delta\theta\to 0} \frac{1}{\Delta\theta} \left(f(\theta + \Delta\theta) - f(\theta) \right). \tag{4.216}$$

The Grassmann parity of $\Delta\theta$ is the same as that of θ. Let now $f(\theta) = g(\theta'(\theta))$, then under the same definition:

$$\frac{\partial}{\partial\theta} f(\theta) = \lim_{\Delta\theta\to 0} \frac{1}{\Delta\theta} \left(g(\theta'(\theta + \Delta\theta)) - g(\theta'(\theta)) \right). \tag{4.217}$$

We expand in the same way $\theta'(\theta + \Delta\theta) = \theta' + \Delta\theta'$, leading to:

$$\frac{\partial}{\partial\theta} f(\theta) = \lim_{\Delta\theta'\to 0,\, \Delta\theta\to 0} \frac{\Delta\theta'}{\Delta\theta} \frac{1}{\Delta\theta'} \left(g(\theta' + \Delta\theta') - g(\theta') \right) = \frac{\partial\theta'}{\partial\theta} \frac{\partial g}{\partial\theta'}. \tag{4.218}$$

This is structured NW-SE. Simple examples confirm this rule[10].

Due to the coordinate invariant (NW-SE) structure, a coordinate transformation acts on a contravariant vector $\dot{Z}^M$ from the right:

$$\boxed{\dot{Z}^{M\prime} = \dot{Z}^N \frac{\partial Z^{M\prime}}{\partial Z^N}}. \tag{4.220}$$

$\frac{\partial Z^{M\prime}}{\partial Z^N}$ is defined as the inverse of $\frac{\partial Z^N}{\partial Z^{M\prime}}$ within the NW-SE structure:

$$\frac{\partial Z^{M\prime}}{\partial Z^A} \frac{\partial Z^B}{\partial Z^{M\prime}} = \delta^A_B. \tag{4.221}$$

Note that these agreements need to be fixed beforehand since the structure $U_M V^M$ is not mutually coordinate invariant if $U^M V_M$ has been fixed beforehand due to the appearance of the *relative* sign factors in the summation:

$$U_M V^M = g_{MN} U^N V^M = (-)^{N(1+M)} U^N g_{MN} V^M = (-)^M U^N g_{NM} V^M \neq U^M V_M \tag{4.222}$$

We also have e.g.

$$U_M V^M = (-)^M V^M U_M \neq (-)^M V^{M\prime} U'_M.$$

[10] We convince ourselves with a simple yet general example for two variables. Take $f(\theta_1, \theta_2)$ a bosonic function of two fermionic variables θ'_1, θ'_2:

$$f(\theta'_1, \theta'_2) = \alpha\theta'_1 + \beta\theta'_2 + m\theta'_1\theta'_2$$

, with α, β constant fermionic variables and m a scalar constant. This leads to:

$$\frac{\partial f}{\partial \theta'_1} = -\alpha + m\theta'_2, \qquad \frac{\partial f}{\partial \theta'_2} = -\beta - m\theta'_1.$$

Next we consider a transformation to two new fermionic variables θ_1, θ_2

$$\theta'_1(\theta_1, \theta_2) = a\,\theta_1 + b\theta_2 + \mu\theta_1\theta_2, \qquad \theta'_2(\theta_1, \theta_2) = c\,\theta_1 + d\theta_2 + \nu\theta_1\theta_2.$$

a, b, c, d should be bosonic, while μ, ν should be fermionic to obey the fermionic parity of both θ_1, θ_2 and θ'_1, θ'_2. This leads to:

$$\frac{\partial \theta'_1}{\partial \theta_1} = a - \mu\theta_2, \quad \frac{\partial \theta_1}{\partial \theta_2} = b + \mu\theta_1, \qquad \frac{\partial \theta'_2}{\partial \theta_1} = c - \nu\theta_2, \quad \frac{\partial \theta'_2}{\partial \theta_2} = d + \nu\theta_1$$

, and

$$f(\theta_1, \theta_2) = a\alpha\theta_1 + b\alpha\theta_2 + \alpha\mu\theta_1\theta_2 + \beta c\theta_1 + \beta d\theta_2 + \beta\nu\theta_1\theta_2 + mad\theta_1\theta_2 + mbc\theta_2\theta_1.$$

We can consider the derivative with respect to θ_1:

$$\frac{\partial f}{\partial \theta_1} = -a\alpha + \alpha\mu\theta_2 - \beta c + \beta\nu\theta_2 + mad\theta_2 - mbc\theta_2. \tag{4.219}$$

On the other hand, we can consider this result from the NW-SE chain rule:

$$\begin{aligned} \frac{\partial f}{\partial \theta_1} &= \frac{\partial \theta'_1}{\partial \theta_1}\frac{\partial f}{\partial \theta'_1} + \frac{\partial \theta'_2}{\partial \theta_1}\frac{\partial f}{\partial \theta'_2} = (a - \mu\theta_2)(-\alpha + m\theta'_2) + (c - \nu\theta_2)(-\beta - m\theta'_1) \\ &= -a\alpha + am\,(\cancel{c\theta_1} + d\theta_2 + \cancel{\nu\theta_1\theta_2}) + \mu\theta_2\alpha - \cancel{\mu\theta_2 mc\theta_1} - c\beta - mc\,(\cancel{a\theta_1} + b\theta_2 + \cancel{\mu\theta_1\theta_2}) + \nu\theta_2\beta + \cancel{m\nu\theta_2 a\theta_1}. \end{aligned}$$

Taking into account the appropriate Grassmann parities, this result indeed coincides with Eq 4.219. The same conclusion holds for the derivative with respect to θ_2.

In particular, once these conventions are fixed, we cannot define a covector from $\dot{Z}_N \neq \dot{Z}^M g_{MN}$ despite earlier intuition.
On the other hand, interchanging U^M and V^M within the NW-SE structure in the inner product $U^M V_M$, is manifestly invariant due to the antisymmetric convention of Eq 4.211;

$$\begin{aligned} U^M V_M &= U^M g_{MN} V^N = V^N (-)^N U^M g_{MN} = V^N (-)^{N+M+MN} g_{MN} U^M \\ &= V^N g_{NM} U^M = V^N U_N. \end{aligned}$$

4.6.2 Geodesics in superspace

EOW branes are ultimately defined in terms of their geodesic trajectory. In the bosonic action Eq 3.1, the dilaton field acts as a Lagrange multiplier enforcing the constraint $K \equiv 0$, which we have argued to be equivalent to an extremal geodesic solution in section 3.3.2. We should therefore add a similar term to the free particle action that fixes the free particle action to describe geodesic trajectories in superspace. Somewhat surprisingly, the mathematical and physical literature on geodesics in superspace is relatively scarce. For example, the definition of the extrinsic curvature in superspace is not immediately clear. We could opt to choose the same K appearing in the boundary action Eq 4.204. However, this is defined in terms of a superderivative acting on a $1|1$-dimensional sheet, whereas we consider a proper $1|0$-dimensional curve. Compared to the former, the analysis is facilitated due to the single bosonic affine parameter, whose proper derivative conveniently acts as a bosonic "time" derivative along the curve. We opt to be pragmatic and define the extrinsic curvature a posteriori, with the soul purpose of localizing onto geodesics.
A natural choice would be to simply generalize the definition of the bosonic extrinsic curvature directly to superspace Eq 3.43:

$$K = U^\mu U^\alpha \nabla_\alpha n_\mu \quad \rightarrow \quad K = U^M U^N \nabla_N n_M, \qquad \left(U^M \equiv \dot{Z}^M = \frac{dZ^M}{ds} \right) \tag{4.223}$$

, not the least because the contractions appear in the natural NW-SE structure. We define the normal vector in superspace $n_M(s)$ as the normal vector field along the entire tangent vector field $U^M(s) = \dot{Z}^M$ at every point described by s, by analogy to the bosonic case:

$$U^M(s)\, n_M(s) \equiv 0. \tag{4.224}$$

However, it is not immediately clear if this definition is consistent with the geodesic equations of motion in superspace, and which choice of the covariant derivative in superspace naturally preserves coordinate-invariant structures. To resolve these issues, we develop the relevant different geometry in superspace by hand, and generalize the familiar development of geodesics in general relativity to (super)geodesics in superspace.
We start by writing the free particle action in superspace symbolically as

$$I = \mu \int ds \sqrt{\dot{Z}^A g_{AB} \dot{Z}^B} = \mu \int \sqrt{dZ^A g_{AB} dZ^B} \tag{4.225}$$

, by formally absorbing the measure of the affine parameter ds inside the square root. This is only possible by virtue of the bosonic nature of s. Consequently varying the action should yield the classical equations of motion in superspace;

$$\begin{aligned}\delta I &= \mu \int \delta\left(\sqrt{dZ^A g_{AB} dZ^B}\right) \\ &= \mu \int \frac{1}{2\sqrt{dZ^A g_{AB} dZ^B}} \left(\delta dZ^A\, g_{AB} dZ^B + dZ^A\, \delta g_{AB}\, dZ^B + dZ^A\, g_{AB}\, \delta dZ^B\right) \\ &= \frac{\mu}{2} \int ds\, \left((\delta \dot{Z}^A)\, g_{AB} \dot{Z}^B + \dot{Z}^A\, \delta g_{AB}\, \dot{Z}^B + \dot{Z}^A g_{AB}\, (\delta \dot{Z}^B)\right).\end{aligned}$$

We have used the standard chain rule in the bosonic line element $ds = \sqrt{dZ^A g_{AB} dZ^B}$. Furthermore, according to standard practice in supergravity, the variation δ obeys the bosonic product rule in the convention that it acts from the left[11]. In the third line, we have reintroduced the affine line element ds due to its appearance in the denominator, and we have used the commutativity between δ and $\frac{d}{ds}$.
Due to our choice Eq 4.211, the first and last term are in fact equal:

$$\dot{Z}^A g_{AB}\, (\delta \dot{Z}^B) = (-)^{B+A+AB} (\delta \dot{Z}^B)\, g_{AB} \dot{Z}^A = (\delta \dot{Z}^B)\, g_{BA} \dot{Z}^A$$

Furthermore, using the natural chain rule within the NW-SE convention, we may write $\delta g_{AB} = \delta Z^C \partial_C g_{AB}$;

$$\begin{aligned}\delta I &= \mu \int ds\, \left((\delta \dot{Z}^A)\, g_{AB} \dot{Z}^B + \frac{1}{2} \dot{Z}^A\, \delta Z^C\, \partial_C g_{AB} \dot{Z}^B\right) \\ &= \mu \int ds\, \left((\delta \dot{Z}^A)\, g_{AB} \dot{Z}^B + (-)^{A(1+B)} \frac{1}{2} \delta Z^C\, \partial_C g_{AB} \dot{Z}^A \dot{Z}^B\right) \\ &\simeq -\mu \int ds\, \left(\delta Z^A\, g_{AB} \ddot{Z}^B + \delta Z^A\, \dot{Z}^C \partial_C g_{AB} \dot{Z}^B - (-)^{A(1+B)} \frac{1}{2} \delta Z^C\, \partial_C g_{AB} \dot{Z}^A \dot{Z}^B\right)\end{aligned}$$

, where we have partially integrated in the last line in the bosonic derivative with respect to s. Separating out $\delta Z^A g_{AB}$, and using our definition of the inverse metric $g_{AB} g^{BC} = \delta_A^C$;

$$\begin{aligned}\delta I &= -\mu \int ds\, \delta Z^A g_{AB} \left(\ddot{Z}^B + g^{BC} \dot{Z}^M \partial_M g_{CN} \dot{Z}^N - \frac{1}{2} (-)^{M(1+N)} g^{BC} \partial_C g_{MN} \dot{Z}^M \dot{Z}^N\right) \\ &= -\mu \int ds\, \delta Z^A g_{AB} \left(\ddot{Z}^B + \frac{1}{2} g^{BC} \left(2(-)^{M(1+C)} \partial_M g_{CN} - (-)^M \partial_C g_{MN}\right) \dot{Z}^N \dot{Z}^M\right).\end{aligned}$$

Symmetrizing the first term within brackets;

$$\begin{aligned}2(-)^{M(1+C)} \partial_M g_{CN} \dot{Z}^N \dot{Z}^M &= (-)^{M(1+C)} \partial_M g_{CN} \dot{Z}^N \dot{Z}^M + (-)^{N(1+C)} \partial_N g_{CM} \dot{Z}^M \dot{Z}^N \\ &= \left((-)^{M(1+C)} \partial_M g_{CN} + (-)^{N(1+C+M)} \partial_N g_{CM}\right) \dot{Z}^N \dot{Z}^M\end{aligned}$$

[11] We imagine that we transmute every term in a product to the left first, and consequently apply a variation δ to the first term. By transmuting the variational term back to its original position in the product reabsorbs all minus encountered in the original transmutation. E.g. consider the variation of the metric:

$$\delta_g\left(dZ^A g_{AB} dZ^B\right) = (-)^{A(1+B)} \delta_g\left(g_{AB} dZ^A dZ^B\right) = (-)^{A(1+B)}\left(\delta g_{AB} dZ^A dZ^B\right) = dZ^A \delta g_{AB} dZ^B.$$

, eventually yields:

$$\delta I = -\mu \int ds\, \delta Z^A g_{AB} \Big(\ddot{Z}^B + \frac{1}{2} g^{BC} \Big((-)^{M(1+C)} \partial_M g_{CN} + (-)^{N(1+C+M)} \partial_N g_{CM} - (-)^M \partial_C g_{MN} \Big) \dot{Z}^N \dot{Z}^M \Big)$$
$$\equiv -\mu \int ds\, \delta Z^A g_{AB} \left(\ddot{Z}^B + \Gamma^B_{MN} \dot{Z}^N \dot{Z}^M \right). \tag{4.226}$$

In the last line, we introduced an appropriate definition of the Christoffel symbol in superspace:

$$\boxed{\Gamma^A_{MN} \equiv \frac{1}{2} g^{AC} \left((-)^{M(1+C)} \partial_M g_{CN} + (-)^{N(1+C+M)} \partial_N g_{CM} - (-)^M \partial_C g_{MN} \right)}. \tag{4.227}$$

This definition of the Christoffel symbol matches with the one introduced in a different context in [46]. As a consistency check, it reduces to the familiar definition in bosonic GR by taking all $\mathbb{Z}_2$-sign labels to zero. We can write the last line in a more suggestive coordinate invariant NW-SE ordered way;

$$\delta I = -\mu \int ds \left(\ddot{Z}^B + \Gamma^B_{MN} \dot{Z}^N \dot{Z}^M \right) (-)^B \delta Z^A g_{AB} = \mu \int ds \left(\ddot{Z}^B + \Gamma^B_{MN} \dot{Z}^N \dot{Z}^M \right) (-)^{B+A+AB} g_{AB}\, \delta Z^A$$
$$= -\mu \int ds \left(\ddot{Z}^A + \Gamma^A_{MN} \dot{Z}^N \dot{Z}^M \right) g_{AB}\, \delta Z^B$$
$$= -\mu \int ds \left(\ddot{Z}^A + \Gamma^A_{MN} \dot{Z}^N \dot{Z}^M \right) \delta Z_A \tag{4.228}$$

, by our definition of the metric tensor Eq 4.211, and the covector Eq 4.212.
To proceed, we rewrite the geodesic equation more compactly by introducing a covariant derivative in superspace in terms of the superspace Christoffel symbol. Acting on a contravariant vector U^B, we propose

$$\boxed{\nabla_A U^B \equiv \partial_A U^B + (-)^{A(B+1)}\, \Gamma^B_{AC} U^C} \tag{4.229}$$

, and check that this is indeed consistent:

$$U^A \nabla_A U^B = \dot{U}^B + \Gamma^B_{AC} U^C U^A. \tag{4.230}$$

In the first term, we have used the natural NW-SE chain rule for the (possibly) Grassmann derivative[12]: $U^A \partial_A U^B = \dot{Z}^A \partial_A U^B = \frac{d}{ds} U^B = \dot{U}^B$. Since this involves a chain rule with respect to a bosonic param-

[12] We prove that this is the correct ordening for Grassmann derivatives by a proof of induction. Take a function of two Grassmann variables $\theta_1(s)$, $\theta_2(s)$ that describe a trajectory in superspace. The most general fermionic function can be expanded in terms of four bosonic numbers:

$$F(\theta_1(s), \theta_2(s))(s) = a + b\theta_1(s) + c\theta_2(s) + d\, \theta_1(s)\theta_2(s).$$

Since the chain rule obviously works on functions of one Grassmann variable: $\frac{d}{ds}(\theta_1(s)) = \dot{\theta}_1(s) \equiv \dot{\theta}_1(s) \frac{\partial \theta_1}{\partial \theta_1}$, we refrain our attention to the function involving the product of Grassmann numbers $G_2(s) = \theta_1(s)\theta_2(s)$. Since $\frac{\partial G_2}{\partial \theta_1} = \theta_2$ and $\frac{\partial G_2}{\partial \theta_2} = -\theta_1$, this is again consistent with the chain rule acting on the left:

$$\frac{d}{ds} G_2 = \frac{d}{ds} (\theta_1(s)\theta_2(s)) = \dot{\theta}_1 \theta_2 + \theta_1 \dot{\theta}_2 = \dot{\theta}_1 \frac{\partial G_2}{\partial \theta_1} - \frac{\partial G_2}{\partial \theta_2} \dot{\theta}_2$$
$$= \dot{\theta}_1 \frac{\partial G_2}{\partial \theta_1} + \dot{\theta}_2 \frac{\partial G_2}{\partial \theta_2}.$$

eter, this requires a separate proof, which we demonstrate in footnote 12 by the rule of induction (c.f. Eq 4.231). We may therefore write the variation of the action more suggestively as:

$$\delta I = -\mu \int ds \ \left(U^B \nabla_B U^A\right) \delta Z_A \tag{4.232}$$

, where all contractions appear in a manifestly coordinate-invariant NW-SE ordening. Since the free particle action that we started from is manifestly coordinate invariant, its variation should obey this same property. This unambiguously fixes the transformation of $\nabla_A U^B$ under general coordinate transformations in order to preserve this structure,

$$\nabla_B U^A \quad \rightarrow \quad \nabla'_B U^{A\prime} = \frac{\partial Z^C}{\partial Z'^B} \left(\nabla_C U^D\right) \frac{\partial Z'^A}{\partial Z^D}. \tag{4.233}$$

In this sense, our definition of the covariant derivative is consistent. Moreover, it reduces to the standard transformation rule in bosonic GR when all fermions are turned off. Note that this trick allows us to deduce the transformation rule under general coordinate transformations without actually calculating the explicit transformations of the Christoffel symbols.

4.6.3 Extrinsic curvature in superspace

We generalize the construction in section 3.3.2 and introduce an appropriate definition of the extrinsic curvature in superspace. In particular, since any variation can be decomposed into its tangential and normal direction, we may characterize the general variation entirely along the normal vector field $\delta Z_A = n_A$;

$$\delta I = -\mu \int ds \ \left(U^B \nabla_B U^A\right) n_A. \tag{4.234}$$

We introduce a covariant derivative acting on covectors by demanding that the covariant derivative acting on a scalar structure reduces to the standard (possibly Grassmann) derivative: $\nabla_A X \equiv \partial_A X$. Acting on the scalar product $U^A n_A$, it should obey the standard product rule:

$$\nabla_B \left(U^A n_A\right) \equiv \partial_B \left(U^A n_A\right) = \left(\partial_B U^A\right) n_A + (-)^{AB} U^A \left(\partial_B n_A\right). \tag{4.235}$$

Since the chain rule seems to work on functions of one and two Grassmann variables, we can prove that it also works on functions with an arbitrary number of Grassmann variables by the rule of **induction**. Take $G_{n+1}(s)$ a function of $n+1$ fermionic variables that describe a curve in superspace: $G_{n+1}(s) = \theta_1(s)\dots\theta_n(s)\theta_{n+1}(s) = G_n(s)\theta_{n+1}(s)$. If the chain rule works on functions of n fermionic variables G_n, it must also work on the function of $n+1$ fermionic variables:

$$\frac{d}{ds} G_{n+1} = \frac{d}{ds}\left(\theta_1(s)\dots\theta_n(s)\theta_{n+1}(s)\right) = \frac{d}{ds}\left(\theta_1\dots\theta_n\right)\theta_{n+1} + \left(\theta_1\dots\theta_n\right)\dot{\theta}_{n+1}.$$

The first term already has the correct structure by the premise that functions of n fermionic variables $G_n(s)$ satisfy the natural chain rule. To check the second term, we need to take into account the Grassmann parity of G_n: $\frac{d}{d\theta_{n+1}}\left(G_n\theta_{n+1}\right) = (-)^n G_n$, to obtain

$$\begin{aligned}\frac{d}{ds} G_{n+1} &= \dot{\theta}_1 \frac{dG_{n+1}}{d\theta_1} + \dots + \dot{\theta}_n \frac{dG_{n+1}}{d\theta_n} + (-)^n \frac{dG_{n+1}}{d\theta_{n+1}}\dot{\theta}_{n+1} \\ &= \dot{\theta}_1 \frac{dG_{n+1}}{d\theta_1} + \dots + \dot{\theta}_n \frac{dG_{n+1}}{d\theta_n} + \dot{\theta}_{n+1} \frac{dG_{n+1}}{d\theta_{n+1}}.\end{aligned} \tag{4.231}$$

In the last line, we have used that the Grassmann parity of G_n coincides with the Grassmann parity of $\frac{dG_{n+1}}{d\theta_{n+1}}$.

On the other hand, we *define* the covariant derivative on covectors such that it obeys a similar product rule;

$$\nabla_B \left(U^A n_A\right) \equiv \left(\nabla_B U^A\right) n_A + (-)^{AB} U^A \left(\nabla_B n_A\right) \tag{4.236}$$
$$= \left(\partial_B U^A + (-)^{B(A+1)}\, \Gamma^A_{BC} U^C\right) n_A + (-)^{AB} U^A \left(\nabla_B n_A\right). \tag{4.237}$$

Compared to the previous line Eq 4.235, this fixes the covariant derivative on a general covector n_A:

$$\boxed{\nabla_B n_A \equiv \partial_B n_A - \ (-)^{B(1+C)+A(1+C)}\ \Gamma^C_{BA} n_C} \tag{4.238}$$

in terms of the Christoffel symbol Eq 4.227. Since the LHS in Eq 4.237 and the first term on the RHS are manifestly covariant under the NW-SE convention, the transformation rule of the covariant derivative acting on covectors under general coordinate transformations is a posteriori fixed to:

$$\nabla_B n_A \quad \to \quad \nabla'_B n'_A \ = \ \frac{\partial Z^D}{\partial Z'^B}\, \nabla_D \left(\frac{\partial Z^E}{\partial Z'^A} n_E\right) \ \equiv \ (-)^{D(E+A)} \frac{\partial Z^D}{\partial Z'^B}\, \frac{\partial Z^E}{\partial Z'^A}\, \nabla_D n_E \tag{4.239}$$

This transformation is fine-tuned to keep the second term invariant in Eq 4.236;

$$\begin{aligned}(-)^{AB} U^A \left(\nabla_B n_A\right) \ &\to (-)^{AB+D(E+A)} U^G\, \frac{\partial Z'^A}{\partial Z^G}\, \frac{\partial Z^D}{\partial Z'^B}\, \frac{\partial Z^E}{\partial Z'^A}\, \nabla_D n_E \\ &= (-)^{EB} U^G\, \frac{\partial Z'^A}{\partial Z^G} \frac{\partial Z^E}{\partial Z'^A} \frac{\partial Z^D}{\partial Z'^B}\, \nabla_D n_E = (-)^{EB} U^E \frac{\partial Z^D}{\partial Z'^B} \nabla_D n_E \\ &= (-)^{ED} \frac{\partial Z^D}{\partial Z'^B} U^E \nabla_D n_E.\end{aligned} \tag{4.240}$$

As a consistency check, the definition of the covariant derivative acting on covectors again reduces to the bosonic case for soley bosonic sign factors.

Since the normal vector field is defined to be orthogonal to the tangent vector at all times $U^A n_A \equiv 0$, we may take the covariant derivative on both sides of this equation to characterize the variation of the tangent vector equivalently in terms of the variation of the normal vector

$$\begin{aligned}\nabla_M \ \left(U^A n_A\right) &= \nabla_M U^A\ n_A + (-)^{AM} U^A\ \left(\nabla_M n_A\right) \equiv 0 \\ \leftrightarrow \qquad \nabla_M U^A\ n_A &= -(-)^{AM} U^A\ \nabla_M n_A.\end{aligned} \tag{4.241}$$

Inserted in the action Eq 4.234 yields

$$\delta I = -\mu \int ds\ \left(U^B \nabla_B U^A\right) n_A \ = \ \mu \int ds\ (-)^{AB} U^B U^A \nabla_B n_A \tag{4.242}$$

, where we acknowledge that the last structure is coordinate invariant due to the transformation rule Eq 4.240. We may absorb the relative minus signs by swapping U^B and U^A, and obtain:

$$\delta I = \mu \int ds\ U^A U^B \nabla_B n_A \equiv \mu \int ds\ K. \tag{4.243}$$

We have defined the extrinsic curvature along the $1|0$-dimensional curve as:

$$\boxed{K = U^A U^B \nabla_B n_A} \tag{4.244}$$

This characterizes completely the variation of the free particle action in superspace, and reduces naturally to the definition of the pulled-back extrinsic curvature in bosonic GR. It furthermore admits the natural NW-SE structure, and is thereby a proper coordinate scalar. This is indeed the anticipated result Eq 4.223, where we have now also unambiguously defined the covariant derivative acting on n_A in Eq 4.238 and its transformation rule. In any case, by imposing

$$K \equiv 0 \tag{4.245}$$

along the curve, the variation of the free particle vanishes $\delta I = 0$, and the curve should follow its classical geodesic trajectory in superspace.

4.6.4 Ends of the World in superspace

We may finally write down the total Euclidean action of $\mathcal{N} = 1$ JT supergravity in the presence of an EOW brane:

$$\boxed{I_{JT}^{\mathcal{N}=1} = \frac{1}{4}\left[\int_\Sigma d^2z d^2\theta \; E\Phi(R_{+-}+2) + 2\int_{\partial\Sigma} d\tau d\vartheta \; \Phi K\right] + \int_{EOW} ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N} \; (\mu - \phi K)} \tag{4.246}$$

, where ϕ is defined as the bottom component of the dilaton superfield Eq 4.28. This is the natural generalization of the bosonic action Eq 3.1 written in [36]. We stress again that, although both denoted by K, the extrinsic curvature along the AdS-boundary is different to the extrinsic curvature along the EOW brane. The former is defined along the $1|1$-dimensional boundary curve Eq 4.205 in [39], while we have defined the latter along the $1|0$-dimensional brane in Eq 4.244.

Also note that we did not need to specify the number of superspace coordinates to define both the extrinsic curvature and the free particle action. We may thus suspect that this appearance naturally generalizes to theories of JT supergravity involving any number of supercurrents. However, we will see in section 5.1.1 that this argument still needs some fine-tuning.

4.7 Wilson lines as probe particles in superspace

To calculate the full-fledged free-particle path integral, we need an analogous identification between Wilson lines in the OSp$(1|2,\mathbb{R})$ BF theory, and the free particle path integral (c.f. 3.4). A first attempt was already established in [40], using the same reasoning of [23] for the bosonic case. Let me work it out more in detail within the current conventions. The derivation is nearly identical to section 3.3.1, with only minor modifications for the additional spin structure.

We start by introducing a gauge field, and expand it into $\mathfrak{osp}(1|2,\mathbb{R})$-generators in terms of the first order superframe fields and super spin connection,

$$\mathbf{A}_M = E_M{}^A J_A + \Omega_M P_2, \qquad A = 0,1,+,- \tag{4.247}$$

, where now letters at the beginning of the alphabet denote Lorentz frame indices, while letters in the middle of the alphabet denote Einstein superspace indices. The $\mathfrak{osp}(1|2,\mathbb{R})$-generators J_A are defined in section 4.3.1, where $J_A = P_A$, $(A = 0,1,2)$ denote the bosonic generators Eq 4.75, while $J_\alpha = Q_\alpha$ denote the fermionic generators Eq 4.78. We summarize their normalization in terms of the Cartan-Killing metric κ_{IJ} $(I,J = 0,1,2,+,-)$:

$$\mathrm{STr}(J_A J_B) = \frac{\kappa_{AB}}{2}, \qquad \kappa_{AB} = \eta_{AB}, \quad (A,B = 0,1,2), \qquad \kappa_{\alpha\beta} = \epsilon_{\alpha\beta}, \quad (\alpha,\beta = +,-,\ \epsilon_{+-} = -1) \tag{4.248}$$

, with $\eta_{AB} = \mathrm{diag}(1,1,-1)$. We note that the restriction of the Cartan-Killing metric to the directions $A = 0,1,+,-$ coincides with the local Lorentz metric Eq 4.4. Adopting the conventions of [40] for consistency with the developments from earlier in this chapter, spinor-indices are contracted in the SW-NE (south-west - north-east) convention: $\bar{\psi}_\alpha \chi^\alpha \equiv \psi^\beta \epsilon_{\beta\alpha}\chi^\alpha$, with the Majorana conjugate spinor defined as $\bar{\psi}_\alpha = \psi^\beta \epsilon_{\beta\alpha}$. Note that due to the Majorana flip symmetry in 2d (Eq 4.19), one encounters no relative minus signs for spinor interchanges in this ordening: $\overline{\psi}\chi = \overline{\chi}\psi$. The covector for both fermionic and bosonic Lorentz frame coordinates is therefore defined as (compare to Eq 4.212 for the definition of the covector in Einstein indices, where the metric acts on the left):

$$U_A = U^B \kappa_{BA}. \tag{4.249}$$

This fixes the raising of Lorentz covectors by acting with the inverse Cartan-Killing metric from the right:

$$U^B = U_A \kappa^{AB}. \tag{4.250}$$

The Cartan-Killing metric contains no Grassmann entries for both the fermionic- or bosonic blocks, and we may freely commute the Cartan-Killing metric through the spinor entries. The antisymmetry properties of the Cartan-Killing metric are summarized as:

$$\kappa_{AB} = (-)^{AB}\kappa_{BA} = (-)^A \kappa_{BA} = (-)^B \kappa_{BA} \tag{4.251}$$

, where the last two identities hold since the Cartan-Killing metric is block diagonal. The latter ensures that both Lorentz indices always share the same Grassmann parity. This antisymmetry property should be contrasted to the second-order metric that may itself contain Grassmann entries Eq 4.211. The Majorana flip symmetry can be combined with the bosonic symmetry to:

$$V_A W^A = V^B \kappa_{BA} W^A = (-)^{AB} W^A \kappa_{BA} V^B = W^A \kappa_{AB} V^B = W_B V^B. \tag{4.252}$$

In contrast, we have Eq 4.6:

$$V_A W^A = (-)^A W^A V_A \neq W^A V_A. \tag{4.253}$$

, and Eq 4.17:

$$V_A W^A = V^B \kappa_{BA} W^A = (-)^B V^B W^A \kappa_{AB} = (-)^B V^B W_B \neq V^A W_A. \tag{4.254}$$

On the other hand, we have demonstrated in Eqs 4.13, 4.14 that both structures $V_A W^A$ and $V^A W_A$ are Lorentz invariant given the specific form of the Lorentz transformation rule in 2d (see Eq 4.8), which is specified in terms of a single bosonic number L. This is not the case for general coordinate contractions $\dot{Z}^M \dot{W}_M$ since the form of the coordinate transformations Eqs 4.220 and 4.214 might contain fermionic entries.

Using similar reasoning to section 3.3.1, we write a Wilson loop in the discrete series representation labeled by j as a path integral over the first order action $S_\alpha[g, \mathbf{A}]$ (c.f. Eq 3.7);

$$\mathcal{W}_j(\mathcal{C}) = \int \mathcal{D}\alpha g \; e^{-S_\alpha[g,\mathbf{A}]}. \tag{4.255}$$

The first order action is minimally coupled to a gauge field in superspace $\mathbf{A}_M = A_M^A J_A$ (c.f. Eq 3.8):

$$S_\alpha[g, \mathbf{A}] = -\int_{\mathcal{C}} ds \; \mathrm{STr}\left(\alpha g^{-1} D_A g\right) \tag{4.256}$$

with the covariant derivative defined symbolically as:

$$D_A = \partial_s + \mathbf{A}_s. \tag{4.257}$$

The gauge field along the curve is defined as $\mathbf{A}_s = \dot{Z}^M \mathbf{A}_M(Z(s))$. This ensures that the action is invariant under left multiplication by elements of $\mathrm{OSp}(1|2,\mathbb{R})$. α is the highest-weight vector of the spin-j representation in the $\mathfrak{osp}(1|2,\mathbb{R})$-algebra[13]:

$$\alpha = \alpha^a P_a + \Xi^\alpha Q_\alpha, \qquad a = 0, 1. \tag{4.258}$$

Note that the P_2-component is omitted in the summation, since the P_2-component is not independent from the $P_{0,1}$-components due to the torsion constraints [40].
$g(s)$ are maps $\mathcal{C} \to \mathrm{OSp}(1|2,\mathbb{R})$ that parameterize the orbit of α in the algebra. To fix the representation on the left- and right-hand side of the identity Eq 4.255, the length of the highest-weight vector α is constrained by the eigenvalue of the quadratic Casimir upon combining the constraints in Eqs 3.11 and 3.20:

$$\begin{aligned}\frac{1}{2}\mathrm{STr}\left(\alpha^2\right) &= \frac{1}{4}\alpha^I \kappa_{IJ}\alpha^J = \frac{1}{4}\left(\alpha^a \eta_{ab}\alpha^b + \Xi^\alpha \epsilon_{\alpha\beta}\Xi^\beta\right) = \frac{1}{4}\left(\alpha_a\alpha^a + \overline{\Xi}\Xi\right) \\ &\equiv -\mathcal{C}_2(j) = j(j+1/2) \equiv m^2\end{aligned} \tag{4.259}$$

, where the eigenvalue of the quadratic Casimir is defined in Eq B.13: $\mathcal{C}_2(j) = -j(j+1/2)$.

We argue for this identification by parameterizing the left action of $g(s) : \mathcal{C} \to \mathrm{OSp}(1|2,\mathbb{R})$ along the curve $\mathcal{C}$

[13] Remember that the $\mathfrak{osp}(1|2,\mathbb{R})$ generators, defined in section 4.3.1 contain only bosonic entries.

in local coordinates:

$$g(s) \;\rightarrow\; e^{-Z^A(s)P_A}g(s_0) \equiv U(s)g(s_0). \tag{4.260}$$

Since we consider the STr operation, supermatrices with fermionic entries can be cyclically permuted[14], thus $-\mathrm{STr}(\alpha g^{-1}D_A g) = -\mathrm{STr}(D_A g\ \alpha g^{-1})$. Writing this Lagrangian as $L = \pi_A \dot{Z}^A - H$, the derivative term contains the conjugate momentum[15]:

$$\pi_A = (-)^A \frac{\partial L}{\partial \dot{Z}^A} = (-)^A \mathrm{STr}(P_A g\alpha g^{-1}). \tag{4.261}$$

Expanding $g\alpha g^{-1} = (g\alpha g^{-1})^B J_B$;

$$\pi_A = (-)^A(g\alpha g^{-1})^B \mathrm{STr}(J_A J_B) = (-)^A(g\alpha g^{-1})^B\frac{\kappa_{AB}}{2} = \frac{(g\alpha g^{-1})^B\kappa_{BA}}{2} = \frac{(g\alpha g^{-1})_A}{2}. \tag{4.262}$$

In the quantum theory, the conjugate momenta become the generators of the left action of g along the curve $g(s) \rightarrow U(s)g(s_0)$. The additional sign factors ensure that the Lagrangian can indeed be written as:

$$\begin{aligned} L &= \pi_A \dot{Z}^A - H = (-)^A(g\alpha g^{-1})^B\frac{\kappa_{AB}}{2}\dot{Z}^A - H = \dot{Z}^A\mathrm{STr}(P_A P_B)(g\alpha g^{-1})^B - H \\ &= \mathrm{STr}(\dot{Z}^A P_A\ g\alpha g^{-1}) - H \\ &= -\mathrm{STr}(\partial_s g\ \alpha g^{-1}) - H \end{aligned} \tag{4.263}$$

, from which we identify the Hamiltonian upon comparison with the total Lagrangian Eq 4.256:

$$\begin{aligned} H &= \mathrm{STr}(\mathbf{A}g\alpha g^{-1}) = A^A(g\alpha g^{-1})^B\mathrm{STr}(J_A J_B) = (-)^A(g\alpha g^{-1})^B\frac{\kappa_{AB}}{2}A^A \\ &= (g\alpha g^{-1})^B\frac{\kappa_{BA}}{2}A^A = \pi_A A^A = \mathbf{A}. \end{aligned} \tag{4.264}$$

In the quantum theory, the conjugate momenta become operators that statisfy the (Euclidean) commutation relations

$$[\hat{\pi}_B(s), Z^A(t)]_\pm = -(-)^{AB}\delta^B_A\delta(s-t) \tag{4.265}$$

, where we take an anti-commutator for fermionic indices and commutator for bosonic indices. These relations are realized as $\hat{\pi}_A(s) = -(-)^{AB}\frac{\delta}{\delta Z^A(s)}$. These act on the Hilbert space of functions that are invariant under the global right action that stabilizes the orbit of α, and are left-parameterized by $g(s) = e^{-Z^A(s)P_A}g(s_0)$. The Hamiltonian is thus diagonalized on the elements of $g(s)$ by the expansion of $\mathbf{A}$ in terms of the matrix generators P_A;

$$H\ g = A^A_v P_A\ g. \tag{4.266}$$

The identification of the path integral over closed loops $g(s)$ with the trace of a Hamiltonian evolution operator Eq D.26

$$\mathrm{Tr}_j(e^{-\beta H}) \simeq \oint \mathcal{D}\phi\ e^{-I[\phi]} \tag{4.267}$$

settles the identification between the first order path integral over the configurational fields $g(s)$ and the Wilson

[14] The generators of the supermatrices J_A with only bosonic entries cannot be cyclically permuted in the STr!

[15] The extra sign factor is present from the SW-NE convention in the first order Lagrangian: $\pi_A = (-)^A\frac{\partial}{\partial \dot{Z}^A}(\pi_A\dot{Z}^A - H) = (-)^A\frac{\partial L}{\partial \dot{Z}^A}$.

loop Eq 4.255.

To identify the quadratic Casimir, we start from the definition of the *opposite* $\mathfrak{osp}(1|2,\mathbb{R})$ algebra Eq B.34. This is because, just like the Borel-Weil generators, the fermionic generators π_A along the path $\mathcal{C}$ are themselves fermionic (as opposed to the fermionic generators $Q_\pm$ which contain solely bosonic entries). The definition of the quadratic Casimir follows from the inverse Cartan-Killing metric Eq B.13 with the additional flip in the sign of the fermionic block generators:

$$\mathcal{C}_2 = -(-)^A \pi_A \kappa^{AB} \pi_B = -\pi_A \pi_B \kappa^{BA}. \tag{4.268}$$

Inserting the explicit form of the generators π_A Eq 4.262, and raising and lowering the Lorentz index in the first and second factor respectively according to the rules Eqs 4.250, 4.249, we obtain:

$$\mathcal{C}_2 = -\frac{(g\alpha g^{-1})_A (g\alpha g^{-1})_B \kappa^{BA}}{4} = -\frac{(g\alpha g^{-1})^B \kappa_{BA} (g\alpha g^{-1})^A}{4}. \tag{4.269}$$

Using the definition of the Cartan-Killing metric according to the normalization of the J_A-generators $\mathrm{STr}(J_A J_B) = \kappa_{AB}/2$:

$$\begin{aligned}\mathcal{C}_2 &= -\frac{1}{2}(g\alpha g^{-1})^B \mathrm{STr}(J_B J_A)(g\alpha g^{-1})^A = -\frac{1}{2}\mathrm{STr}(\alpha^2)\\ &= -\frac{1}{4}\alpha^A \kappa_{AB} \alpha^B.\end{aligned} \tag{4.270}$$

This proves the identification between the Casimir eigenvalue and the length of the adjoint vector α Eq 4.259. By fixing the length of this vector, we fix the representation unambiguously to:

$$\frac{1}{4}\alpha^I \kappa_{IJ} \alpha^J = -\mathcal{C}_2(j) = j(j+1/2) = m^2. \tag{4.271}$$

For a lowest-weight module with $\ell = -j$, the relation between mass and conformal weight is equivalent to:

$$\ell(\ell - 1/2) = m^2. \tag{4.272}$$

Since the adjoint action of $\mathrm{OSp}(1|2,\mathbb{R})$ is transitive on all Lie algebra-valued vectors α with the same length constraint, we introduce a functional integral over the latter

$$\int \mathcal{D}\alpha \mathcal{D}\Theta \mathcal{D}_\alpha g \; e^{-S_\alpha[g,\mathbf{A}]} \tag{4.273}$$

, where Θ is a scalar bosonic Lagrange multiplier enforcing the constraint $(\alpha_a \alpha^a + \bar{\Xi}\Xi) \equiv 4m^2$ (c.f. Eq 3.23):

$$S_\alpha[\alpha, g, \mathbf{A}, \Theta] = \int_{\mathcal{C}} ds \left[-\mathrm{STr}(\alpha g^{-1} D_A g) + \frac{i}{4}\Theta(\alpha_a \alpha^a + \bar{\Xi}\Xi - 4m^2) \right]. \tag{4.274}$$

Note that we again integrate over adjoint elements α^A that live in a sub vector space of the algebra, by excluding the P_2-spin component in the expansion Eq 4.258. This does not pose a problem as long as the total length of the vector is still constrained by Eq 4.259.

We fix the gauge redundancy in g, which amounts to right multiplying g by the stabilizer of α in $\mathrm{OSp}(1|2,\mathbb{R})$, by setting $g \equiv 1$ along the entire curve $\mathcal{C}$ and smoothly extending this gauge into the bulk. This expresses the Lagrangian in terms of the expansion coefficients of the gauge field (Eq 4.247) with:

$$g^{-1}D_A g = \mathbf{A}_s = \dot{Z}^M (E_M{}^A J_A). \tag{4.275}$$

The total action thus becomes:

$$S_\alpha[\alpha, g, \mathbf{A}] = \frac{1}{2}\int_{\mathcal{C}} ds \left[-\alpha_A \dot{Z}^M E_M{}^A + \frac{i}{2}\Theta(\alpha_a\alpha^a + \overline{\Xi}\Xi - 4m^2)\right]. \tag{4.276}$$

As a consequence of the general Majorana flip symmetry in 2d (Eq 4.252), the equations of motion with respect to α^A are convenient to deduce from the variations that act on the left:

$$\begin{aligned} -\delta\alpha_A \dot{Z}^M E_M{}^A + \frac{i}{2}\Theta\delta\alpha_A\alpha^A + \frac{i}{2}\Theta\alpha_A\delta\alpha^A = \; & \delta\alpha_A\left(-\dot{Z}^M E_M{}^A + i\Theta\alpha^A\right) \equiv 0 \\ \leftrightarrow \quad & \alpha^A = -\frac{i}{\Theta}\dot{Z}^M E_M{}^A. \end{aligned}$$

Inserted in the total action yields:

$$S_\alpha[\alpha, g, \mathbf{A}] = i\int_{\mathcal{C}} ds \; \left[\frac{1}{4\Theta}\dot{Z}^N E_N{}^A \kappa_{AB} \dot{Z}^M E_M{}^B - \Theta m^2\right]. \tag{4.277}$$

Consequently integrating out the Lagrange multiplier Θ, and choosing the upper branch solution of Θ

$$\Theta = \frac{i}{2m}\sqrt{\dot{Z}^N E_N{}^A \kappa_{AB} \dot{Z}^M E_M{}^B} \tag{4.278}$$

gives (c.f. Eq 3.25):

$$S_\alpha[\alpha, g, \mathbf{A}] = m\int_{\mathcal{C}} ds \sqrt{\dot{Z}^N E_N{}^A \kappa_{AB} \dot{Z}^M E_M{}^B}. \tag{4.279}$$

While this is exact off-shell, we should again modify the integration measure to make this a proper Gaussian integral in Θ [97].

We characterize the metric tensor in terms of the frame fields as:

$$\boxed{g_{NM} = (-)^M E_N{}^A \; \kappa_{AB} \; E^B{}_M}. \tag{4.280}$$

, where the restriction of the Cartan-Killing metric κ_{IJ} Eq 4.248 to the $P_{0,1}$ and fermionic $Q_\pm$ directions coincides with the local Lorentz metric Eq 4.4. This definition is in particular compatible with Eq 4.53. Note that we took the transpose of the second frame matrix in this definition. It turns out that this matrix should be antisymmetric in its double fermionic block to obey the required symmetries:

$$E_M{}^B = (-)^{MB} E^B{}_M. \tag{4.281}$$

This is conform with the antisymmetry properties in our definition of the metric tensor in Eqs 4.211, 4.251:

$$\begin{aligned}
g_{MN} &= (-)^{N} E_M{}^A \, \kappa_{AB} \, E^B{}_N \\
&= (-)^{N+NM+AB+BM+AN} E^B{}_N \kappa_{AB} E_M{}^A \\
&= (-)^{N+NM} (-)^{A(N+M)+B(N+M)} E_N{}^B \kappa_{BA} E^A{}_M \\
&= (-)^{N+NM+M} (-)^{M} E_N{}^B \kappa_{BA} E^A{}_M \\
&= (-)^{N+M+NM} g_{NM}.
\end{aligned}$$

The first line is our definition of the metric g_{MN}. In the second line, I have interchanged the frame matrices $E_M{}^A$ and $E_N{}^B$. Essentially, since the Cartan-Killing metric contains no Grassmann entries, there are no additional sign switches when pulling the frame fields through the Cartan-Killing metric. In the third line, I have used the antisymmetry in the frame matrices and the Cartan-Killing metric. These yield an overall sign factor $(-)^{(A+B)(N+M)}$. This factor essentially vanishes in our case since the Cartan-Killing metric is block diagonal in the fermionic and bosonic entries. Therefore, if A has an odd parity, so does B and vice versa. In the fourth line, I have isolated a factor $(-)^M$ and recognized the definition of the metric g_{NM} up to the required sign factors defined in Eq 4.211.
The inverse frame fields Eqs 4.7 $E_A{}^M E_M{}^B = \delta_A^B$, $E_N{}^A E_A{}^M = \delta_N^M$, define the inverse metric according to:

$$\boxed{g^{MN} = E^M{}_A \kappa^{BA} E_B{}^N}. \tag{4.282}$$

This choice satisfies the required antisymmetry property Eq 4.211:

$$\begin{aligned}
g^{NM} &= E^N{}_A \kappa^{BA} E_B{}^M \\
&= (-)^{AB+MN+BN+AM} E_B{}^M \kappa^{BA} E^N{}_A \\
&= (-)^{MN+M(A+B)+N(A+B)} E^M{}_A \kappa^{BA} E_B{}^N \\
&= (-)^{MN} g^{MN}.
\end{aligned}$$

Crucially, we needed that κ^{AB} is block diagonal in the fermionic and bosonic entries. Thereby A and B always share the same parity, and $(-)^{M(A+B)+N(A+B)} = 1$. It is also the orthogonal partner to g^{MN} defined in Eq 4.280:

$$\begin{aligned}
g^{MN} g_{NP} &= (-)^{P} E^M{}_A \kappa^{BA} E_B{}^N E_N{}^C \kappa_{CD} E^D{}_P \\
&= (-)^{P} E^M{}_A \kappa^{BA} \kappa_{BD} E^D{}_P \\
&= (-)^{P+A} E^M{}_A \kappa^{AB} \kappa_{BD} E^D{}_P \\
&= (-)^{P+A} E^M{}_A E^A{}_P \\
&= (-)^{P+AP+AM+MP} E^A{}_P E^M{}_A \\
&= (-)^{P+MP} E_P{}^A E_A{}^M \\
&\equiv (-)^{P+MP} \delta_P^M \simeq \delta_P^M.
\end{aligned}$$

, again since the Killing metric is block diagonal $\kappa^{AB} = (-)^{AB} \kappa^{BA} = (-)^A \kappa^{BA}$.

Using these definitions of the spacetime metric, the term inside the square root of the action Eq 4.279 is simply the line element defined in the action of the EOW branes Eq 4.246:

$$\begin{aligned}\dot{Z}^N E_N{}^A \kappa_{AB} \dot{Z}^M E_M{}^B &= (-)^{M+MB} \dot{Z}^N E_N{}^A \kappa_{AB} E_M{}^B \dot{Z}^M = (-)^M \dot{Z}^N E_N{}^A \kappa_{AB} E^B{}_M \dot{Z}^M \\ &= \dot{Z}^N g_{NM} \dot{Z}^M .\end{aligned}$$

Recognizing that flat field gauge transformations in the BF path integral are equivalent to superdiffeomorphisms in the metric formulation finally proves the equivalence between a Wilson loop operator insertion in the BF path integral, and the free-particle path integral in the metric formulation:

$$\mathcal{W}_j(\mathcal{C}) \simeq \int_{\mathcal{C}} \mathcal{D}Z \; e^{-m\int_{\mathcal{C}} ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N}} . \tag{4.283}$$

Note that we should interpret the equality as an operator equivalence *inside* the BF path integral to account for the superdiffeomorphic symmetries.

We distinguish two separate cases; when either the path $\mathcal{C}_{\tau_1\tau_2}$ connects two points at the asymptotic boundary with fixed proper time, or when $\mathcal{C}$ forms a closed loop inside the bulk. In the case of the latter, there are no further restrictions on the probe field $g(s)$ except that it should be single valued around the circle. Taking periodic boundary conditions for the fermions, taking a (super)trace over closed paths associated to the probe yields after gauge fixing the previously established result:

$$\mathcal{W}_j(\mathcal{C}) \;=\; \text{STr}_j \left(\mathcal{P} \exp - \oint_{\mathcal{C}_{\tau_1\tau_2}} \mathbf{A} \right) \;=\; \oint_{\mathcal{C}} \mathcal{D}Z \; e^{-m\int_{\mathcal{C}} ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N}} . \tag{4.284}$$

The evaluation amounts to taking a character in the spin-j lowest-weight discrete-series module.

In the case of the former, the path integral over a probe with both endpoints anchored to the thermal disk at proper times $\tau_1 - \tau_2$ prepares an evolution operator between two states on the boundary. Both the right and left parabolic labels of the asymptotic Hartle-Hawking states are constrained by the coset boundary conditions to a predefined weight $\nu_- = \nu_+ = 1$. By current conservation of the Clebsch-Gordan coefficients, the Wilson line operator is constrained to the lowest-weight states $\nu_- = \nu_+ = 0$. This boundary data should be incorporated in the boundary conditions of the probe field $g(s)$ in the evaluation of the path integral. The precise implementation is yet to be figured out. We thus have the following identification:

$$\mathcal{W}_{j,00}(\mathcal{C}_{\tau_1\tau_2}) \;=\; \langle j, 0_- \mid \mathcal{P} \exp \left(- \int_{\mathcal{C}_{\tau_1\tau_2}} \mathbf{A} \right) \mid j, 0_+ \rangle \;=\; \int_{\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}Z \; e^{-m\int_{\mathcal{C}} ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N}} . \tag{4.285}$$

4.8 Super-Gravitational amplitudes involving EOW branes

Having the correct boundary action Eq 4.246 in hand, the procedure is identical to the bosonic case. Path integrating over the dilaton field imposes the extrinsic supercurvature to vanish:

$$K \equiv 0. \tag{4.286}$$

By construction of the extrinsic supercurvature, the free particle path integral localizes onto geodesics;

$$\int_{\text{paths}\sim\mathcal{C}} \mathcal{D}Z\, e^{-\int_{\mathcal{C}} ds\sqrt{\dot{Z}^M g_{MN}\dot{Z}^N}(\mu-\Phi K)} \xrightarrow{\text{Integrate over }\Phi} \int_{\text{geodesics}\sim\mathcal{C}} \mathcal{D}Z\, e^{-\mu\int_{\mathcal{C}} ds\sqrt{\dot{Z}^M g_{MN}\dot{Z}^N}}. \tag{4.287}$$

This localization is achieved by taking μ to be large (compared to the underlying Planck scale). Equivalently, due to the relation between the tension parameter μ and the conformal weight ℓ, the geodesic approximation instructs us to identify $\ell^2 \approx \mu^2$, leading to

$$\boxed{\ell \approx \mu} \tag{4.288}$$

, where we took the plus sign in the root since discrete lowest-weight modules are defined for $\ell > 1/2$ as we will see momentarily.

To relate the gravitational to the group theoretical solutions, we ought to relate the relevant group parameters to geometrical quantities (such as geodesic length). The latter are calculated in [46], which expresses the bosonic solution Eq 3.57 into $\mathrm{OSp}(1|2,\mathbb{R})$-invariant quantities. They have computed the geodesic distance between two endpoints (τ_1, ϑ_1) and (τ_2, ϑ_2) on the boundary of the super-Poincaré-upper-half-plane (SUHP) metric (Eq 4.46) to be:

$$\cosh d = 1 + \frac{|\tau_1' - \tau_2' - \theta_1'\theta_2'|^2}{2(z_1' - \bar{z}_1' - \theta_1'\bar{\theta}_1')(z_2' - \bar{z}_2' - \theta_2'\bar{\theta}_2')}. \tag{4.289}$$

Coupled to the holographic wiggly boundary curve, the endpoints satisfy the single bosonic constraint Eq 4.50. To leading order in ϵ, we write

$$\cosh d = 1 + \frac{|\tau_1' - \tau_2' - \theta_1'\theta_2'|^2}{-8\epsilon^2 |D\theta_1'|^2 |D\theta_2'|^2} \tag{4.290}$$

, which approximates to:

$$d \approx \log \frac{|\tau_1' - \tau_2' - \theta_1'\theta_2'|^2}{|D\theta_1'|^2 |D\theta_2'|^2} - \log(-8\epsilon^2) \tag{4.291}$$

in the asymptotic regime $\epsilon \to 0$. This is the $\mathrm{OSp}(1|2,\mathbb{R})$-invariant equivalent form of Eq 3.57, whose bottom component b exactly coincides with the latter (c.f. Eq 4.52).

To proceed, we relate the bottom component of the lowest-weight super-Wilson line insertion Eq 4.168 to the holographic description in terms of a bilocal operator in the super-Schwarzian theory Eq 4.165. Using the explicit expression of the geodesic length in SUHP coordinates, we readily relate the bosonic geodesic length b to the hyperbolic group parameter ϕ :

$$\boxed{b = -2\phi}. \tag{4.292}$$

This is the same identification for the bosonic case 3.60, which is unsurprising since the bosonic submetric of both cases coincides.

4.8.1 Half-moon super-gravitational amplitudes

The super-gravitational calculation of half-disks ending on an EOW brane proceeds in complete parallel to the discussion in section 3.4.1, to which I refer for a more in-depth discussion.
In particular, we use an open Cauchy slicing of the Hilbert space anchored on both sides to the asymptotic boundary. Since the free particle path integral of a probe field moving between two proper time coordinates on the boundary $\mathcal{C}_{\tau_1\tau_2}$ is ultimately identified with a boundary-anchored Wilson line, we have the identification in superspace:

$$\int \mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A}\ \mathcal{W}_\ell(\mathcal{C}_{\tau_1\tau_2})e^{-I_{BF}^{\mathfrak{osp}(1|2,\mathbb{R})}[\mathbf{B},\mathbf{A}]} = \int \mathcal{D}g\mathcal{D}\Phi \int_{\text{geodesic}\sim\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}x\ e^{-\mu\int_{\mathcal{C}_{\tau_1\tau_2}} ds\sqrt{\dot{Z}^M g_{MN}\dot{Z}^N}}\ e^{-I_{JT}^{\mathcal{N}=1}[g,\Phi]} \tag{4.293}$$

, where path integrating over the dilaton along the EOW brane fixes the regime

$$\mu \sim \ell, \qquad \mu \gg 1. \tag{4.294}$$

Anchored to the asymptotic boundary, we have argued that the free particle path integral evaluates to a lowest-weight Wilson line insertion:

$$R_{\ell,0_-,0_+}(\phi) = \langle \ell, 0_- \mid \mathcal{P}\exp\left(-\int_{\mathcal{C}_{\tau_1\tau_2}}\mathbf{A}\right) \mid \ell, 0_+\rangle. \tag{4.295}$$

Specifying to the bottom component of the diagonal super-Wilson line matrix element in the lowest-weight representation specified by $j=-\ell$, we know the solution in the regime $\ell > 1/2$ (Eq 4.166) in terms of the modified Bessel function of the first kind, defined as $J_\alpha(iz) \simeq I_\alpha(z)$

$$R_{j,\nu_-\nu_+}(\phi) = e^{\phi}J_{2j+1}\left(2\sqrt{-\nu\nu}e^{\phi}\right) \propto e^{\phi}I_{|-2\ell+1|}(2\nu e^{\phi}) = e^{\phi}I_{2\ell-1}(2e^{\phi}). \tag{4.296}$$

Note that as an operator insertion, we do not in general care about normalization factors independent of the group label. This is the same form of the bosonic Wilson line insertion Eq 2.278. Specifying to the lowest-weight yields the exponential behaviour Eq 2.279:

$$R_{j=-\ell,00}(\phi) \sim e^{2\ell\phi}. \tag{4.297}$$

On the other hand, within the geodesic approximation, any diagonal mixed parabolic state will yield the same amplitude up to some ϕ-independent prefactor due to the asymptotic approximation Eq 3.66:

$$R_{j=-\ell,\nu_-\nu_+}(\phi) \sim e^{2\ell\phi}. \tag{4.298}$$

Using the group parameter identification Eq 4.292, and irrep identification $\mu \approx \ell$, this matrix element is written gravitationally as:

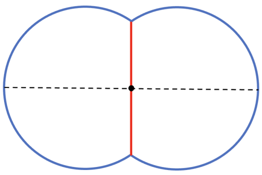
$$R_{\mu,0_-0_+}(b) = e^{-\mu b}. \tag{4.299}$$

We recognize this form as the on-shell (geodesic) approximation of the bottom component of free-particle path integral in superspace:

$$\int_{\text{paths}\sim\mathcal{C}_{\tau_1\tau_2}} \mathcal{D}x\; e^{-\mu\int_{\mathcal{C}_{\tau_1\tau_2}} ds} \;\approx\; e^{-\mu b}. \tag{4.300}$$

Of course, this identity holds only inside the path integral, resulting in an integration over geodesic lengths b, since the boundary data is unspecified on the wiggly boundary curve. This demonstrates consistency between the gravitational and the group theoretic approach since we a priori expect the geodesic approximation of the free particle path integral to be one-loop exact in this limit. On the one hand, we can immediately write down Eq 4.300 as the on-shell approximation of the free particle path integral $e^{-\mu\oint_{\mathcal{C}} ds\sqrt{\dot{Z}^M g_{MN}\dot{Z}^N}} \approx e^{-\mu b}$. On the other hand, we see that by taking a suitable limit of the exact answer obtained in group theory leads to the same result.

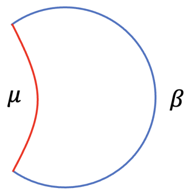
The EOW brane amplitude is calculated from the doubled Euclidean solution with the insertion of a Wilson line Eq 4.169:



$$= \langle \mathbf{1} \mid e^{-\beta H}\mathcal{W}_{\ell,00} \mid \mathbf{1}\rangle = \int_{-\infty}^{\infty} d\phi\, \left(\frac{e^{-\phi}}{2}\right) \langle \mathbf{1} \mid e^{-\beta_2 H} \mid \phi\rangle\, R_{-\ell,00}(\phi)\, \langle \phi \mid e^{-\beta_1 H} \mid \mathbf{1}\rangle$$

$$= \int_{-\infty}^{\infty} d\phi\, \left(\frac{1}{2}e^{-\phi}\right) Z_{\text{Hartle}}(\phi,\beta_1)^*\, (R_{-\ell,00}(f))\, Z_{\text{Hartle}}(\phi,\beta_2). \tag{4.301}$$

Performing a $\mathbb{Z}_2$-quotient along the geodesic EOW brane fixed points removes one asymptotic Hartle-Hawking state. Inserting the explicit amplitude of the remainder (Eq 4.150) yields:



$$= \int_{-\infty}^{\infty} d\phi\, \left(\frac{e^{-\phi}}{2}\right) R_{j=-\ell,0_-0_+}(\phi)\, \langle \phi \mid e^{-\beta H} \mid \mathbf{1}\rangle$$

$$\simeq \int_0^{\infty} dk\; \cosh(2\pi k) e^{-\beta k^2} \int_{-\infty}^{\infty} d\phi\, \left(K_{1/2+2ik}(2e^{\phi}) + \epsilon_-\epsilon_+ K_{1/2-2ik}(2e^{\phi})\right) e^{2\ell\phi}. \tag{4.302}$$

We may interpret this result as the supergravitational amplitude of a pure state in Euclidean signature.

4.8.2 Trumpet gravitational amplitudes

The result for boundary-anchored EOW branes does not teach us much about the underlying group-theoretic structure since the matrix element is, by consistency, just the on-shell geodesic saddle of the full free-particle path integral. EOW branes that do not reach the asymptotic boundary, but describe a closed loop at the neck of a supersymmetric trumpet are structurally more interesting. This is because the closed loops can contribute one-loop corrections to the classical saddle, which can be easily pinned down by group-theoretical considerations.

In particular, Eq 4.284 demonstrates that the full-fledged path integral is the character evaluated in a lowest-weight discrete series representation module labeled in terms of the tension parameter μ, by the relation Eq 4.288. For Wilson lines in the lowest-weight discrete series representation, this is the character of some hyperbolic class element.

Discrete series representation

The lowest-weight discrete series representations of $\mathrm{OSp}(1|2,\mathbb{R})$ are constructed in much the same way as those of $\mathrm{SL}(2,\mathbb{R})$ [40]. For $\mathrm{OSp}(1|2,\mathbb{R})$, basis states are constructed with respect to the Borel-Weil generators Eq B.32. These should diagonalize the Cartan element $iH = -x\partial_x - \frac{1}{2}\vartheta\partial_\vartheta + j$. The eigenvalue under iH is called the weight of the eigenstate. As a consequence of the $\mathfrak{osp}(1|2,\mathbb{R})$-algebra Eq B.34, weights are raised and lowered by application of the pairs iF_-, iE_- and iF_+, iE_+ respectively[16]. In particular applying iE_-, iE_+ to a basis state with weight j raises, respectively lowers the eigenvalue by one unit under the commutation relations:

$$[iH, iE_-] = iE_-, \qquad [iH, iE_+] = -iE_+. \tag{4.303}$$

On the other hand, the $\mathfrak{osp}(1|2,\mathbb{R})$ algebra is equipped with fermionic generators iF_-, iF_+ that raise, respectively lower the weight by $1/2$ under the commutation relations:

$$[iH, iF_-] = \frac{1}{2}iF_-, \qquad [iH, iF_+] = -\frac{1}{2}iF_+. \tag{4.304}$$

Highest-weight modules are constructed by acting indefinitely with the lowering operator $iF_+ = -\frac{1}{2}x\partial_\vartheta - \frac{1}{2}x\vartheta\partial_x + j\vartheta$ on some highest-weight state ψ_{HW}. This highest-weight state should be annihilated under the raising operator $iF_- = \frac{1}{2}(\partial_\vartheta + \vartheta\partial_x)$. Its weight under iH is called the weight of the highest-weight module. We realize the discrete representation on the superline $\mathbb{R}^{1|1}$ by the monomial representation. The highest-weight state is the constant function

$$\psi_{HW}(x,\vartheta) = 1 \tag{4.305}$$

, whose weight under iH is simply j. The successive application of iF_+ on this highest-weight state generates an entire module $j, j-\frac{1}{2}, j-1, \ldots$

Lowest-weight representations are constructed by acting successively with the raising operator iF_- on some

[16] Beware again the perhaps misleading notation of the generators. P.S. [40] uses throughout an isomorphism of the $\mathfrak{osp}(1|2,\mathbb{R})$ that is structurally more intuitive. On the other hand, I have opted to match the conventions used in the bosonic case as much as possible for clarity.

lowest-weight state ψ_{LW}. This state should be annihilated by the lowering operator iF_+. On $\mathbb{R}^{1|1}$, one chooses

$$\psi_{LW}(x,\vartheta) = x^{2j} \tag{4.306}$$

, whose weight under iH is $iH = -j$. When $2j \in \mathbb{N}$, the representation becomes finite for some $j < \infty$. We therefore introduce a positive $\ell > 1/2$ with $j = -\ell$, consistent with $2\ell \in \mathbb{N}$. The lowest-weight state in this case is:

$$\psi_{LW}(x,\vartheta) = \frac{1}{x^{2\ell}} \tag{4.307}$$

, with corresponding weight $iH = \ell$. Successive action of iF_- generates an infinite module $j, j+\frac{1}{2}, j+1, \dots$. Also note that within the universal covering group $\tilde{\mathrm{OSp}}(1|2,\mathbb{R})$, the weights $2j$ are not necessarily constrained to integer values, and can admit a continuous range.

Note that in both lowest- and highest-weight modules, all states are simultaneous eigenvalues of the sCasimir $(-)^F = 1 - 2\vartheta\partial_\theta$ (Eq B.35), since both the lowest- and highest-weight states are bosonic with eigenvalue $(-)^F = 1$. Successive application of the fermionic raising and lowering operators alternates the $\mathbb{Z}_2$ eigenvalue $(+,-)$ under $(-)^F$ [40].

Since we imagine the EOW loops at the neck of a supergravitational trumpet, the relevant group elements are part of the hyperbolic conjugacy class. In the Gauss parametrization, the hyperbolic class elements are parameterized by the single group element $g(\phi) = e^{2\phi iH}$, labeled by the hyperbolic group parameter ϕ (c.f. Eq 4.183):

$$g(\phi) = e^{2\phi iH} = \left(\begin{array}{cc|c} e^{-\phi} & 0 & 0 \\ 0 & e^{\phi} & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right). \tag{4.308}$$

The character in the **NS** sector $(-)$ is simply given by the value of the STr (since the definition of the STr compensates the minus sign for $e = -1$). The lowest-weight discrete series representation $j = -\ell$ reads:

$$\chi_\ell^{\mathbf{NS}}(\phi) = \mathrm{STr}\left(e^{2\phi iH}\right) = \sum_{n=2\ell}^{\infty} e^{n\phi}. \tag{4.309}$$

Note that in order for the geometric series to converge, we need $\phi = -\frac{b}{2} < 0$, and we should therefore constrain to positive geodesic lengths. The novel difference between the evaluation of the hyperbolic $\mathrm{SL}(2,\mathbb{R})$ character (c.f. Eq 3.83) is that we take the sum in steps of $1/2$ instead of single units.

Also note that the multiplicity of each state is still unity in the general definition Eq 3.81, although it might seem that linearly independent states with the same integer weight might be created by either raising in integer steps iE_-, or with doubly as many half-integer steps iF_-. However, the algebra anticommutation relation $\{iF_-, iF_-\} = \frac{1}{2}iE_-$ demonstrates that the successive action of iF_- squares to iE_-, and that these sates are not linearly independent. In any case, the multiplicity of each weight is still one.

A geometric series immediately learns:

$$\boxed{\chi_\ell^{\mathbf{NS}}(\phi) = \frac{e^{2\ell\phi-\phi/2}}{2\sinh\left(-\phi/2\right)}}. \tag{4.310}$$

Modulo different conventions, this result coincides with the result of [40].

Within the **R** sector, the hyperbolic matrix should first be multiplied by the sCasimir $(-)^F$ in the evaluation of the STr to generate a genuine trace [40]. Since every successive state alternates between $(-)^F = 1, -1, 1, -1, \ldots$, starting with 1 for the lowest-weight state, the evaluation of the lowest-weight hyperbolic group character in the **R** sector is readily achieved by adding an additional sign factor $(-)^{n-2\ell}$:

$$\chi_\ell^{\mathbf{R}}(\phi) = \mathrm{STr}\left((-)^F\, e^{2\phi iH}\right) = \sum_{n=2\ell}^{\infty} (-)^{-2\ell+n}\, e^{n\phi} \tag{4.311}$$

In order for the geometric series to converge, we again require $\phi = -\frac{b}{2} < 0$, thereby constraining the regime to positive geodesic lengths. The result is readily worked out:

$$\boxed{\chi_\ell^{\mathbf{R}}(\phi) = \frac{e^{2\ell\phi-\phi/2}}{2\cosh(-\phi/2)}} \tag{4.312}$$

, which again coincides with [40] up to the current conventions.

Gluing along the trumpet partition function

The identity Eq 4.284 instructs us to identify the free particle path integral over closed loops superdiffeomorphic to $\mathcal{C}$ with a discrete series character insertion in the BF path integral. According to the familiar cutting-and-gluing axioms, we simply glue the relevant discrete series character along the geodesic ends of the single trumpet amplitude in superspace. Since the trumpet geometry is achieved by inserting a hyperbolic character of the continuous series representation in the disk partition function, we glue along the hyperbolic characters in the discrete series representation derived above.

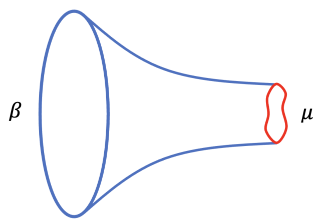
The transition from the group theoretical language to gravity is achieved by replacing $2\phi \rightarrow -b$, (where b corresponds to the bottom component of the geodesic length). This is true for Wilson lines attached to the boundary, but also holds for the geodesic at the neck of the trumpet. Indeed, according to the discussion of section 4.5, we extrapolate the monodromy relations of a defect insertion into the bulk by using the same bosonic submetric corresponding to defects in bosonic JT gravity in section 2.9. We may therefore immediately extrapolate the bosonic identification $2\phi = -b$ (Eq 2.313) to the bottom component of the geodesic length in superspace.

We further identify the mass tension $\mu \approx \ell \gg 1$ with the weight of the discrete series representation, and identify the closed loop path integral in each of the different spin structures with the character insertions Eqs 4.310 4.312 *inside* the total path integral:

$$\mathbf{NS} \oint_{\mathcal{C}} \mathcal{D}Z \; e^{-\mu \oint_{\mathcal{C}} ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N}} \simeq \frac{e^{-\mu b}}{2\sinh(b/4)}, \tag{4.313}$$

$$\mathbf{R} \oint_{\mathcal{C}} \mathcal{D}Z \; e^{-\mu \oint_{\mathcal{C}} ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N}} \simeq \frac{e^{-\mu b}}{2\cosh(b/4)}. \tag{4.314}$$

Within the geodesic approximation, we have neglected the linear terms in the exponent. It is interesting to note that the denominator can be interpreted as a one-loop correction to the classical saddle (geodesic) approximation. Gluing the free particle amplitudes along the geodesic length ($\phi < 0 \;\; \rightarrow b > 0$) at the neck of the relevant spin-structured trumpets (Eqs 4.197, 4.198) finally yields:



$$Z^{\mathbf{NS}}_{EOW}(\beta) = \int_0^\infty dk \; e^{-\beta k^2} \int_0^\infty db \cos(bk) \frac{e^{-\mu b}}{2\sinh(b/4)}, \tag{4.315}$$

$$Z^{\mathbf{R}}_{EOW}(\beta) = \int_0^\infty dk \; e^{-\beta k^2} \int_0^\infty db \sin(bk) \frac{e^{-\mu b}}{2\cosh(b/4)}. \tag{4.316}$$

An immediate realization is that the spurious UV divergence of the bosonic result Eq 3.87 for small $b \rightarrow 0$ is only present in the **NS** sector. On the other hand, the **R** sector is perfectly regular in the UV.
In accordance with the bosonic solution [36], we keep the Weyl denominator of the discrete series character since the free particle path integral evaluates to a genuine character in group theory. In contrast, the hyperbolic defect of the gravitational trumpet is only formally identified with only the numerator of the continuous series character.

Chapter 5

Outlook and Future Developments

5.1 Future developments

5.1.1 EOW branes in $\mathcal{N} = 2$ JT supergravity

We can try to generalize this further to $\mathcal{N} = 2$ supersymmetry and higher. JT supergravity with two supercharges is characterized by the breaking $\mathrm{SL}(1,1|1) \supset \mathrm{SL}(2,\mathbb{R}) \times U(1)$. The global super-conformal group $\mathrm{SU}(1,1|1) \simeq \mathrm{OSp}(2|2,\mathbb{R}) \simeq \mathrm{SL}(1|2,\mathbb{R})$ is 4|4-dimensional with four bosonic and four fermionic generators. The former are labeled by $H, Z, E_{\pm}$, the latter by $F_{\pm}, \overline{F}_{\pm}$.
The $\mathfrak{sl}(1|2,\mathbb{R}) \simeq \mathfrak{osp}(2|2,\mathbb{R})$ algebra is realized in the fundamental representation by the a set of 8 $2|2 \times 2|2$-dimensional supermatrices [109]:

$$
H = \left(\begin{array}{cc|cc} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), E^{+} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), E^{-} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), Z = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \end{array}\right),
$$
$$
F^{+} = \left(\begin{array}{cc|cc} 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \end{array}\right), \bar{F}^{+} = \left(\begin{array}{cc|cc} 0 & 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 \end{array}\right), F^{-} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{i}{2} \\ \hline \frac{1}{2} & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 \end{array}\right),
$$
$$
\bar{F}^{-} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} \\ \hline -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 \end{array}\right). \tag{5.1}
$$

These satisfy the $\mathfrak{sl}(1|2,\mathbb{R}) \simeq \mathfrak{osp}(2|2,\mathbb{R})$ superalgebra [109]:

$$\begin{aligned}
&[H,E^{\pm}] = \pm E^{\pm}, && [H,F^{\pm}] = \pm\frac{1}{2}F^{\pm}, && [H,\bar{F}^{\pm}] = \pm\frac{1}{2}\bar{F}^{\pm},\\
&[Z,H] = [Z,E^{\pm}] = 0, && [Z,F^{\pm}] = \frac{1}{2}F^{\pm}, && [Z,\bar{F}^{\pm}] = -\frac{1}{2}\bar{F}^{\pm},\\
&[E^{\pm},F^{\pm}] = [E^{\pm},\bar{F}^{\pm}] = 0, && [E^{\pm},F^{\mp}] = -F^{\pm}, && [E^{\pm},\bar{F}^{\mp}] = \bar{F}^{\pm},\\
&\{F^{\pm},F^{\pm}\} = \{\bar{F}^{\pm},\bar{F}^{\pm}\} = 0, && \{F^{\pm},F^{\mp}\} = \{\bar{F}^{\pm},\bar{F}^{\mp}\} = 0, && \{F^{\pm},\bar{F}^{\pm}\} = E^{\pm},\\
&[E^{+},E^{-}] = 2H, && \{F^{\pm},\bar{F}^{\mp}\} = Z \mp H. &&
\end{aligned} \tag{5.2}$$

The Cartan subalgebra now consists of two commuting operators H, Z, whose eigenvalues j and b label the distinct representations. Typical irreducible representations labeled by j and b are denoted by $R_{j,b}$, and can be decomposed into the bosonic $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$ subalgebra according to the Branching rule [109]:

$$R_{j,b} \simeq (j,b) \oplus (j-\frac{1}{2}, b-\frac{1}{2}) \oplus (j-\frac{1}{2}, b+\frac{1}{2}) \oplus (j-1,b). \tag{5.3}$$

Due to a modification of the conventions of the bosonic $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in Eq 5.2 with respect to Eq A.5, the definition of the corresponding Borel-Weil $\mathfrak{sl}(2,\mathbb{R})$-generators is modified to

$$H = x\partial_x - j, \qquad E^{+} = -x^2\partial_x + 2jx, \qquad E^{-} = \partial_x. \tag{5.4}$$

Instead of lowest-weight modules, the monomial $|j,0\rangle = x^{2j}$ correspond to a highest-weight state, for which the successive action of E_{-} generates an entire highest-weight module. Explicitly, we know that the relevant highest-weight state of $\mathfrak{sl}(2,\mathbb{R})$ is constructed by acting with $H = x\partial_x - j$ on x^{2j}, producing a weight under H of $H = j$. This state is also annihilated by the raising operator $E^{+} = -x^2\partial_x + 2jx$. Due to the $\mathfrak{sl}(2,\mathbb{R})$-algebra relation $[H,E^{-}] = -E^{-}$, an infinite module of weights $j - \mathbb{N}$ is generated. Since the multiplicity of each weight is one, the highest-weight character of a hyperbolic class element is readily calculated using a geometric series:

$$\chi_j^{\mathfrak{sl}(2,\mathbb{R})}(e^{2\phi H}) = \sum_{n=-\infty}^{j} e^{2\phi n} = \frac{e^{\phi+2\phi j}}{2\sinh\phi}. \tag{5.5}$$

The corresponding $\mathfrak{u}(1)$ character is the exponential

$$\chi_b^{\mathfrak{u}(1)}(\theta) = e^{2ib\theta}. \tag{5.6}$$

Using the Branching rule Eq 5.3, the highest-weight $\mathcal{N} = 2$ character of the hyperbolic class element $g = e^{2H\phi}e^{2i\theta Z}$, evaluated in representation (j, b), is readily:

$$\begin{aligned}\chi^{\mathfrak{osp}(2|2)}_{j,b}(\phi,\theta) &= \mathrm{Tr}_{j,b}(e^{2H\phi}e^{2i\theta Z}) \qquad (5.7)\\ &= \frac{e^{\phi(1+2j)}}{2\sinh\phi}e^{2ib\theta} + \frac{e^{\phi(-1+2j)}}{2\sinh\phi}e^{2ib\theta} - \frac{e^{2j\phi}}{2\sinh\phi}e^{2i(b-1/2)\theta} - \frac{e^{2j\phi}}{2\sinh\phi}e^{2i(b+1/2)\theta}\\ &= e^{2j\phi}e^{2ib\theta}\left(\frac{\cosh\phi-\cos\theta}{2\sinh\phi}\right)\\ &= \frac{e^{2j\phi}e^{2ib\theta}}{\sqrt{\Delta(\phi,\theta)}}. \qquad (5.8)\end{aligned}$$

The minus signs are for fermionic states, which in the supertrace automatically get a minus sign. The denominator $\sqrt{\Delta(\phi,\theta)}$ corresponds to the Weil-denominator of the group $\mathrm{OSp}(2|2,\mathbb{R})$ [109]:

$$\Delta(\phi,\theta) = \frac{\sinh^2\phi}{(\cosh\phi-\cos\theta)^2}. \qquad (5.9)$$

For gravitational applications, we restrict to highest-weight states characterized by a negative $j = -\ell < 0$, with the corresponding character

$$\chi^{\mathfrak{osp}(2|2,\mathbb{R}}_{j=-\ell,b}(\phi,\theta) = \frac{e^{-2\ell\phi}e^{2ib\theta}}{\sqrt{\Delta(\phi,\theta)}}. \qquad (5.10)$$

In the same way, we may find the hyperbolic character corresponding to the principal series representations. The continuous series $\mathfrak{sl}(2,\mathbb{R})$ character of spin j is given by Eq 2.322. Taking into account the modified bosonic $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in Eq 5.2 we simply shift $\phi \to -\phi$:

$$\chi^{\mathfrak{sl}(2,\mathbb{R})}_{j}(\phi) = \frac{\cosh(2j+1)\phi}{\sinh\phi}. \qquad (5.11)$$

We deduce the relevant $\mathfrak{osp}(2|2,\mathbb{R})$ character using again the branching rule 5.3:

$$\begin{aligned}&\chi^{\mathcal{N}=2}_{j,b}(\phi,\theta) \qquad (5.12)\\ &= \frac{\cosh(2j+1)\phi}{\sinh\phi}e^{2ib\theta} - \frac{\cosh(2j)\phi}{\sinh\phi}e^{2i(b-1/2)\theta} - \frac{\cosh(2j)\phi}{\sinh\phi}e^{2i(b+1/2)\theta} + \frac{\cosh(2j-1)\phi}{\sinh\phi}e^{2ib\theta}\\ &= \frac{2\cosh(2j\phi)\,(\cosh\phi-\cos\theta)e^{2ib\theta}}{\sinh\phi} = \frac{2\cosh(2j\phi)\,e^{2ib\theta}}{\sqrt{\Delta(\phi,\theta)}} \qquad (5.13)\end{aligned}$$

, with $\cosh(2j+1)\phi = \cosh 2j\phi\cosh\phi + \sinh 2j\phi\sinh\phi$ and $\cosh(2j-1)\phi = \cosh 2j\phi\cosh\phi - \sinh 2j\phi\sinh\phi$. Now finally setting $j = ik$ for unitarity [109], we obtain the principal series character:

$$\chi^{\mathcal{N}=2}_{k,b}(\phi,\theta) = \frac{2\cos(2k\phi)\,e^{2ib\theta}}{\sqrt{\Delta(\phi,\theta)}}. \qquad (5.14)$$

This is again manifestly orthogonal with respect to the Weil denominator Eq 5.9. The numerator serves as a hyperbolic defect insertion to obtain a single trumpet amplitude between an asymptotic boundary of length β and a geodesic with length related to ϕ:

$$Z_{\text{trump}}(\phi,\theta) = \int dk dq e^{-\beta(k^2-q^2)} \cos(2k\phi) e^{2iq\theta}. \tag{5.15}$$

The quadratic Casimir in the propagator is given by $\mathcal{C}_2 = k^2 - b^2$. In contrast to the $\mathcal{N}=1$ case, there are no two disconnected solutions depending on the periodicity sector of the gravitinos around the circle. Instead, all periodicity sectors are continuously connected by the θ degree of freedom, for which the fermions pick up the $\mathfrak{u}(1)$ phase factor $e^{i\theta}$ upon travelling around the circle. For $\theta = 0$ this is the periodic R sector, while $\theta = \pi$ corresponds to the antiperiodic NS sector.

Denoting the collection of all generators J_I, we may calculate the Cartan-Killing metric associated to these states following the usual definition:

$$\text{STr}(J_I J_J) = \frac{\kappa_{IJ}}{2}. \tag{5.16}$$

In the order $\{H, Z, E^+, E^-, F^+, F^-, \bar{F}^+, \bar{F}^-\}$, we readily calculate the metric to be:

$$\kappa_{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}. \tag{5.17}$$

Indices are raised and lowered with respect to this metric. Its inverse is given by

$$\kappa^{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \end{pmatrix}. \tag{5.18}$$

This leads to the definition of the quadratic Casimir:

$$\mathcal{C}_2 = \kappa^{IJ} J_I J_J \tag{5.19}$$
$$= H^2 - Z^2 + \frac{1}{2}\left(E^+E^- + E^-E^+\right) + \frac{1}{2}\left(F^+\bar{F}^- + F^-\bar{F}^+\right) - \frac{1}{2}\left(\bar{F}^+F^- + \bar{F}^-F^+\right). \tag{5.20}$$

Using the algebra relations Eq 5.2, this readily reduces to:

$$
\begin{aligned}
\mathcal{C}_2 &= H^2 - Z^2 + E^- E^+ + H + \frac{1}{2}\left(Z - H - \bar{F}^- F^+ + F^- \bar{F}^+\right) - \frac{1}{2}\left(Z + H - F^- \bar{F}^+ + \bar{F}^- F^+\right) \\
&= H^2 - Z^2 + E^- E^+ - \bar{F}^- F^+ + F^- \bar{F}^+.
\end{aligned}
$$

The eigenvalue of the quadratic Casimir is again parameterized by the eigenvalues j, b as [109]:

$$\mathcal{C}_2 = j^2 - b^2. \tag{5.21}$$

We would now like to repeat the usual analysis of section 3.3, and obtain an action of an EOW brane whose quantum amplitude is described by precisely a Wilson loop. Thereto, we again define linear combinations of the bosonic generators Eq 2.21:

$$P_0 = -H, \qquad P_1 = \frac{1}{2}(E_- + E_+), \qquad P_2 = \frac{1}{2}(E_- - E_+) \tag{5.22}$$

, and expand the gauge field $\mathbf{A}$ as

$$\mathbf{A} = E^A P_A + \Omega J_2. \tag{5.23}$$

P_A denote all the remaining $\mathrm{OSp}(2|2,\mathbb{R})$ generators except J_2. We may therefore embed the frame fields into a $3|4$-dimensional Lorentz manifold. By metric invertibility, this is translated into a $3|4$-dimensional Einstein manifold.

By the same arguments around Eq 4.255, we write a Wilson loop, evaluated in a highest-weight discrete series representation labeled by j and b as a path integral in the BF theory:

$$\mathcal{W}_{j,b}(\mathcal{C}) = \int \mathcal{D}_\alpha g e^{-S_\alpha[g,\mathbf{A}]} \tag{5.24}$$

, where the first order action is given by:

$$S_\alpha = -\int_{\mathcal{C}} ds\ \mathrm{Tr}(\alpha g^{-1} D_A g). \tag{5.25}$$

$\alpha \in \mathfrak{osp}(2|2,\mathbb{R})$ is a vector in the $\mathfrak{osp}(2|2,\mathbb{R})$ algebra whose length is constrained by the eigenvalue of the quadratic Casimir:

$$\mathcal{C}_2 = \frac{1}{2}\mathrm{Tr}(\alpha^2) = j^2 - b^2 \equiv m^2. \tag{5.26}$$

The last equality associates the quadratic Casimir of the representation with the total energy m^2. However, this does not fix the representation unambiguously since the Cartan subalgebra contains two commuting elements Z and H. To fix the representation, we also need to fix the eigenvalue of the cubic Casimir, defined in terms of the cubic Killing form [95]:

$$\mathrm{STr}(J_I J_J J_K) = \frac{h_{IJK}}{8}. \tag{5.27}$$

This tensor determines the cubic Casimir[1];

$$\mathcal{C}_3 = h_{IJK} J^I J^J J^K. \tag{5.28}$$

Using the conjugate momenta defined in Eq 4.261, the cubic length of the vector α is constrained by the value of the cubic Casimir (c.f. Eq 4.259):

$$\mathrm{STr}(\alpha^3) = \frac{1}{8} h_{IJK} \alpha^I \alpha^J \alpha^K = \mathcal{C}_3 \equiv \frac{1}{8} Q^3. \tag{5.29}$$

We fix the cubic Casimir by a numerical charge Q, whose relation with the representation labels j, b is yet to be determined.

We may enforce these constraints directly in the path integral by introducing two scalar Lagrange multipliers Θ_1, Θ_2, and functionally integrating over a family of vectors $\alpha \in \mathfrak{osp}(2|2, \mathbb{R})$ whose quadratic and cubic lengths are constrained according to the above prerequisites:

$$\alpha = \alpha^A P_A. \tag{5.30}$$

We again neglect the spin component corresponding to J_2 in the sum over the generators.

$$\mathcal{W}_{j,b}(\mathcal{C}) = \int \mathcal{D}\alpha \mathcal{D}\Theta_1 \mathcal{D}\Theta_2 \mathcal{D}_\alpha g \; e^{-S_\alpha[\alpha, g, \mathbf{A}, \Theta_1, \Theta_2]}. \tag{5.31}$$

The appropriate first order action is

$$S_\alpha[\alpha, g, \mathcal{A}, \Theta_1, \Theta_2] = \int_{\mathcal{C}} ds \left[-\mathrm{STr}(\alpha g^{-1} D_A g) + \frac{i}{4} \Theta_1 (\alpha_A \alpha^A - 4m^2) + i\Theta_2 (h_{ABC} \alpha^A \alpha^B \alpha^C - Q^3) \right]. \tag{5.32}$$

Fixing the gauge redundancy in g by setting $g = 1$ along the entire curve amounts to setting $g^{-1} D_A g = \mathbf{A}_s = \dot{Z}^M (E_M{}^A J_A)$, leading to:

$$\frac{1}{2} \int_{\mathcal{C}} ds \left[-\alpha_A \dot{Z}^M E_M{}^A + \frac{i}{2} \Theta_1 \left(\alpha_A \alpha^A - 4m^2 \right) + 2i\Theta_2 (h_{ABC} \alpha^A \alpha^B \alpha^C - Q^3) \right]. \tag{5.33}$$

The action integral is now non-Gaussian in the α^A vector. On the other hand, integrating out Θ_2 leads to the same integrand corresponding to the free particle action Eq 4.276. The constraint $\alpha^A \alpha^B \alpha^C h_{ABC} = Q^3$ is now implicit in the domain of integration.

To make this constraint visible in the second order form, we define the inverse frame fields Eq 4.7, satisfying:

$$E_A{}^M E_M{}^B = \delta_A^B, \qquad E_N{}^A E_A{}^M = \delta_N^M. \tag{5.34}$$

We now define the conjugate momenta and metric from the frame fields and α-components:

$$k_M \equiv \frac{\partial L}{\partial \dot{Z}^M} = -\frac{1}{2} (-)^{AM} \alpha_A E_M{}^A = -\frac{1}{2} \alpha_A E^A{}_M \tag{5.35}$$

[1] This is the SUSY version of the cubic Casimir in [95]. It is not clearly settled by now if this definition naturally generalizes to the $\mathfrak{osp}(2|2, \mathbb{R})$ superalgebra.

, where I have used the frame field transpose property. We isolate α_A from k_M by contracting k_M with $E_A{}^M$:

$$\begin{aligned}\alpha_A &= -(-)^A 2E_A{}^M k_M = (-)^A E_A{}^M \alpha_B E^B{}_M = (-)^{A+MB} E_A{}^M \alpha_B E_M{}^B \\ &= (-)^{A+B} E_A{}^M E_M{}^B \alpha_B \simeq (-)^{A+B} \delta_A^B \alpha_B \simeq \alpha_A .\end{aligned} \tag{5.36}$$

Indices of the Lorentz vectors are lowered from the right with respect to the Cartan-Killing metric: $\alpha_A = \alpha^B \kappa_{BA}$. By consistency, indices are raised from the right $\alpha^B = \alpha_A \kappa^{AB}$. Explicitly:

$$\begin{aligned}\alpha^B &= \alpha_A \kappa^{AB} = -(-)^A 2E_A{}^M k_M \kappa^{AB} = -(-)^A 2\kappa^{AB} E_A{}^M k_M \\ &= -2\kappa^{BA} E_A{}^M k_M\end{aligned} \tag{5.37}$$

, again using that $\kappa^{AB} = (-)^A \kappa^{BA}$. Remember that the Cartan-Killing metric contains no fermionic entries, and therefore commutes freely to the left.
The first constraint $\alpha_A \alpha^A$ may therefore be recast in the metric formalism as:

$$\begin{aligned}\alpha_A \alpha^A &= (-)^A 4E_A{}^M k_M \kappa^{AB} E_B{}^N k_N = (-)^{A+AB+AN} 4E_B{}^N \kappa^{AB} E_A{}^M k_M k_N \\ &= (-)^{A+AB+BN} 4E_B{}^N \kappa^{AB} E_A{}^M k_M k_N = (-)^{A+AB} 4E^N{}_B \kappa^{AB} E_A{}^M k_M k_N \\ &= 4g^{NM} k_M k_N = 4k^M k_M\end{aligned} \tag{5.38}$$

, where we have used that the Cartan-Killing metric is fermionic block diagonal, thereby identifying the Grassmann parity of A and B. In the last line, I have used the definition of the inverse metric Eq 4.282 and $(-)^{A+AB} = 1$. Remember that opposed to Lorentz indices, Einstein indices are raised and lowered from the left. The first constraint $\alpha_A \alpha^A = 4m^2$ therefore physically constrains the total energy of the probe via the conjugate momenta:

$$k^M k_M \equiv m^2 . \tag{5.39}$$

In the same way, we can cast the second constraint in terms of the conjugate momenta:

$$h_{ABC} \kappa^{AD} E_D{}^M k_M \kappa^{BE} E_E{}^N k_N \kappa^{CF} E_F{}^P k_P \ \equiv -\frac{1}{8} Q^3 . \tag{5.40}$$

The physical interpretation of this constraint in the second order metric formulation is much less clear than the quadratic constraint, and we leave it open for now. We may therefore write a Wilson line in terms of a path integral over the second order action:

$$\begin{aligned}S_\alpha[g, \Theta_1, \Theta_2, k] \equiv \int_{\mathcal{C}} ds \, \Big[& \dot{Z}^M k_M + i\Theta_1 (k^M k_M - m^2) \\ & + i\Theta_2 (\kappa^{AD} E_D{}^M k_M \kappa^{BE} E_E{}^N k_N \kappa^{CF} E_F{}^P k_P \, h_{ABC} + \frac{1}{8} Q^3) \Big]\end{aligned} \tag{5.41}$$

The first two terms agree with the first order metric formulation of the free particle action on the world-line $\mathcal{C}$. The third term makes the integral non-Gaussian, and we cannot cast it in the closed second order form. The action of an EOW brane in superspace for $\mathcal{N} = 2$ and higher should thus be formulated in the first order form over the phase space coordinates Z^M and k_M, with suitable constraints on the conjugate momenta. The first being just the Einstein relation between energy and mass. The physical origin of the second is still unclear by the time of writing.

The free particle path integral with these extra constraints thus prepares a Wilson loop in the bulk, evaluated in a highest-weight representation $(j = -\ell, b)$. These parameters are related to the physical EOW tension and charge parameters μ, Q, whose explicit dependence is left implicit for now. Using the explicit form of the $\mathcal{N} = 2$ trumpet partition function Eq 5.15 and that of the highest-weight $\mathrm{OSp}(2|2, \mathbb{R})$ discrete series character Eq 5.10 we readily deduce:

$$\boxed{Z_{EOW}^{\mathcal{N}=2}(\ell, b) = \int dk dq e^{-\beta(k^2-q^2)} \int d\phi d\theta \cos(2k\phi) e^{i2q\theta} \frac{\cosh\phi - \cos\theta}{\sinh\phi} e^{-2\ell\phi} e^{2ib\theta}}. \tag{5.42}$$

In accordance with the bosonic solution of the boundary particle formalism [36], we keep the Weyl denominator of the discrete series character since the full free particle path integral evaluates to a genuine character in group theory. In contrast, the hyperbolic defect preparing the gravitational trumpet is only formally identified with the numerator of the continuous series character.

Note that these results are still preliminary. In particular, the precise interpretation of Eq 5.40 is not yet understood in gravity. Furthermore, we still have to obtain the precise form and eigenvalue of the cubic Casimir in $\mathrm{OSp}(2|2, \mathbb{R})$. Additionally, it has been quite a puzzle to obtain consistent definitions for the geometric quantities in superspace. Further puzzling might reveal some more elegant set of conventions.

To obtain the amplitudes of an EOW brane anchored to the asymptotic boundary, we should still work out the Hartle-Hawking states starting from the mixed parabolic Whitakker functions of the $\mathrm{OSp}^+(2|2, \mathbb{R})$ supergroup.

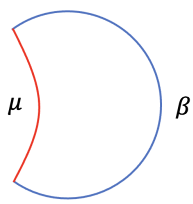
5.2 Summary and results

Starting from the total action of JT gravity in the presence of an end-of-the-world (EOW) brane Eq 3.1:

$$S = \frac{1}{2} \int \Phi \sqrt{|g|}(R+2) + \int_{AdS} du\, \Phi \sqrt{-g_{uu}}(K-1) + \int_{EOW} dv \sqrt{-g_{vv}}(\Phi K - \mu) \tag{5.43}$$

, [34] and [36] have obtained various quantum amplitudes involving EOW branes in different topologies using the boundary particle formalism. The former [34] considered EOW branes anchored to the holographic boundary, resulting in:

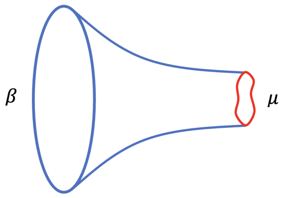
$Z_{EOW}(\beta) =$



$$= \int_0^\infty dk\, k \sinh 2\pi k\, e^{-\beta k^2} 2^{2\mu-2} |\Gamma(\mu - \frac{1}{2} + ik)|^2. \tag{5.44}$$

The latter [36] considered EOW loops attached to the neck of a single trumpet amplitude, characterized by:

$$Z_{EOW}(\beta) =$$



$$= \int dk\, e^{-\beta k^2} \int_0^\infty \frac{db}{\sinh(b/2)} \cos(kb) e^{-\mu b}. \tag{5.45}$$

In both cases, β denotes the length along the asymptotic boundary, while μ denotes the tension along the brane. The first amplitude corresponds to a purification of the Euclidean disk, and has been used to model pure states in the black hole evaporation process in [34]. This argument is reviewed in section 3.5.
In this thesis, we have obtained these amplitudes using the first-order gauge theoretic formulation of JT quantum gravity, by linking the missing pieces of the puzzle that were largely present throughout the literature.
To start, section 1.5 reviews the discussion due to Maldacena et al. [17], pinning down the holographic description of JT gravity in terms of the Schwarzian boundary theory. In particular, by path integrating over the dilaton field in the bulk term of the action Eq 5.43, the bulk contribution vanishes and the geometry is described by patches of pure AdS_2. The remaining degrees of freedom are captured completely in terms of the conformal reparameterization modes of the wiggly boundary curve. The pulled-back extrinsic curvature along the boundary curve is in this case equivalent to the Schwarzian derivative (c.f. Eq 1.88). The one-loop exact partition function has been obtained by Stanford et al. [20], while the exact quantum amplitudes of multiple bilocal operator insertions in the Schwarzian perspective are calculated by Mertens et al. [21].

Blommaert et al. [22] have obtained the same disk amplitudes of JT gravity with multiple Wilson line insertions in the gauge theoretical BF perspective. Wilson lines in the bulk can consequently be regarded as the bulk duals to bilocal operators of the Schwarzian boundary theory. The disk amplitudes with Wilson line insertions match precisely with the holographic Schwarzian correlators of bilocal operators. The result is a set of diagrammatic rules Eqs 2.284, 2.285, 2.286 that are exact to all order in perturbation theory. This was the first approach towards an exact quantization of JT gravity directly from the bulk perspective, complementing quantization approaches in the dual Schwarzian perspective.
Although the on-shell equivalence between JT gravity and an $\mathfrak{sl}(2,\mathbb{R})$ BF theory had already been established before [78], [79], the exact quantization requires explicit knowledge of the precise exponentiation of the algebra. Blommaert et al. [22] argue that JT gravity is described in terms of the subsemigroup $\text{SL}^+(2,\mathbb{R})$. The quantum amplitudes with multiple Wilson line insertions follow directly from an open-channel perspective that is found to be in harmony with the closed channel perspective of YM_2. The holographic particle-on-a-group description of the BF formulation reduces to the holographic Schwarzian description of JT gravity in a constrained setup. The precise constraints are the coset boundary conditions reviewed in section 2.183.
Blommaert et al. [24] fine-tuned the equivalence between JT gravity and a constrained $\text{SL}^+(2,\mathbb{R})$ BF theory further, by demonstrating that this choice naturally constrains to smooth geometries in the path integral. On a technical level, a motivation in favour of the subsemigroup is that the continuous series representations are the only ones appearing in the Plancherel decomposition.

Using a modification of the analysis of Iliesiu et al. [23], we have related the free particle action in the second order metric formulation along paths diffeomorphic to $\mathcal{C}$ to a Wilson line (loop) insertion in the path integral (c.f. Eqs 3.31, 3.30). This relates the full-fledged path integral of a free particle with mass m to a group theoretical operator insertion evaluated in a lowest-weight discrete series representation of spin-ℓ. The mass and representation label (related to the conformal dimension of the dual bilocal operator) are related according to the standard holographic dictionary 3.5.

When $\mathcal{C}$ is anchored on both sides to the asymptotic boundary, the boundary coset restrictions apply Eq 3.63, and we evaluate the free particle path integral as the preparation of a Wilson line between two lowest-weight states. For $\mathcal{C}$ describing closed loops in the interior, the weight labels are unconstrained and the general Gibbons-Hawking prescription of thermal QFT instructs us to take a trace over the holonomy instead.

We further note that the extrinsic curvature in Eq 5.43 vanishes along the entire trajectory by integrating over the dilaton field, which acts as a Lagrange multiplier in the total path integral. Using the variation of the free particle action Eq 3.44, we argue that this effectively constrains all possible paths in the path integral to the collection of geodesic trajectories Eq 3.45 between two points. Since the boundary conditions on the wiggly boundary are non-unique, we still consider a remaining integration over all possible geodesic lengths.

To effectively localize on geodesics, we ought to take μ large compared to the effective quantum scales in the underlying theory. This identifies the tension one-to-one with the representation label Eq 3.46. Using the identification between the hyperbolic group label ϕ and the geodesic length b Eq 3.60, the boundary-anchored EOW brane path integral with coset boundary conditions in the geodesic limit is, by consistency, identified with the on-shell approximation of the free particle path integral (c.f. Eq 3.67). This is to be expected since the free particle path integral is one-loop exact.

We obtain the half-moon amplitude of an EOW brane attached to the asymptotic boundary by considering the standard result of the two-sided Euclidean black hole with the insertion of a Wilson line Eq 2.276. Performing a $\mathbb{Z}_2$-quotient removes one asymptotic Hartle-Hawking state and identifies the remaining amplitude with the result of [34] (Eq 5.44). Interestingly, we may interpret the Gamma function in the latter as an overlap between matrix elements of the discrete- and continuous series representation.

On the other hand, an EOW brane attached to the neck of a single trumpet partition function exhibits an interesting correction to the classical saddle. First of all, the single trumpet partition function is achieved by inserting a hyperbolic defect in the disk partition function [37]. In the BF formulation, this is equivalent to inserting a suitably normalized continuous series character of a hyperbolic conjugacy class element (c.f. Eq 2.289). We have calculated the relevant continuous series hyperbolic character in $\mathrm{SL}(2,\mathbb{R})$ representation theory in analogy to the techniques of [90] [40] (c.f. 2.323). Earlier calculations in the holographic Schwarzian perspective Eq 2.317 instruct us to consider only the numerator of this character. By removing the denominator, we consequently glue with a flat integration measure in Teichmüller space.

The EOW brane itself is equivalent to a Wilson loop attached to the neck of this trumpet. The equivalence between the free particle path integral and a Wilson loop insertion instructs us to consider the hyperbolic character of an $\mathrm{SL}(2,\mathbb{R})$ lowest-weight discrete series module. Within the geodesic approximation, the correction to the on-shell geodesic approximation Eq 3.85 coincides exactly with the EOW wavefunction of the free-particle formalism [36] (c.f. 5.45).

Although obtaining the EOW brane amplitudes from a group theory perspective might obscure some of the immediate gravitational features of the free particle perspective, it does allow us to generalize the notion of EOW branes directly to applications of JT supergravity, whose action in $2|2$-dimensional superspace is given by [39] (c.f. Eq 4.1):

$$I_{JT}^{\mathcal{N}=1} = \frac{1}{4}\left[\int_{\Sigma} d^2z d^2\theta \; E\Phi(R_{+-}+2) + 2\int_{\partial\Sigma} d\tau d\vartheta \; \Phi K\right]. \tag{5.46}$$

Essentially, the holographic description is given by the super-Schwarzian theory in terms of the superconformal reparametrization modes of the $1|1$-dimensional asymptotic boundary curve. The latter being infinitesimally thickened in the fermionic ϑ-direction. In its first order form, the extrinsic curvature along the asymptotic curve is defined in terms of the superderivative with the additional spin connections (c.f. Eq 4.57).
The procedure to obtain the holographic equivalence with the super-Schwarzian theory, is again to path integrate over the superdilaton field, which cancels the bulk term and imposes the bulk metric to describe patches of the super Poincaré upper-half plane. The remaining degrees of freedom are located entirely along the asymptotic boundary, and should preserve the asymptotic form of the metric (c.f. Eq 4.53). This leads to a wiggly boundary curve parameterized in terms of a single bosonic and fermionic proper time coordinate (c.f. Eq 4.50). Along this wiggly boundary curve, the definition of the extrinsic curvature reduces to the definition of the super-Schwarzian derivative (c.f. Eq 4.67), establishing full off-shell equivalence between $\mathcal{N}=1$ JT SUGRA and the super-Schwarzian boundary theory. The partition function and quantum amplitudes with insertions of bilocal operators in the boundary perspective have been obtained in [20] and [21] respectively.

One can also prove the on-shell equivalence between $\mathcal{N}=1$ JT SUGRA and an $\mathfrak{osp}(1|2,\mathbb{R})$ BF theory (c.f. section 4.3.1). Fan et al. [40] generalized the discussion of Blommaert et al. [22] [24] to obtain the same quantum amplitudes from this bulk first order perspective. First of all, only in a constrained setup does the holographic particle-on-a-group theory reduce to the super-Schwarzian theory, as demonstrated in section 4.3.2 (see in particular the Brown-Henneaux boundary condition Eq 4.101). By again constraining to the subsemisupergroup OSp$^+(1|2,\mathbb{R})$, the Plancherel measure matches precisely with the gravitational density of states Eq 4.137. Moreover, only the continuous series representation appears in the Plancherel decomposition Eq 4.139. The precise quantum amplitudes follow from the open-channel slicing developed for the bosonic case in [22]. The insertion of a hyperbolic defect leads to the creation of a single trumpet amplitude. In particular, from a group-theoretical perspective, the insertion of a hyperbolic defect is equivalent to inserting a hyperbolic character evaluated in the continuous series representation of OSp$(1|2,\mathbb{R})$. Since the group structure for the fermionic coordinates is disconnected into a periodic Ramond (**R**) and an anti-periodic Neveu-Schwarz (**NS**) sector, the hyperbolic class elements are characterized by different values of the supertrace STr (c.f. Eq 4.184). The calculation proceeds slightly differently in both cases, leading to two different spin-structured hyperbolic characters Eqs 4.186, 4.187. One can argue that the Weyl denominator is again irrelevant in a gravitational setting. This leads to two different spin structured single-trumpet amplitudes in each sector Eqs 4.197, 4.198. As a follow up to [40], we argue that the **NS** trumpet is one-loop exact, while the **R** trumpet is only exact in a proper perturbative expansion in increasing order of the inverse coupling $1/\beta$ (c.f. Eq 4.203).

An important realization to extend the notion of EOW branes to superspace is that the geodesics describe

proper $1|0$-dimensional curves in the $2|2$-dimensional superspace manifold Eq 4.207, as opposed to the holographic $1|1$-dimensional UV boundary curves. Therefore, we cannot readily use the definition of the extrinsic supercurvature along the $1|1$-dimensional boundary curve given in Eq 5.46. We set out to find it from first principles following the textbook development of bosonic GR. To obtain a natural coordinate invariant definition, we have opted to consider NW-SE invariant contractions in superspace. We deduce that this necessarily fixes the definition of the (inverse) metric, covector and coordinate transformations to respectively Eqs 4.211, 4.212, 4.214, 4.220.
Within these conventions, a consistent coordinate-invariant definition of the free-particle Lagrangian is given by:

$$L = \dot{Z}^M g_{MN} \dot{Z}^N \tag{5.47}$$

, where Z^M label the collection of bosonic and fermionic coordinates in superspace. Variation of the action leads to the classical geodesic equations of motion Eq 4.228 with a modified Christoffel symbol in superspace Eq 4.227. The novel difference between the bosonic Christoffel symbols is the appearance of $\mathbb{Z}_2$-sign factors that take into account the Grassmann parity of the superspace coordinates. By writing the geodesic equation manifestly coordinate-covariant, the Christoffel symbols readily define a super covariant derivative according to Eq 4.229. Its coordinate transformation rule is fixed by demanding that the variation of the coordinate invariant action Eq 5.47 should remain coordinate invariant after variation. By additionally demanding that the covariant derivative on a scalar reduces to an ordinary (possibly Grassmann) derivative, the covariant derivatives on covectors are also fixed by imposing the natural fermionic Leibnitz rule to hold, according to Eq 4.238. Their correct transformation rule is also fixed under this definition.
We may then use a trick to characterize the variation of the tangent vector field in terms of the variation of the normal vector field, to obtain a proper definition of the pulled-back extrinsic supercurvature along the $1|0$-dimensional EOW brane Eq 4.244:

$$K = \dot{Z}^M \dot{Z}^N \nabla_N n_M . \tag{5.48}$$

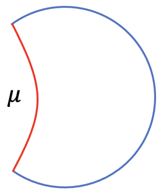
Since this procedure did not need to specify the number of fermionic coordinates on the supermanifold, this is the correct definition in any number of supersymmetry. By construction, a vanishing extrinsic curvature along the EOW brane localizes onto geodesic trajectories in superspace.
We propose that using this definition for the extrinsic supercurvature, the total Euclidean action of an EOW brane in superspace is given by:

$$\boxed{I_{EOW} = \int ds \sqrt{\dot{Z}^M g_{MN} \dot{Z}^N} (\mu - \phi K)} . \tag{5.49}$$

ϕ serves as the bottom component of the dilaton superfield.

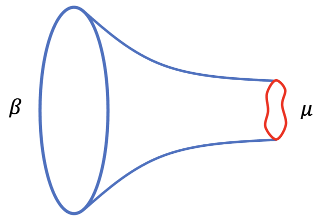
Using a generalization of the set-up in [23], we obtain an operator equivalence between the free particle action in superspace and a Wilson operator insertion in the $\mathcal{N} = 1$ BF path integral, conform with our earlier conventions of the metric. The result Eq 4.283 again depends on the topology of the EOW brane. Anchored to the asymptotic boundary, the coset restrictions apply and by consistency, we find that the exact quantum operator reduces to the on-shell value of the free-particle path integral in the geodesic approximation (c.f. Eq 4.300).

This leads to the final half-moon amplitude Eq 4.302:



$$\simeq \int_0^\infty dk \ \cosh(2\pi k) e^{-\beta k^2} \int_{-\infty}^{\infty} d\phi \left(K_{1/2+2ik}(2e^\phi) + \epsilon_- \epsilon_+ K_{1/2-2ik}(2e^\phi) \right) e^{2\ell\phi}.$$

Once again, we find that the trumpet partition functions are structurally more interesting. For closed loops of EOW branes in the interior, the identity Eq 4.283 instructs us to consider the lowest-weight discrete series character of a hyperbolic class element. We again find two disconnected sectors depending on the periodicity of the fermionic coordinates around the circle Eq 4.310, 4.312, leading to a correction to the classical saddle in the total path integral Eqs 4.313, 4.314. Gluing along the respective single trumpet amplitudes immediately leads to a generalization of the bosonic results Eqs 4.315, 4.316 to $\mathcal{N}=1$ JT SUGRA Eq 5.45:



$$Z^{\mathbf{NS}}_{EOW}(\beta) = \int_0^\infty dk \ e^{-\beta k^2} \int_0^\infty db \cos(bk) \frac{e^{-\mu b}}{2\sinh(b/4)},$$

$$Z^{\mathbf{R}}_{EOW}(\beta) = \int_0^\infty dk \ e^{-\beta k^2} \int_0^\infty db \sin(bk) \frac{e^{-\mu b}}{2\cosh(b/4)}.$$

Notably, the result for the **R** sector does not yield the spurious UV divergence for small geodesic lengths $b \to 0$.

In section 5.1.1, we have made some first steps to extend this discussion to applications of $\mathcal{N}=2$ JT SUGRA. The analysis is slightly more complicated since the relevant $\mathfrak{osp}(2|2,\mathbb{R})$ superalgebra contains two commuting Cartan generators that label the representations. Thus, apart from fixing the quadratic Casimir, we also have to fix the cubic Casimir in the identification between the free particle action and the Wilson operator insertion. The resulting integral is non-Gaussian, and cannot be reduced to the standard form Eq 4.246.
On the other hand, we have rewritten the Wilson operator in terms of a first order metric formulation in Eq 5.41. This is formulated in terms of the configurational Z^M and adjoint momentum k_M coordinates. The quadratic Casimir constraint is translated in terms of the energy-momentum relation $k^M k_M = m^2$. The cubic Casimir constraint is translated in terms of an a priori undetermined relation in gravity Eq 5.40. It would be very interesting to shed further light on this in future work.

Preliminary results in the representation theory of the $\mathrm{OSp}(2|2,\mathbb{R})$ determine the continuous series and highest-weight discrete series character of a hyperbolic group element Eqs 5.14, 5.10 in terms of the Branching rule of the bosonic subalgebra Eq 5.3. It is interesting to note that this group does not decompose into two disconnected subgroups depending on the periodicity of the fermionic coordinates. Instead, all periodicity states are connected by spectral flow.
Using the hyperbolic continuous series character as an operational defect, we obtain the $\mathcal{N}=2$ single trumpet

partition function Eq 5.15. Using the equivalence between a Wilson loop operator in the BF path integral and an $\mathcal{N} = 2$ EOW brane, we obtain the partition function of an EOW loop attached to the neck of this single trumpet amplitude [109]:

$$Z_{EOW}^{\mathcal{N}=2}(\ell, b) = \int dk dq e^{-\beta(k^2-q^2)} \int d\phi d\theta \cos(2k\phi) e^{i2q\theta} \frac{\cosh\phi - \cos\theta}{\sinh\phi} e^{-2\ell\phi} e^{2ib\theta}. \tag{5.50}$$

To deduce the half-moon amplitudes of a single EOW brane anchored to the holographic boundary, we should still find the mixed parabolic Whitakker matrix elements from first principles. This would directly lead to a description of the asymptotic Hartle-Hawking states.

For now, we take comfort in the fact that we have not only obtained an alternative perspective on the bosonic EOW brane quantum amplitudes of both [34] and [36] via group theory, we have also been able to exploit this perspective and extrapolate the notion of EOW branes directly to superspace. This yields the explicit quantum amplitudes in $\mathcal{N} = 1$ JT supergravity. The above preliminary results seem promising to extend this to $\mathcal{N} = 2$ supergravity (and higher).

To be continued

Appendix A

Representation theory of SL$(2,\mathbb{R})$

A.1 Representation theory of SL$(2,\mathbb{R})$

In this appendix, I will delve deeper in the representation theory of SL$(2,\mathbb{R})$ and SL$^+(2,\mathbb{R})$. This will closely follow the approaches of [24] [22] [40], which itself was based largely on [90].

A.1.1 Borel-Weil realization on $L^2(\mathbb{R})$

To develop the representation theory of SL$(2,\mathbb{R})$, we use a realization of the group elements in terms of 2×2 matrices, satisfying the constraint $\det g = 1$;

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \equiv 1. \tag{A.1}$$

The eigenvalues λ of a generic element $g \in$ SL$(2,\mathbb{R})$ are determined by the characteristic equation

$$\begin{aligned} &(a-\lambda)(d-\lambda) - bc = 0 \\ \leftrightarrow \quad &\lambda^2 - \mathrm{Tr}(g)\lambda + 1 = 0 \end{aligned} \tag{A.2}$$

, where we used the defining relation $ad - bc = 1$ and $\mathrm{Tr}(g) = a + d$. The eigenvalues are thereby given by the roots

$$\lambda_\pm = \frac{\mathrm{Tr}(g) \pm \sqrt{\mathrm{Tr}(g)^2 - 4}}{2}. \tag{A.3}$$

Since the eigenvalues are preserved under conjugation, we discriminate three conjugacy classes depending on the sign of the discriminant:
elliptic: $\mathrm{Tr}(g) < 2$, *parabolic*: $\mathrm{Tr}(g) = 2$, and *hyperbolic*: $\mathrm{Tr} > 2$.
Infinitesimally expanding the group elements into the generators $g = 1 + i\epsilon^a J_a$, the constraint $\det g \equiv 1$ is translated to the ambient $\mathfrak{sl}(2,\mathbb{R})$ algebra as the vanishing of the trace of the 2×2 matrices $\mathrm{Tr}(iJ_a) \equiv 0$. Of the

four free parameters, this leaves a three dimensional algebra, spanned in terms of the fundamental generators Eq 2.146:

$$iJ_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad iJ_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad iJ_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{A.4}$$

, satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra[1] Eq 2.147:

$$\boxed{[J_0, J_\pm] = \pm i J_\pm, \qquad [J_+, J_-] = 2iJ_0}. \tag{A.5}$$

The Cartan-Killing metric is defined as before (in the order $0, -, +$):

$$\kappa_{ab} = 2\mathrm{Tr}[(iJ_a)(iJ_b)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{A.6}$$

Indices are raised and lowered using this metric. However, in the following, I will mostly restrict to the lower indices. Taking the inverse of the Cartan-Killing metric κ^{ab} defines the quadratic Casimir in Eq 2.93: $\mathcal{C}_2 = -\kappa^{ab} iJ_a iJ_b$. We construct representations by diagonalizing one of the generators iJ_a and the quadratic Casimir[2], which we can work out explicitly as:

$$\boxed{\mathcal{C}_2 = -\kappa^{ab} iJ_a iJ_b = J_0^2 + \frac{1}{2}\{J_+, J_-\} \equiv -j(j+1)}. \tag{A.7}$$

The ansatz $-j(j+1)$ defines the eigenvalues of the quadratic Casimir in terms of a (possibly complex) parameter j. This label is often denoted as the spin of the representation in analogy to SU(2).
The eigenvalues of the generator to choice are denoted as $J_a = \nu_a$. We readily calculate the quadratic Casimir of the above matrix generators Eq A.4 to form a fundamental spin-1/2 representation.

Apart from other finite-dimensional representations, more general spin-j representations can be projected on the real number line in terms of a square integrable function $f^j_\nu(x) = \langle x \mid j, \nu\rangle \in L^2(\mathbb{R})$, where $|x\rangle$ is introduced as a complete set of states in configuration space, defined in terms of the inner product on $L^2(\mathbb{R})$:

$$\langle j\nu \mid l\mu\rangle = \int_{\mathbb{R}} dx\ \langle j\nu \mid x\rangle\langle x \mid l\mu\rangle = \delta_{jl}\int_{\mathbb{R}} f^j_\nu(x)^* f^l_\mu(x). \tag{A.8}$$

[1] A word of caution; when inserting the imaginary unit in the commutation relations, we obtain $[iJ_0, iJ_\pm] = \mp iJ_\pm$, $[iJ_-, iJ_+] = 2iJ_0$. This is the familiar $\mathfrak{sl}(2,\mathbb{R})$ algebra, with the understanding that iJ_+ corresponds to a lowering operator, while iJ_- corresponds to a raising operator, despite their notation. This should not pose any difficulties, but should be kept in mind.

[2] The quadratic Casimir commutes with all generators of the algebra. Indeed, the Cartan-Killing metric satisfies the important invariance property $\kappa([iJ_a, iJ_b], iJ_c) = \kappa(iJ_a, [iJ_b, iJ_c])$. We define the structure constants of the algebra as $[iJ_a, iJ_b] = f_{ab}{}^c iJ_c$. From the commutator, these are anti-symmetric in the first pair of indices (not necessarily in all three indices!). In terms of the structure constants, the invariance property becomes $f_{ab}{}^d\kappa_{dc} = f_{bc}{}^d\kappa_{ad}$. Lowering indices with respect to the Cartan-Killing metric $f_{abc} = f_{bca}$, we see that they are completely anti-symmetric for all indices lowered. The quadratic Casimir is now readily seen to commute with all generators: $[\mathcal{C}_2, iJ_c] = -\kappa^{ab}[iJ_a iJ_b, iJ_c] = -\kappa^{ab}(iJ_a f_{bc}{}^d iJ_d + f_{ac}{}^d iJ_d iJ_b) = -iJ^b(f_{bcd} + f_{dcb})iJ^d = 0$. From antisymmetry of the lowered generators the commutator vanishes.

The action of the SL$(2, \mathbb{R})$ matrices Eq A.1 on $L^2(\mathbb{R})$ is defined as:

$$\boxed{\langle x \mid g \mid j\nu\rangle = (g \cdot f_\nu^j)(x) = ((\hat{T}_j(g)) f_\nu^j)(x) = |bx+d|^{2j} f_\nu^j \left(\frac{ax+c}{bx+d}\right)}. \tag{A.9}$$

The above realization of SL$(2, \mathbb{R})$ on $L^2(\mathbb{R})$ is refered to as the *principal series representation*[3]. It acts projectively by its transpose as $X^T g$, with $X^T = (x \quad z)$. Taking the transpose ensures that the group action composes naturally under group multiplication. We will see that it defines a representation on the group elements, acting on the space of square integrable functions $L^2(\mathbb{R})$:

$$\hat{T}_j : g \in SL(2, \mathbb{R}) \mapsto \hat{T}_j(g).$$

We can interpret the coefficients $\langle x \mid g \mid j\nu\rangle$ as the Fourier components corresponding to the action of $g \in$ SL$(2, \mathbb{R})$ on $f \in L^2(\mathbb{R})$, according to:

$$\langle j\nu \mid g \mid l\nu\rangle = \delta_{jl} \int_{\mathbb{R}} f_\nu^j(x)^* (g \cdot f_\mu^l)(x). \tag{A.10}$$

Infinitesimally expanding into the generators according to $g = \mathbf{1} + \epsilon^a i J_a$ leads to the *Borel-Weil* realization of the $\mathfrak{sl}(2, \mathbb{R})$ algebra[4] [24]:

$$\boxed{iJ_- = \partial_x, \qquad iJ_0 = -x\partial_x + j, \qquad iJ_+ = -x^2\partial_x + 2jx}. \tag{A.11}$$

We readily check that the above generators actually correspond to a spin-j representation by calculating explicitly the quadratic Casimir;

$$\begin{aligned} \mathcal{C}_2 \equiv J_0^2 + \frac{1}{2}\{J_+, J_-\} &= -(-x\partial_x + j)(-x\partial_x + j) - \frac{1}{2}\partial_x(-x^2\partial_x + 2jx) - \frac{1}{2}(-x^2\partial_x + 2jx)\partial_x \\ &= -x\partial_x - x^2\partial_x^2 + xj\partial_x - j^2 + x\partial_x + \frac{x^2}{2}\partial_x^2 - j + \frac{x^2}{2}\partial_x^2 - jx\partial_x \\ &= -j(j+1). \end{aligned} \tag{A.12}$$

They furthermore satisfy the algebra Eq A.5. Last but not least, the Borel-Weil realization on SL$(2, \mathbb{R})$ satisfies the defining property of representation matrices $R(g_1 \cdot g_2) = R(g_1)R(g_2)$ for $g_1, g_2 \in$ SL$(2, \mathbb{R})$. This can be

[3]Note that the above realization is in fact a realization of the projective group PSL$(2, \mathbb{R})$, where the results are invariant under $g \to -g$. I will continue to denote it SL$(2, \mathbb{R})$ however.

[4]For example, working out iJ_0 explicitly, we know that the group action $g = \mathbf{1} + \epsilon i J_0 = \begin{pmatrix} 1-\epsilon/2 & 0 \\ 0 & 1+\epsilon/2 \end{pmatrix}$ leads to the projective action on $f^j(x) \to |1+\epsilon/2|^{2j} f^j\left(\frac{(1-\epsilon/2)x}{1+\epsilon/2}\right) = (1+\epsilon/2)^{2j} f((1-\epsilon)x)$. Taylor expanding gives $f^j(x) \to f^j(x) + \epsilon\left(jf^j(x) - x\partial_x f^j(x)\right)$, from which we identify $iJ_0 = j - x\partial_x$.

verified explicitly by calculating the action of two subsequent 2×2 SL$(2,\mathbb{R})$ matrices Eq A.1 on $L^2(\mathbb{R})$:

$$\begin{aligned}\langle x \mid g_1 \cdot g_2 \mid f\rangle &= g_1 \cdot \left(|b_2 x + d_2|^{2j} f\left(\frac{a_2 x + c_2}{b_2 x + d_2}\right)\right), & g_2 &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= |b_1 x + d_1|^{2j} \left|b_2 \frac{a_1 x + c_1}{b_1 x + d_1} + d_2\right|^{2j} f\left(\frac{a_2 \frac{a_1 x + c_1}{b_1 x + d_1} + c_2}{b_2 \frac{a_1 x + c_1}{b_1 x + d_1} + d_2}\right), & g_1 &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \\ &= |x(a_1 b_2 + b_1 d_2) + (d_1 d_2 + b_2 c_1)|^{2j} f\left(\frac{x(b_1 c_2 + a_1 a_2) + (c_2 d_1 + a_2 c_1)}{x(a_1 b_2 + b_1 d_2) + (c_1 b_2 + d_1 d_2)}\right).\end{aligned}$$

Of course, the latter is precisely the transformation law of f under the composition

$$g_1 g_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

Unitary of the group representations further constrains the allowed values of j. Indeed, unitary of the representation matrices demands the generators iJ_a to be anti-Hermitian (operators J_a Hermitian): $g^\dagger g = (1 - i\epsilon J^\dagger)(1 + i\epsilon J) \equiv 1$. Checking the anti-hermiticity of iJ_0 on the inner product of $L^2(\mathbb{R})$ yields:

$$\begin{aligned}\int_{\mathbb{R}} dx\ f(x)^*(-x\partial_x + j)g(x) &= \int_{\mathbb{R}} dx\ g(x)\partial_x(xf(x)^*) + jg(x)f(x)^* \\ = \int_{\mathbb{R}} dx\ g(x)f(x)^* + g(x)x\partial_x f(x)^* + jg(x)f(x)^* &\equiv -\left(\int_{\mathbb{R}} dx\ g(x)^*(-x\partial_x + j)f(x)\right)^*\end{aligned}$$

, where we assumed that the relevant functions decay fast enough as $|x| \to \infty$ to ignore the contribution from possible boundary terms. The last equality is just the definition of an anti-Hermitian inner product. It is readily seen that in order for both lines to be consistent, we should have that $j + 1 \equiv -j^*$, or:

$$\boxed{j = -\frac{1}{2} + ik} \qquad k \in \mathbb{R}. \tag{A.13}$$

This defines the principal continuous series representation.
Changing variables to $x = \frac{x'd - c}{-bx' + a}$ shows that the adjoint action is given by g^{-1} [40]:

$$\int_{\mathbb{R}} dx\ f(x)^*|bx + d|^{2j} g\left(\frac{ax + c}{bx + d}\right) \to \int_{\mathbb{R}} dx' \left(|-bx' + a|^{2j} f\left(\frac{dx' - c}{-bx' + a}\right)\right)^* g(x')$$

, where we recognize

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{A.14}$$

Indeed, the Jacobian under the transformation above is given by[5] $dx = \frac{dx'}{|-bx'+a|^2}$, while the prefactor transforms as $|bx + d|^{2j} \to \frac{1}{|-bx'+a|^{2j}}$. Using $j + 1 = -j^*$ readily yields the above result. Therefore, the adjoint action is indeed obtained by the inverse group element iff $j + 1 \equiv -j^*$ or $j =$ A.13.
A more pragmatic approach to construct the unitary principal series representation on SL$(2,\mathbb{R})$, is to show that

[5] Using the SL$(2,\mathbb{R})$ constraint $ad - bc = 1$.

the non-compact picture descends from a normalized induced representation by a parabolic subgroup. This representation will then automatically be contained within the left regular representation of the group. More details for the case of both SL$(2,\mathbb{R})$ and OSp$(1|2,\mathbb{R})$ can be found in appendix E of [40].

A.1.2 Principal Series Representation

Matrix elements and the Plancherel measure

Focusing on the principal continuous series representation for now with $j = -\frac{1}{2} + ik$, we would like to calculate the Plancherel measure using the orthogonality of the matrix elements in the grand-orthogonality theorem Eq 2.200. To this end, we should specify a basis of the k-representation, obtained by diagonalizing one of the generators iJ_a. Ultimately, this choice is arbitrary in the Peter-Weyl decomposition. However, with the gravitational coset constraints in mind, we choose to diagonalize the mixed parabolic basis of iJ_-, iJ_+. Specifically, the group element corresponding to $iJ_\pm$ is obtained by exponentiating $h_\pm = \exp(itJ_\pm)$, yielding respectively:

$$h_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad h_+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{A.15}$$

The action of $h_-(t)$ on $f^k_{\nu_-}(x)$ is:

$$\langle x \mid h_-(t) \mid f^k_{\nu_-}\rangle = (h_-(t) \cdot f^k_{\nu_-})(x) = f^k_{\nu_-}(x+t). \tag{A.16}$$

The action is readily diagonalized in terms of a plane wave basis $f^k_{\nu_-}(x) = e^{i\nu_- x}$, with the associated eigenvalue under $J_- = \nu_-$: $(h_-(t) \cdot f^k_{\nu_-})(x) = e^{i\nu_- t} f^k_{\nu_-}(x)$. We denote this state on $L^2(\mathbb{R})$ as:

$$\boxed{\langle x \mid \nu_-\rangle = e^{i\nu_- x}}. \tag{A.17}$$

On the other hand, the action of $h_+(t)$ on the carrier space is given by:

$$\langle x \mid h_+(t) \mid f^k_\nu\rangle = (h_+(t) \cdot f^k_{\nu_+})(x) = |tx+1|^{2j} f^j_{\nu_+}\left(\frac{x}{tx+1}\right). \tag{A.18}$$

This is readily diagonalized in terms of functions $f^k_{\nu_+}(x) = |x|^{2ik-1} e^{i\nu_+/x}$, with eigenvalue $J_+ = \nu_+$. Indeed, using $j = -\frac{1}{2} + ik$ yields the action on $f^k_{\nu_+}$:

$$\begin{aligned}(h_+(t) \cdot f^k_{\nu_+})(x) &= |tx+1|^{2j} \left|\frac{x}{tx+1}\right|^{2j} e^{\frac{i\nu_+}{x}(tx+1)} \\ &= e^{i\nu_+ t}(|x|^{2ik-1} e^{i\nu_+/x}).\end{aligned}$$

The corresponding state vector on $L^2(\mathbb{R})$ is:

$$\boxed{\langle x \mid \nu_+\rangle = |x|^{2j} e^{i\nu_+/x} = |x|^{2ik-1} e^{i\nu_+/x}}. \tag{A.19}$$

The eigenstates of $iJ_\pm$ are not independent, and one can transform an iJ_- eigenstate into an iJ_+ eigenstates and vice versa by applying the following SL$(2,\mathbb{R})$ transformation

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.20}$$

, where we see from $\langle x \mid \omega \mid \nu_- \rangle = |x|^{2j} e^{-i\nu_-/x} = \langle x \mid -\nu_+ \rangle$ that:

$$\omega \left| \nu_- \right\rangle = \left| -\nu_+ \right\rangle . \tag{A.21}$$

Matrix elements in the mixed parabolic basis $\langle \nu_- \mid g(\phi, \gamma_-, \gamma_+) \mid \nu_+ \rangle$ are convenient to parameterize in the Gauss decomposition in terms of real parameters ϕ, γ_+, γ_- (c.f. Eq 2.148):

$$g(\phi, \gamma_-, \gamma_+) = e^{\gamma_- iJ_-} \; e^{\phi 2iJ_0} \; e^{\gamma_+ iJ_+}.$$

Being eigenvectors of respectively $J_\pm$, we can write directly (using the hermiticity of J_-)

$$\langle \nu_- \mid g(\phi, \gamma_-, \gamma_+) \mid \nu_+ \rangle = e^{i\gamma_- \nu_-} e^{i\gamma_+ \nu_+} \langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+ \rangle . \tag{A.22}$$

The matrix element in the mixed parabolic basis diagonalizes up to the evaluation of the hyperbolic matrix element. The eigenvectors $|\nu_\pm\rangle$ are the so-called the Whittaker vectors, while the hyperbolic matrix element is called the Whittaker function.
Since the generator iJ_0 defined as Eq A.4 is diagonal, it readily exponentiates to:

$$e^{2i\phi J_0} = \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} . \tag{A.23}$$

This always resides in the hyperbolic conjugacy class by the standard inequality $e^{-\phi} + e^{\phi} \geqslant 2$. This has an action on $f^k_{\nu_+}(x)$ in terms of:

$$\begin{aligned} f^k_{\nu_+}(x) \to (g \cdot f^k_{\nu_+})(x) &= |e^{\phi}|^{2j} f^k_{\nu_+}\left(e^{-2\phi} x\right) \\ &= |e^{\phi}|^{2j} \; |e^{-2\phi} x|^{2j} e^{i\nu_+ e^{2\phi}/x}. \end{aligned} \tag{A.24}$$

The overlap $\langle \nu_- \mid \nu_+ \rangle$ can be calculated by inserting a complete set of states $\mathbf{1} = \int_{\mathbb{R}} dx \;\; |x\rangle\langle x|$ and splitting the integral over $\mathbb{R}^-$ and $\mathbb{R}^+$:

$$\langle \nu_- \mid \nu_+ \rangle = \int_{-\infty}^{+\infty} dx \; |x|^{2ik-1} e^{-i\nu_- x} e^{i\nu_+/x} = \int_0^{+\infty} dx \; x^{2ik-1} e^{i\nu_- x} e^{-i\nu_+/x} + \int_0^{+\infty} dx \; x^{2ik-1} e^{-i\nu_- x} e^{i\nu_+/x}.$$

We use the integral representation of the modified Bessel function of the second kind (for $\nu_\pm > 0$) [24]:

$$\int_0^{+\infty} dx \; x^{2ik-1} e^{-\nu_- x} e^{-\nu_+/x} = \left(\frac{\nu_+}{\nu_-}\right)^{ik} K_{2ik}\left(\sqrt{\nu_- \nu_+}\right). \tag{A.25}$$

Analytically continuing $\nu_- \to -i\nu_-$, $\nu_+ \to i\nu_+$ in the above integral identity to yield the first integral and

$\nu_- \to i\nu_-$, $\nu_+ \to -i\nu_+$ to yield the second integral, eventually gives [24]:

$$\begin{aligned}\langle \nu_- \mid \nu_+ \rangle &= \left(\left(\frac{e^{i\pi/2}}{e^{-i\pi/2}} \right)^{ik} + \left(\frac{e^{-i\pi/2}}{e^{i\pi/2}} \right)^{ik} \right) \left(\frac{\nu_+}{\nu_-} \right)^{ik} K_{2ik} \left(\sqrt{\nu_- \nu_+} \right) \\ &\simeq \cosh(\pi k) \left(\frac{\nu_+}{\nu_-} \right)^{ik} K_{2ik} \left(\sqrt{\nu_- \nu_+} \right). \end{aligned} \tag{A.26}$$

The Whittaker function is then conveniently calculated from this overlap by considering the action of the hyperbolic group element Eq A.24 and shifting $x \to e^\phi x$ in the integral:

$$\langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+ \rangle = \int_{\mathbb{R}} dx\, e^{-i\nu_- x} |e^\phi|^{2j} |e^{-2\phi} x|^{2j} e^{i\nu_+ e^{2\phi}/x} \to e^\phi \int_{\mathbb{R}} dx\, e^{-i\nu_- e^\phi x} |x|^{2j} e^{i\nu_+ e^\phi / x}. \tag{A.27}$$

This is the same matrix element as the overlap $\langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+ \rangle = e^\phi \langle \nu_- e^\phi \mid \nu_+ e^\phi \rangle$, yielding the total SL$(2, \mathbb{R})$ matrix element in the mixed parabolic basis, up to normalization:

$$\boxed{\langle \nu_- \mid g(\phi, \gamma_-, \gamma_+) \mid \nu_+ \rangle = e^{i\gamma_- \nu_-} e^{i\gamma_+ \nu_+}\ e^\phi \cosh(\pi k) \left(\frac{\nu_+}{\nu_-} \right)^{ik} K_{2ik} \left(\sqrt{\nu_- \nu_+} e^\phi \right)}. \tag{A.28}$$

From the orthonormality relation [40]

$$\int_0^\infty \frac{dx}{x} K_{2ik}(x) K_{2ik'}(x)^* = \int_{-\infty}^{+\infty} d\phi K_{2ik}(e^\phi) K_{2ik'}(e^\phi)^* = \frac{\pi^2}{8k \sinh(2\pi k)} \delta(k - k') \tag{A.29}$$

, and the Fourier representation of the delta function $\frac{1}{2\pi} \int_{-\infty}^{+\infty} d\gamma_-\ e^{i(\nu_- - \nu_-')\gamma_-} \simeq \delta(\nu_- - \nu_-')$, we conclude that up to some k-independent prefactor,

$$\int dg\, \langle k\nu_- \mid g \mid k\nu_+ \rangle \langle k'\nu_-' \mid g \mid k'\nu_+' \rangle^* \simeq \frac{\delta(k - k')}{\rho(k)} \delta(\nu_- - \nu_-') \delta(\nu_+ - \nu_+') \tag{A.30}$$

with

$$\boxed{\rho(k) = \frac{k \sinh(2\pi k)}{\cosh^2(\pi k)} \simeq k \tanh(\pi k)}. \tag{A.31}$$

We may identify $\rho(k)$ with the Plancherel measure in the grand-orthogonality theorem Eq 2.200. The measure dg in the integral is the usual Haar measure $dg = e^{-2\phi} d\phi d\gamma_- d\gamma_+$, deduced in Eq 2.150.

There is a subtlety however, since the Gauss decomposition only covers the Poincaré patch and not the entire SL$(2, \mathbb{R})$ manifold, where in general $\exp(\gamma_- iJ_- + 2i\phi J_0 + \gamma_+ iJ_+) \neq e^{\gamma_- iJ_-}\ e^{\phi 2iJ_0}\ e^{\gamma_+ iJ_+}$ due to the Campbell-Baker-Hausdorff formula. It turns out that the entire SL$(2, \mathbb{R})$ manifold is covered by four patches, obtained by shifting the Poincaré matrix element by either

$$\begin{aligned} g(\phi, \gamma_-, \gamma_+) &\to g(\phi, \gamma_-, \gamma_+) \cdot \pm\omega \\ &\to g(\phi, \gamma_-, \gamma_+) \cdot (\pm \mathbf{1}). \end{aligned}$$

In the PSL$(2, \mathbb{R})$ representation theory, these are two-by-two the same. Summing the contributions of all patches gives a factor of four [24], yielding up to normalization the same Plancherel measure as before.

Hyperbolic basis

We could have also constructed eigenfunctions in the hyperbolic basis, associated to diagonalizing iJ_0. More specifically, since iJ_0 in Eq A.4 is diagonal, the group element obtained by exponentiating $g = \exp(2itJ_0)$ is immediately:

$$g = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}. \tag{A.32}$$

The properly normalized eigenvectors on $\mathbb{R}^{\pm}$ are (with $s \in \mathbb{R}$):

$$f_s^k(x) = \langle x \mid s, \pm \rangle = \frac{1}{\sqrt{2\pi}} (\pm x)^{is-1/2}, \qquad \langle s, \pm \mid x \rangle = \frac{1}{\sqrt{2\pi}} (\pm x)^{-is-1/2}, \quad \pm x > 0. \tag{A.33}$$

Its eigenvalue under g is given by considering the group action:

$$(g \cdot f_s^k)(x) = \frac{1}{\sqrt{2\pi}} e^{2jt} |e^{-2t} x|^{is-1/2} = \frac{1}{\sqrt{2\pi}} e^{2jt} e^{t} e^{-2ist} |x|^{is-1/2}.$$

Inserting $j = -\frac{1}{2} + ik$, the eigenvalue under iJ_0 is $iJ_0 = i(k-s)$. They are orthogonal on either $\mathbb{R}^+$, $\mathbb{R}^-$ from the delta function identity [40]

$$\langle s_1, \pm \mid s_2, \pm \rangle = \pm \int_0^{\pm\infty} dx \, \langle s_1, \pm \mid x \rangle \langle x \mid s_2, \pm \rangle = \frac{1}{2\pi} \int_0^{\pm\infty} \frac{dx}{x} (\pm x)^{-i(s_1 - s_2)} = \delta(s_1 - s_2) \tag{A.34}$$

, and are complete in the sense that

$$\langle x \mid x' \rangle = \int_{-\infty}^{+\infty} ds \, \langle x \mid s, \pm \rangle \langle s, \pm \mid x' \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds (\pm x)^{is-1/2} (\pm x')^{-is-1/2} = \delta(x - x'). \tag{A.35}$$

Therefore, the hyperbolic basis naturally decomposes into a basis of states associated to either $\mathbb{R}^{\pm}$. This allows us to write generic matrix elements as a 2×2 matrix

$$\mathbf{K}(g) = \begin{pmatrix} K^{++} & K^{+-} \\ K^{-+} & K^{--} \end{pmatrix} \tag{A.36}$$

, whose components are defined as:

$$K^{\pm\pm}_{s_1 s_2}(g) \equiv \langle s_1, \pm \mid g \mid s_2, \pm \rangle. \tag{A.37}$$

This matrix composes under group multiplication in terms of matrix multiplication $\mathbf{K}(g_1 \cdot g_2) = \mathbf{K}(g_1)\mathbf{K}(g_2)$ [22], and with inverse $\mathbf{K}(g^{-1}) = \mathbf{K}(g)^{-1}$ [24].

A.2 Representation theory of SL$^+(2,\mathbb{R})$

The representation theory of SL$^+(2,\mathbb{R})$ is very similar to the one developed for SL$(2,\mathbb{R})$, so I will be brief. Group elements $g \in$ SL$^+(2,\mathbb{R})$ are still represented by sets of 2×2 matrices with unit determinant:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(g) = 1 \tag{A.38}$$

, with the additional restriction that all matrix entries are strictly positive $a,b,c,d > 0$. This satisfies closure, the existence of the identity element and associativity. However, there exists no proper inverse in SL$^+(2,\mathbb{R})$, therefore lacking the requirements of a proper group structure. One therefore refers to SL$^+(2,\mathbb{R})$ as a *semi*group. Despite the lack of an inverse in SL$^+(2,\mathbb{R})$, we can define the inverse g^{-1} for every $g \in$ SL$^+(2,\mathbb{R})$ from the parent SL$(2,\mathbb{R})$ manifold. The latter contains SL$^+(2,\mathbb{R})$ as a subset, and hence a better nomenclature would be to refer to it as a subsemigroup.
Either way, the algebra corresponding to the infinitesimal expansion into generators $g = \mathbf{1} + i\epsilon^a J_a$ still corresponds to the conventional $\mathfrak{sl}(2,\mathbb{R})$ algebra Eq A.5.
We again define the *principal series representation* of SL$^+(2,\mathbb{R})$ on $L^2(\mathbb{R}^+)$ in terms of the *Borel-Weil* action Eq A.9, but restricted to positive $x > 0$:

$$\langle x \mid g \mid j\nu\rangle = (g \cdot f_\nu^j)(x) = |bx+d|^{2j} f_\nu^j\left(\frac{ax+c}{bx+d}\right), \qquad x > 0. \tag{A.39}$$

The corresponding infinitesimal generators $iJ_0, iJ_\pm$ are still given by Eq A.11. The action of this representation is closed on $\mathbb{R}^+$ due to the positivity of the matrix entries, and therefore well-defined. An attractive feature of restricting to SL$^+(2,\mathbb{R})$ is that the Gauss parametrization Eq 2.148 (with $\gamma_\pm > 0$) now covers the entire manifold [22], and we may generically write for every $g \in$ SL$^+(2,\mathbb{R})$:

$$g = e^{\gamma_- iJ_-}\, e^{\phi 2iJ_0}\, e^{\gamma_+ iJ_+} = \begin{pmatrix} 1 & 0 \\ \gamma_- & 1 \end{pmatrix}\begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix}\begin{pmatrix} 1 & \gamma_+ \\ 0 & 1 \end{pmatrix}, \quad \gamma_+, \gamma_- > 0. \tag{A.40}$$

Note that there is no restriction on ϕ since the hyperbolic group element is strictly positive for any ϕ.
An important distinction however, is that the inner product on the carrier space $L^2(\mathbb{R}^+)$ is restricted to $x > 0$: $\langle f \mid g\rangle = \int_0^\infty dx\ f(x)^* g(x)$. Therefore, only the hyperbolic generator iJ_0 is anti-hermitian on $\mathbb{R}^+$, since the power of x in $iJ_\pm$ either grows too fast or too slow as $x \to 0$ and $x \to +\infty$ [40]. As a consequence, only the hyperbolic vectors furnish a complete orthogonal basis on SL$^+(2,\mathbb{R})$. Restricting to $x > 0$, these are Eq A.33:

$$f_s^k(x) = \langle x \mid s\rangle = \frac{1}{\sqrt{2\pi}}\, x^{is-1/2}, \qquad f_s^k(x)^* = \langle s \mid x\rangle = \frac{1}{\sqrt{2\pi}} x^{-is-1/2}. \tag{A.41}$$

The corresponding matrix elements on $\mathbb{R}^+$ are now restricted to $K^{++}_{s_1,s_2}(g)$:

$$K^{++}_{s_1 s_2}(g) = \langle s_1 \mid g \mid s_2\rangle = \frac{1}{2\pi}\int_0^{+\infty} dx\ x^{-is_1-1/2}(g \cdot x^{is_2-1/2}) \tag{A.42}$$

, since for positive $g > 0$, the action of g on $f \in L^2(\mathbb{R}^2)$ cannot change the sign in the integral. As a consequence, the overlaps vanish in this case $\langle s_1, \pm \mid g \mid s_2, \mp\rangle \equiv 0$ [44].
The matrix composition law $\mathbf{K}(g_1 \cdot g_2) = \mathbf{K}(g_1)\mathbf{K}(g_2)$ is now constrained to K^{++}, and yields a proper representation of irreps on SL$^+(2,\mathbb{R})$:

$$K^{++}_{ab}(g_1 \cdot g_2) = \int_{-\infty}^{+\infty} ds\ K^{++}_{as}(g_1)K^{++}_{sb}(g_2).$$

Matrix elements can be evaluated by inserting a complete set of states on $\mathbb{R}^+$:

$$K^{++}_{s_1 s_2}(g) = \langle s_1 \mid g \mid s_2\rangle = \frac{1}{2\pi}\int_0^{+\infty} dx\ x^{-is_1-1/2}(g \cdot x^{is_2-1/2}). \tag{A.43}$$

Again parameterizing the group manifold in terms of the Gauss decomposition 2.148
$g(\phi,\gamma_+,\gamma_-) = e^{i\gamma_- J_-}e^{2i\phi J_0}e^{i\gamma_+ J_+}$, we may calculate each constituent individually, and take the continuous matrix product[6]

$$K^{++}_{s_1,s_2}(g) = \frac{1}{2\pi}\int_{\mathbb{R}} ds' ds'' K^{++}_{s_1 s'}(\gamma_-)K^{++}_{s's''}(\phi)K^{++}_{s''s_2}(\gamma_+). \tag{A.47}$$

In [24], it was shown that the matrix elements obtained in Eq A.47 are in fact unitary, meaning that

$$\int_{\mathbb{R}} ds\ K^{++}_{s_1 s}(g)K^{++}_{s_2 s}(g)^* = \delta(s_1 - s_2).$$

[6] For the sake of completeness, let me work them out explicitly. Using Eq A.9, we readily evaluate

$$K^{++}_{s's''}(\phi) = \frac{1}{2\pi}\int_0^{\infty} dx\ x^{-is'-1/2}e^{2j\phi}e^{-2\phi(is''-1/2)}x^{is''-1/2} = \frac{1}{2\pi}e^{2i\phi(k-s'')}\int_0^{\infty}\frac{dx}{x}x^{i(s''-s')}$$

, after inserting $j = -\frac{1}{2} + ik$. Using again the delta function identity Eq A.34, this readily becomes

$$K^{++}_{s's''}(\phi) = \frac{1}{2\pi}e^{2i\phi(k-s'')}\delta(s'-s'') \tag{A.44}$$

On the other hand, evaluating $K^{++}_{s_1 s'}(\gamma_-)$ and $K^{++}_{s_1 s'}(\gamma_+)$ requires the relation of the Euler beta-function [40] with the Euler Gamma functions

$$\beta(x,y) = \int_0^{\infty}\frac{t^{x-1}}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Using the action of $g(\gamma_-)\cdot f(x) = f(x+\gamma_-)$, and after substituting $x = \gamma_- t$;

$$\begin{aligned} K^{++}_{s_1 s'}(\gamma_-) &= \frac{1}{2\pi}\int_0^{\infty} dx\ x^{-is_1-1/2}(x+\gamma_-)^{is'-1/2} = \frac{1}{2\pi}\gamma_-^{i(s'-s_1)}\int_0^{\infty}\frac{t^{-1/2-is_1}}{(1+t)^{1/2-is'}} \\ &= \frac{1}{2\pi}\gamma_-^{i(s'-s_1)}\frac{\Gamma(1/2-is_1)\Gamma(is_1-is')}{\Gamma(1/2-is')} \end{aligned} \tag{A.45}$$

Using the action of $g(\gamma_+)\cdot f(x) = |\gamma_+ x + 1|^{2j} f\left(\frac{x}{\gamma_+ x+1}\right)$, and substituting $t = x/\gamma_+$ similarly leads to

$$\begin{aligned} K^{++}_{s''s_2}(\gamma_+) &= \frac{1}{2\pi}\int_0^{\infty} dx\ x^{-is''-1/2}\frac{(\gamma_+ x+1)^{-1+2ik}}{(\gamma_+ x+1)^{is_2-1/2}}x^{is_2-1/2} = \frac{1}{2\pi}\gamma_+^{i(s''-s_2)}\int_0^{\infty} dt\ \frac{t^{i(s_2-s'')-1}}{(1+t)^{1/2-2ik+is_2}} \\ &= \frac{1}{2\pi}\gamma_+^{i(s''-s_2)}\frac{\Gamma(i(s_2-s''))\Gamma(1/2-2ik+is'')}{\Gamma(1/2-2ik+is_2)} \end{aligned} \tag{A.46}$$

In gravitational applications, we will mainly be interested in mixed parabolic matrix elements. The eigenvectors $|\nu_-\rangle$, respectively $|\nu_+\rangle$ of $J_\pm$ are still well-defined; rather they do not constitute a basis since they are not complete nor orthogonal. The left and right parabolic states on $\mathbb{R}^+$, corresponding to eigenfunctions of J_- and J_+, with eigenvalue $J_- = i\nu_-$, and $J_+ = i\nu_+$ respectively, are given by Eqs A.17, A.19 in the region $x > 0$ by shifting the eigenvalues under $J_\pm$: $\nu_- \to i\nu_-$, $\nu_+ \to i\nu_+$:

$$\langle x \mid \nu_-\rangle = f^j_{\nu_-}(x) = e^{-\nu_- x}, \qquad \langle x \mid \nu_+\rangle = f^j_{\nu_+}(x) = x^{2ik-1}e^{-\nu_+/x}. \tag{A.48}$$

The eigenstates above are again refered to as *Whittaker* vectors. The eigenvalue under the right action of iJ_- on the adjoint state $\langle \nu_- \mid x\rangle$ is deduced from the adjoint of J_- (deduced from the parent SL$(2,\mathbb{R})$):

$$\langle \nu_- \mid x\rangle iJ_- = (-iJ_- \langle x \mid \nu_-\rangle)^* = \nu_- \langle \nu_- \mid x\rangle .$$

The imaginary shift to exponentially damped basis states is a more natural choice than the plane wave basis, since the latter is strictly speaking not even in $L^2(\mathbb{R}^+)$. However, note that both the plane-wave, as well as the exponentially damped states are neither complete nor orthogonal on $\mathbb{R}^+$. However, we can expand them in a complete basis of hyperbolic states $|\nu_\pm\rangle = \int_{\mathbb{R}} ds \langle s \mid \nu_\pm\rangle |s\rangle$. The overlaps describe the analogue of the transition from Minkowski to Rindler coordinates, and are worked out explicitly in [24].

Matrix elements in the mixed parabolic basis are readily computed using the Gauss parametrization:

$$R_{k,\nu_-\nu_+}(g) = \langle \nu_- \mid g(\phi,\gamma_+,\gamma_-) \mid \nu_+\rangle = e^{\nu_-\gamma_-}e^{-\nu_+\gamma_+} \langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+\rangle . \tag{A.49}$$

The matrix element is again diagonalized up to the hyperbolic Whittaker function. Again shifting $x \to xe^\phi$, the result can be found immediately without having to split the integral in two parts along the positive and negative axis:

$$\langle \nu_- \mid e^{2i\phi J_0} \mid \nu_+\rangle = \int_{\mathbb{R}^+} dx\, e^{-\nu_- x}(e^\phi)^{2j}(e^{-2\phi}x)^{2j}e^{-\nu_+ e^{2\phi}/x} \to e^\phi \int_{\mathbb{R}^+} dx\, e^{-\nu_- e^\phi x}x^{2j}e^{-\nu_+ e^\phi/x}. \tag{A.50}$$

Inserting $j = -1/2 + ik$, and using the integral identity Eq A.25 yields:

$$\boxed{R_{k,\nu_-\nu_+}(g) = e^{\nu_-\gamma_-}e^{-\nu_+\gamma_+}e^\phi \left(\frac{\nu_+}{\nu_-}\right)^{ik} K_{2ik}(\sqrt{\nu_-\nu_+}e^\phi)}. \tag{A.51}$$

The result is normalized with respect to the Haar measure $dg = e^{-2\phi}d\phi d\gamma_- d\gamma_+$ defined in Eq A.29. The corresponding the Plancherel measure is readily deduced from the normalization identity of the Bessel functions Eq A.29:

$$\boxed{\rho(k) \simeq k \sinh(2\pi k)}. \tag{A.52}$$

Both the evaluation of the matrix elements and the Plancherel measure on SL$^+(2,\mathbb{R})$ considerably simplify with respect to the previous evaluation for SL$(2,\mathbb{R})$ since the integral identities do not need to be split along the positive and negative real axis.

Appendix B

Representation theory of OSp$(1|2,\mathbb{R})$

The following two sections summarize the representation theory of the orthosymplectic supergroup OSp$(1|2,\mathbb{R})$ and of the subsemisupergroup OSp$^+(1|2,\mathbb{R})$, after the broad discussion in appendix E of [40]. Besides summarizing this source, I have worked out a lot of the non-trivial steps in detail for clarity. I HAVE also used an isomorphism of the $\mathfrak{osp}(1|2,\mathbb{R})$ algebra compared to [40], to fit with the bosonic conventions as much as possible.

B.1 Representation theory of OSp$(1|2,\mathbb{R})$

B.1.1 Defining the OSp$(1|2,\mathbb{R})$ supergroup

To begin, the general linear supergroup GL$(1|2,\mathbb{R})$ consists of all invertible 3×3 matrices comprised of five bosonic variables a,b,c,d,e and four fermionic Grassmann variables $\alpha,\beta,\gamma,\delta$, separated into bosonic diagonal and fermionic off-diagonal blocks

$$g = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \gamma \\ \hline \beta & \delta & e \end{array}\right). \tag{B.1}$$

The subgroup OSp$(1|2,\mathbb{R}) \subset$ GL$(1|2,\mathbb{R})$ that preserves the orthosymplectic form Ω, defines the group OSp$(1|2,\mathbb{R})$:

$$\begin{aligned} g^{\text{st}}\Omega g &= \left(\begin{array}{cc|c} a & c & -\beta \\ b & d & -\delta \\ \hline \alpha & \gamma & e \end{array}\right)\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \gamma \\ \hline \beta & \delta & e \end{array}\right) \\ &= \left(\begin{array}{cc|c} 0 & bc-ad-\beta\delta & -a\gamma+c\alpha-\beta e \\ ad-bc-\delta\beta & 0 & -b\gamma+\alpha d-\delta e \\ \hline -\alpha c+a\gamma+e\beta & -\alpha d+b\gamma+e\delta & -\alpha\gamma+\gamma\alpha+e^2 \end{array}\right) \equiv \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right). \end{aligned}$$

Here and below, the supertranspose of an even dimensional supermatrix is defined as

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{\mathrm{st}} = \left(\begin{array}{c|c} A^T & -C^T \\ \hline B^T & D^T \end{array}\right). \tag{B.2}$$

We directly infer the $\mathrm{OSp}(1|2,\mathbb{R})$-constraints:

$$ad - bc - \delta\beta = 1, \quad -a\gamma + c\alpha - \beta e = 0, \quad e^2 + 2\gamma\alpha = 1, \quad -\alpha d + b\gamma + e\delta = 0. \tag{B.3}$$

Note that the st operation is not an involution, but is of order four: $\mathrm{st}^4 = \mathbf{1}$. Imposing the constraint $ad - bc \equiv 1 + \delta\beta$, the other constraints are solved by parameterizing[1]

$$\alpha = \pm(a\delta - b\beta), \qquad \gamma = \pm(c\delta - d\beta), \qquad e = \pm(1 + \beta\delta). \tag{B.4}$$

A general $g \in \mathrm{OSp}(1|2,\mathbb{R})$ matrix can therefore be parameterized as

$$g = \left(\begin{array}{cc|c} a & b & \pm(a\delta - b\beta) \\ c & d & \pm(c\delta - d\beta) \\ \hline \beta & \delta & \pm(1+\beta\delta) \end{array}\right), \qquad ad - bc = 1 + \delta\beta. \tag{B.5}$$

Note that for vanishing fermionic entries, $ad - bc = 1$ is the defining $\mathrm{SL}(2,\mathbb{R})$-constraint. Thereby, the group $\mathrm{OSp}(1|2,\mathbb{R})$ has the non-compact bosonic subgroup $\mathrm{Sp}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R})$ with bosonic entries a, b, c, d. Due to the $\mathrm{OSp}(1|2,\mathbb{R})$ constraints, the inverse matrix is written transparently as:

$$g^{-1} = \left(\begin{array}{cc|c} d & -b & -\delta \\ -c & a & \beta \\ \hline \gamma & -\alpha & e \end{array}\right) = \left(\begin{array}{cc|c} d & -b & -\delta \\ -c & a & \beta \\ \hline \pm(c\delta - d\beta) & \mp(a\delta - b\beta) & \pm(1+\beta\delta) \end{array}\right). \tag{B.6}$$

The Berezinian or superdeterminant of an invertible $\mathrm{GL}(1|2,\mathbb{R})$ matrix $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is defined as [40]:

$$\mathrm{Ber}(M) = \det(A - BD^{-1}C)\det(D)^{-1}. \tag{B.7}$$

Taking into account the anticommutativity between the fermionic blocks B and C, the definition of the Berezinian is naturally invariant under the supertranspose st operation. Applied to the $\mathrm{OSp}(1|2,\mathbb{R})$ matrix

[1] Explicitly; $c\alpha - a\gamma - \beta e = 0$ reads $\pm(ac\delta - bc\beta) \mp (ac\delta - ad\beta) \mp \beta = \pm(ad - bc)\beta \mp \beta = 0$, upon imposing the constraint $ad - bc - \delta\beta = 1$. Analogously, $-d\alpha + b\gamma + e\delta = \mp(ad\delta - bd\beta) \pm (cb\delta - bd\beta) \pm \delta = \mp(ad - bc)\delta \pm \delta = 0$. Furthermore, $e^2 + 2\gamma\alpha = (1 + \beta\delta)^2 + 2(c\delta - d\beta)(a\delta - b\beta) = 1 + 2\beta\delta - 2(ad - bc)\beta\delta = 1$.

g, we readily work out;

$$\begin{aligned} BD^{-1}C &= \frac{1}{\pm(1+\beta\delta)}\begin{pmatrix}\pm(a\delta-b\beta)\\ \pm(c\delta-d\beta)\end{pmatrix}\begin{pmatrix}\beta & \delta\end{pmatrix} = \begin{pmatrix}a\delta\beta & b\delta\beta\\ c\delta\beta & d\delta\beta\end{pmatrix} \\ \leftrightarrow \quad \det(A-BD^{-1}C)\det(D)^{-1} &= \frac{1}{\pm(1+\beta\delta)}\det\begin{pmatrix}a(1+\beta\delta) & b(1+\beta\delta)\\ c(1+\beta\delta) & d(1+\beta\delta)\end{pmatrix} \\ &= \pm\frac{(ad-bc)(1+2\beta\delta)}{1+\beta\delta} = \pm(ad-bc)(1-\delta\beta) = \pm 1. \end{aligned}$$

The group $\mathrm{OSp}(1|2,\mathbb{R})$ naturally decomposes into two disconnected sectors, depending on the sign of the Berezinian, where both sectors are related by applying the sCasimir operator $(-)^F = \mathrm{diag}(1,1 \mid -1)$. Restricting to either sign leads to the projective supergroup $\mathrm{OSp}'(1|2,\mathbb{R}) = \mathrm{OSp}(1|2,\mathbb{R})/\mathbb{Z}_2$.
Just as for $\mathrm{SL}(2,\mathbb{R})$, the different conjugacy class elements are labeled by the value of the supertrace $\mathrm{STr}(g)$, defined in Eq 4.76. Choosing either sign $\pm$ in Eq B.5;

$$\mathrm{STr}(g) = a + d \mp (1+\beta\delta). \tag{B.8}$$

Choosing the **NS**-sector in Eq B.5 $(-)$, group elements with $|\mathrm{STr}(g)| > 3$, $|\mathrm{STr}|(g) = 3$, $|\mathrm{STr}(g)| < 3$ are called hyperbolic, parabolic and elliptic respectively.
In the **R**-sector of Eq B.5 $(+)$, the conjugacy classes are instead: $|\mathrm{STr}(g)| > 1$, $|\mathrm{STr}|(g) = 1$, $|\mathrm{STr}(g)| < 1$, corresponding to hyperbolic, parabolic and elliptic respectively.

B.1.2 Defining the $\mathfrak{osp}(1|2,\mathbb{R})$ superalgebra

The generators iJ_I $(= iH, iE_\pm, iF_\pm)$ of the $\mathfrak{osp}(1|2,\mathbb{R})$ superalgebra are defined in Eq 4.82. Written out explicitly, the 3×3 supermatrices comprise the defining representation of $\mathfrak{osp}(1|2,\mathbb{R})$:

$$\begin{gathered} iH = \frac{1}{2}\left(\begin{array}{cc|c} -1 & 0 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 0\end{array}\right), \quad iE_- = \left(\begin{array}{cc|c} 0 & 0 & 0\\ 1 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right), \quad iE_+ = \left(\begin{array}{cc|c} 0 & 1 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right) \\ iF_- = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 0 & 0\\ 0 & 0 & -1\\ \hline 1 & 0 & 0\end{array}\right), \quad iF_+ = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 0 & 1\\ 0 & 0 & 0\\ \hline 0 & 1 & 0\end{array}\right). \end{gathered} \tag{B.9}$$

We readily check that these matrices indeed satisfy the known $\mathfrak{osp}(1|2,\mathbb{R})$-algebra Eq 4.83:

$$\begin{aligned} &[H,E_\pm] = \pm iE_\pm, \qquad [E_+,E_-] = 2iH, \qquad [H,F_\pm] = \pm\frac{1}{2}iF_\pm, \\ &[E_\pm,F_\mp] = iF_\pm, \qquad \{F_+,F_-\} = \frac{1}{2}iH, \qquad \{F_\pm,F_\pm\} = \mp\frac{1}{2}iE_\pm. \end{aligned} \tag{B.10}$$

The Cartan-Killing metric is defined from the normalization of the generators with respect to the STr operation:

$$\mathrm{STr}((iJ_I)(iJ_J)) \equiv \frac{\kappa_{IJ}}{2}. \tag{B.11}$$

For the generators at hand, we deduce (in the order $iH, iE_+, iE_-, iF_+, iF_-$):

$$\kappa_{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{B.12}$$

We again label the representation by the simultaneous eigenvalue of the quadratic Casimir, defined in terms of the inverse Cartan-Killing metric $\mathcal{C}_2 = -iJ_I\kappa^{IJ}iJ_J \equiv -j(j+1/2)$. Explicitly,

$$\mathcal{C}_2 = -\kappa^{IJ}iJ_IiJ_J = H^2 + \frac{1}{2}(E_+E_- + E_-E_+) - (F_+F_- - F_-F_+) \equiv -j(j+1/2). \tag{B.13}$$

This operator is seen to commute with all the generators in the $\mathfrak{osp}(1|2,\mathbb{R})$-algebra. Using the explicit form of the generators in the defining representation Eq B.9, we readily check that they form a spin-1/2 representation. We can also consider operators that commute with all bosonic generators, while anticommuting with all fermionic generators. For OSp$(1|2,\mathbb{R})$, this operation is given by the sCasimir $\mathcal{Q}$:

$$\mathcal{Q} = (iF_+)(iF_-) - (iF_-)(iF_+) + \frac{1}{8} = -F_+F_- + F_-F_+ + \frac{1}{8} = \left(\frac{j}{2} + \frac{1}{8}\right)(-)^F \tag{B.14}$$

, where $(-)^F = \mathrm{diag}(\mathbf{1}_{2j+1} \mid -\mathbf{1}_{2j})$ is the operator that commutes with all bosonic generators, while anticommuting with the fermionic generators. As a consequence, this operation transforms between the two disconnected components of OSp$(1|2,\mathbb{R})$. The sCasimir has the important property that its square is proportional to the quadratic Casimir[2]:

$$\mathcal{Q}^2 - \frac{1}{64} = -\frac{1}{4}H^2 - \frac{1}{8}(E_+E_- + E_-E_+) + \frac{1}{4}(F_+F_- - F_-F_+) = -\frac{1}{4}\mathcal{C}_2 = \frac{j(j+1/2)}{4}. \tag{B.15}$$

[2]Using the superalgebra relations Eq B.10:

$$\begin{aligned}\mathcal{Q}^2 - \frac{1}{64} &= F_+F_-F_+F_- - F_+F_-F_-F_+ - F_-F_+F_+F_- + F_-F_+F_-F_+ + \frac{1}{4}F_-F_+ - \frac{1}{4}F_+F_- \\ &= -\frac{1}{4}iF_+E_-F_+ + \frac{1}{4}iF_-E_+F_- + \frac{1}{2}iHF_+F_- - F_-F_+F_+F_- + \frac{1}{2}iHF_-F_+ - F_+F_-F_-F_+ + \frac{1}{4}F_-F_+ - \frac{1}{4}F_+F_- \\ &= -\frac{1}{4}iF_+F_+E_- + \frac{1}{4}iF_-F_-E_+ - \frac{1}{4}H^2 + \frac{1}{4}iF_-E_+F_- - \frac{1}{4}iF_+E_-F_+ \\ &= -\frac{1}{4}H^2 - \frac{1}{16}E_+E_- - \frac{1}{16}E_-E_+ - \frac{1}{4}F_-F_+ + \frac{1}{4}iF_-F_-E_+ + \frac{1}{4}F_+F_- - \frac{1}{4}iF_+F_+E_- \\ &= -\frac{1}{4}H^2 - \frac{1}{8}(E_+E_- + E_-E_+) + \frac{1}{4}(F_+F_- - F_-F_+).\end{aligned}$$

B.1.3 Principal series representation

Next to the finite-dimensional representations, one can construct the principal series representation, whose carrier space are the square-integrable functions on $\mathbb{R}^{1|1}$. The principal series representation on $\mathrm{SL}(2,\mathbb{R})$ (c.f. Eq A.9) is realized as the transpose projective action of a group element g on elements X of a two-dimensional vector space: $X^T \to X^T g$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad X^T \to X^T g = \begin{pmatrix} ay + cz & by + dz \end{pmatrix}.$$

This acts projectively on $x \equiv \frac{y}{z}$ as $x \to \frac{ax+c}{bx+d}$. In the case of $\mathrm{OSp}(1|2,\mathbb{R})$, the group acts projectively via the st operation defined in Eq B.2 on a $2|1$-dimensional vector space[3];

$$g = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \gamma \\ \hline \beta & \delta & e \end{array}\right), \quad X = \begin{pmatrix} y \\ z \\ \theta \end{pmatrix},$$

$$X^{\mathrm{st}} = \begin{pmatrix} y & z & | & -\theta \end{pmatrix} \to X^{\mathrm{st}} g = \begin{pmatrix} ay + cz + \beta\theta & by + dz + \delta\theta & | & y\alpha + z\gamma - e\theta \end{pmatrix}. \tag{B.17}$$

This now acts projectively on $(x = \frac{y}{z}, \vartheta = \frac{\theta}{z})$ as:

$$x \to \frac{ax + c + \beta\vartheta}{bx + d + \delta\vartheta}, \qquad \vartheta \to -\frac{x\alpha + \gamma - e\vartheta}{bx + d + \delta\vartheta}. \tag{B.18}$$

To construct a *unitary* representation, one needs to induce a left-regular representation by a parabolic subgroup. The generalization of the method of parabolic induction of $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{OSp}(1|2,\mathbb{R})$ has been worked out in [40]. Barring the details, the (spherical) projective action of $\mathrm{OSp}(1|2,\mathbb{R})$ on the space of square integrable functions[4] $L^2(\mathbb{R}^{1|1})$ on the superline is given by:

$$\boxed{\langle x, \vartheta \mid g \mid f \rangle = (g \cdot f)(x, \vartheta) = \frac{|bx + d + \delta\vartheta|^{2j}}{\mathrm{sgn}(e)^{1/2}\mathrm{sgn}(bx + d + \delta\vartheta)^{1/2}}\, f\left(\frac{ax + c + \beta\vartheta}{bx + d + \delta\vartheta}, -\frac{\alpha x + \gamma - e\vartheta}{bx + d + \delta\vartheta}\right)}. \tag{B.21}$$

The action of the group $\mathrm{OSp}(1|2,\mathbb{R})$ on $L^2(\mathbb{R}^{1|1})$ is called the *principal series representation*. Up to the appearance of the sign factors, this is the natural generalization of the result for $\mathrm{SL}(2,\mathbb{R})$ (Eq A.9). A transparent

[3] Since the st operation is not an involution, but has order four ($\mathrm{st}^4 = \mathbf{1}$), this action is written equivalently according to:

$$X \to g^{\mathrm{st}3} X. \tag{B.16}$$

[4] The space of square integrable functions $L^2(\mathbb{R}^{1|1})$ on the superline is equipped with an inner product

$$\langle F \mid G \rangle = \int_{\mathbb{R}} dx \int d\vartheta\, F^*(x, \vartheta) g(x, \vartheta) \tag{B.19}$$

This defines a complete set of configurational sates $|x, \vartheta\rangle$

$$\int_{\mathbb{R}} dx \int d\vartheta\, |x, \vartheta\rangle\langle x, \vartheta| = \mathbf{1} \tag{B.20}$$

, whose overlap with vectors in $L^2(\mathbb{R}^{1|1})$ is given by: $\langle x, \vartheta \mid F \rangle = F(x, \vartheta)$.

argument to demonstrate that the above action on $L^2(\mathbb{R}^{1|1})$ indeed defines a representation, is to compute the successive action of two group elements $g_1 = \left(\begin{array}{cc|c} a_1 & b_1 & \alpha_1 \\ c_1 & d_1 & \gamma_1 \\ \hline \beta_1 & \delta_1 & e_1 \end{array}\right)$, $g_2 = \left(\begin{array}{cc|c} a_2 & b_2 & \alpha_2 \\ c_2 & d_2 & \gamma_2 \\ \hline \beta_2 & \delta_2 & e_2 \end{array}\right)$ on $f(x,\vartheta)$. The action of g_2 on $f(x,\vartheta)$ is given by Eq B.21:

$$(g_2 \cdot f)(x) = \frac{|b_2x + d_2 + \delta_2\vartheta|^{2j}}{\text{sgn}(e_2)^{1/2}\text{sgn}(b_2x + d_2 + \delta_2\vartheta)^{1/2}} f\left(\frac{a_2x + c_2 + \beta_2\vartheta}{b_2x + d_2 + \delta_2\vartheta}, -\frac{\alpha_2 x + \gamma_2 - e_2\vartheta}{b_2x + d_2 + \delta_2\vartheta}\right).$$

Successive application of g_1 gives:

$$\begin{aligned}(g_1 \cdot (g_2 \cdot f))(x) =& \frac{|x(a_1b_2 + d_2b_1 + \alpha_1\delta_2) + (b_2c_1 + d_1d_2 + \gamma_1\delta_2) + (b_2\beta_1 + d_2\delta_1 + e_1\delta_2)\vartheta|^{2j}}{\text{sgn}(e_1e_2)^{1/2}\ \text{sgn}(x(a_1b_2 + d_2b_1 + \alpha_1\delta_2) + (b_2c_1 + d_1d_2 + \gamma_1\delta_2) + (b_2\beta_1 + d_2\delta_1 + e_1\delta_2)\vartheta)^{1/2}} \\ & \times f\Big(\frac{(a_1a_2 + \alpha_1\beta_2 + c_2b_1)x + (c_1a_2 + d_1c_2 + \gamma_1\beta_2) + (a_2\beta_1 + c_2\delta_1 + e_1\beta_2)\vartheta}{x(a_1b_2 + d_2b_1 + \alpha_1\delta_2) + (b_2c_1 + d_1d_2 + \gamma_1\delta_2) + (b_2\beta_1 + d_2\delta_1 + e_1\delta_2)\vartheta}, \\ & -\frac{(\alpha_2a_1 + b_1\gamma_2 + e_2\alpha_1)x + (\alpha_2c_1 + \gamma_2d_1 + e_2\gamma_1) - (\beta_1\alpha_2 + \delta_1\gamma_2 + e_1e_2)\vartheta}{x(a_1b_2 + d_2b_1 + \alpha_1\delta_2) + (b_2c_1 + d_1d_2 + \gamma_1\delta_2) + (b_2\beta_1 + d_2\delta_1 + e_1\delta_2)\vartheta}\Big).\end{aligned}$$

This is the same action of the composite group element

$$g_1 \cdot g_2 = \left(\begin{array}{cc|c} a_1a_2 + b_1c_2 + \alpha_1\beta_2 & a_1b_2 + b_1d_2 + \alpha_1\delta_2 & a_1\alpha_2 + b_1\gamma_2 + e_2\alpha_1 \\ c_1a_2 + d_1c_2 + \gamma_1\beta_2 & c_1b_2 + d_1d_2 + \gamma_1\delta_2 & c_1\alpha_2 + d_1\gamma_2 + e_2\gamma_1 \\ \hline a_2\beta_1 + c_2\delta_1 + e_1\beta_2 & b_2\beta_1 + d_2\delta_1 + e_1\delta_2 & \beta_1\alpha_2 + \delta_1\gamma_2 + e_1e_2 \end{array}\right). \tag{B.22}$$

A subtlety resides in the sign factor $\text{sgn}(e_1e_2) = \text{sgn}(\beta_1\alpha_2 + \delta_1\gamma_2 + e_1e_2)$. The two factors are in fact equal since positivity of a supernumber is determined entirely by its body (see footnote 2).

$2j$ is the spin label of the representation. Remember that in the Peter-Weyl theorem, we restrict to unitary transformations only. In order for the projective action to realize a unitary representation with respect to the inner product Eq B.19, the spin label should be constrained to the form

$$\boxed{j = \frac{ik}{2} - \frac{1}{4}} \qquad k \in \mathbb{R}. \tag{B.23}$$

We can check this by verifying if indeed

$$\int dx d\vartheta\ F(x,\vartheta)^*(g \cdot G)(x,\vartheta) = \int dx d\vartheta (g^{-1} \cdot F)(x,\vartheta)^* G(x,\vartheta). \tag{B.24}$$

Anticipating that the inverse group element is given by Eq B.6, we make a change of variables

$$x = \frac{du - c + \gamma\theta}{-bu + a - \alpha\theta}, \qquad \vartheta = \frac{\delta u - \beta + e\theta}{-bu + a - \alpha\theta}.$$

This transformation has an easy action on

$$bx + d + \delta\vartheta = \frac{ad - bc - \delta\beta + (\gamma b - \alpha d + e\delta)\,\theta}{-bu + a - \alpha\theta} = \frac{1}{-bu + a - \alpha\theta}$$

, using both the first and last constraint in Eq B.3. Expanded to linear order in the fermionic variable θ:

$$x = \frac{du-c}{-bu+a} + \frac{u(d\alpha - b\gamma) + a\gamma - c\alpha}{(-bu+a)^2}\theta = \frac{du-c}{-bu+a} + e\frac{\delta u - \beta}{(-bu+a)^2}\theta = \frac{du-c}{-bu+a} + \mathrm{sgn}(e)\frac{\delta u - \beta}{(-bu+a)^2}\theta$$

$$\begin{aligned}\vartheta &= \frac{-\beta + u\delta}{a - bu} + \frac{\partial}{\partial\theta}\left((\delta u - \beta + e\theta)\frac{1}{-bu+a-\alpha\theta}\right)\Bigg|_{\theta=0}\theta \\ &= \frac{-\beta + u\delta}{a - bu} + \left(\frac{e}{-bu+a} - (\delta u - \beta)\frac{\partial}{\partial\theta}\frac{1}{-bu+a-\alpha\theta}\Bigg|_{\theta=0}\right)\theta \\ &= \frac{-\beta + u\delta}{-bu+a} + \frac{(-bu+a)(e \mp \beta\delta)}{(-bu+a)^2}\theta = \frac{-\beta+u\delta}{-bu+a} + \mathrm{sgn}(e)\frac{\theta}{-bu+a}\end{aligned}$$

, using again the defining OSp$(1|2,\mathbb{R})$-relations Eq B.4. The transition in the first line from e to sgn(e) is due to the action of the Grassmann numbers β and δ on $e = \pm(1+\beta\delta)$. In the last line, we have used $(\delta u - \beta)\alpha = \pm(\delta u - \beta)(a\delta - b\beta) = \mp(-bu+a)\beta\delta$.
This transformation leads to the super-Jacobian:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial \vartheta}{\partial u} \\ \frac{\partial x}{\partial \theta} & \frac{\partial \vartheta}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{(-bu+a)^2} + \frac{\alpha + \mathrm{sgn}(e)(\delta u - \beta)b}{(-bu+a)^3}\theta & \mathrm{sgn}(e)\frac{\alpha + b\theta}{(-bu+a)^2} \\ \mathrm{sgn}(e)\frac{-\delta u + \beta}{(-bu+a)^2} & \mathrm{sgn}(e)\frac{1}{-bu+a} \end{pmatrix} \tag{B.25}$$

, leading to the Berezinian Eq B.7:

$$\begin{aligned}(A - BD^{-1}C)D^{-1} &= \mathrm{sgn}(e)\left(\frac{ad-bc}{-bu+a} + \frac{\alpha + \mathrm{sgn}(e)(\delta u - \beta)b}{(-bu+a)^2}\theta - \mathrm{sgn}(e)\frac{\alpha + b\theta}{(-bu+a)^2}(-\delta u + \beta)\right) \\ &= \mathrm{sgn}(e)\left(\frac{ad-bc}{-bu+a} + \frac{\alpha\theta}{(-bu+a)^2}\right) - \frac{\alpha}{(-bu+a)^2}(-\delta u + \beta) \\ &= \mathrm{sgn}(e)\left(\frac{1}{-bu+a} + \frac{\delta\beta}{-bu+a} + \frac{\alpha\theta}{(-bu+a)^2}\right) \mp \frac{(a\delta - b\beta)(-\delta u + \beta)}{(-bu+a)^2} \\ &= \mathrm{sgn}(e)\left(\frac{1}{-bu+a} + \frac{\delta\beta}{-bu+a} + \frac{\alpha\theta}{(-bu+a)^2}\right) \mp \frac{(-bu+a)\delta\beta}{(-bu+a)^2} \\ &= \mathrm{sgn}(e)\left(\frac{1}{-bu+a} + \frac{\alpha\theta}{(-bu+a)^2}\right) \\ &= \frac{\mathrm{sgn}(e)}{-bu+a-\alpha\theta}\end{aligned}$$

, using that $\mp\delta\beta = -\mathrm{sgn}(e)\delta\beta$. The Berezinian of the super-Jacobian leads to the correct transformation rule:

$$dx d\vartheta = du d\theta \frac{\mathrm{sgn}(e)}{-bu+a-\alpha\theta} = du d\theta \frac{\mathrm{sgn}(e)\mathrm{sgn}(-bu+a-\alpha\theta)}{|-bu+a-\alpha\theta|}.$$

Using that $(\mathrm{sgn}(e)^{1/2})^* = (\mathrm{sgn}(e))^{-1/2}$, and $(\mathrm{sgn}(-bu+a-\alpha\theta)^{1/2})^* = (\mathrm{sgn}(-bu+a-\alpha\theta))^{-1/2}$, this transformation acts as:

$$
\begin{aligned}
&\int dx d\vartheta\ F(x,\vartheta)^*(g\cdot G)(x,\vartheta)\\
&= \int du d\theta \left(\frac{|-bu+a-\alpha\theta|^{-2j^*-1}}{\operatorname{sgn}(e)^{1/2}\operatorname{sgn}(-bu+a-\alpha\theta)^{1/2}} F\left(\frac{du-c+\gamma\theta}{-bu+a-\alpha\theta}, \frac{\delta u-\beta+e\theta}{-bu+a-\alpha\theta}\right)\right)^* G(x,\vartheta).
\end{aligned}
$$

We see that indeed the adjoint action is given by the group inverse iff $2j = -2j^* - 1$ or $j =$ B.23. This procedure also demonstrates the necessity to introduce the additional sign factors in the definition Eq B.21.

Borel-Weil realization of the (opposite) algebra

Exponentiating the generators in the defining representation of Eq B.9 yields [40]:

$$
\begin{gathered}
e^{2\phi iH} = \left(\begin{array}{cc|c} e^{-\phi} & 0 & 0 \\ 0 & e^{\phi} & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad e^{\gamma^- iE_-} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ \gamma^- & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad e^{\gamma^+ iE_+} = \left(\begin{array}{cc|c} 1 & \gamma^+ & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \\
e^{2\theta^- iF_-} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -\theta^- \\ \hline \theta^- & 0 & 1 \end{array}\right), \quad e^{2\theta^+ iF_+} = \left(\begin{array}{cc|c} 1 & 0 & \theta^+ \\ 0 & 1 & 0 \\ \hline 0 & \theta^+ & 1 \end{array}\right)
\end{gathered} \tag{B.26}
$$

in terms of three scalar Gauss parameters ϕ, γ^-, γ^+ and two Grassmann parameters θ^-, θ^+. The corresponding group action is readily read off using Eq B.21;

$$(e^{2\phi iH}\circ f)(x,\vartheta) = e^{2j\phi} f(e^{-2\phi}x, e^{-\phi}\vartheta) \tag{B.27}$$

$$(e^{\gamma^- iE_-}\circ f)(x,\vartheta) = f(x+\gamma^-,\vartheta) \tag{B.28}$$

$$(e^{\gamma^+ iE_+}\circ f)(x,\vartheta) = \operatorname{sgn}(\gamma^+ x+1)^{-1/2}|\gamma^+ x+1|^{2j} f\left(\frac{x}{\gamma^+ x+1}, \frac{\vartheta}{\gamma^+ x+1}\right) \tag{B.29}$$

$$(e^{2\theta^- iF_-}\circ f)(x,\vartheta) = f(x+\theta^-\vartheta, \vartheta+\theta^-) \tag{B.30}$$

$$(e^{2\theta^+ iF_+}\circ f)(x,\vartheta) = |1+\theta^+\vartheta|^{2j} f\left(\frac{x}{1+\theta^+\vartheta}, \frac{\vartheta-\theta^+ x}{1+\theta^+\vartheta}\right). \tag{B.31}$$

We derive the infinitesimal action of the generators on the space $L^2(\mathbb{R}^{1|1})$ by linearizing with respect to the Gauss parameters. This leads to the *Borel-Weil* realization of $\mathfrak{osp}(1|2,\mathbb{R})$:

$$
\begin{gathered}
iH = -x\partial_x - \frac{1}{2}\vartheta\partial_\vartheta + j, \quad iE_- = \partial_x, \quad iE_+ = -x^2\partial_x - x\vartheta\partial_\vartheta + 2jx \\
iF_- = \frac{1}{2}(\partial_\vartheta + \vartheta\partial_x), \quad iF_+ = -\frac{1}{2}x\partial_\vartheta - \frac{1}{2}x\vartheta\partial_x + j\vartheta.
\end{gathered} \tag{B.32}
$$

Acting on purely bosonic entries, the $\mathfrak{osp}(1|2,\mathbb{R})$ differential operators coincide with the spin-j Borel-Weil realization of $\mathfrak{sl}(2,\mathbb{R})$ (c.f. Eq A.11). Acting on a purely fermionic function $\vartheta f(x)$, the action of the $\mathfrak{osp}(1|2,\mathbb{R})$ Borel-Weil operators inherits a spin-$(j-1/2)$ $\mathfrak{sl}(2,\mathbb{R})$ representation. In particular, the action of iH on $\vartheta f(x)$

yields:

$$iH(\vartheta f(x)) = -\vartheta x\partial_x f(x) - \frac{1}{2}\vartheta f(x) + j\vartheta f(x) = \vartheta\left(-x\partial_x f(x) + \left(j-\frac{1}{2}\right)f(x)\right).$$

This coincides with the spin-$\left(j-\frac{1}{2}\right)$ $\mathfrak{sl}(2,\mathbb{R})$ action of iH on $f(x)$. Therefore, a spin-j representation of $\mathfrak{osp}(1|2,\mathbb{R})$ may be decomposed into a direct sum of irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$ according to:

$$R_j^{\mathfrak{osp}(1|2,\mathbb{R})} = R_j^{\mathfrak{sl}(2,\mathbb{R})} \oplus R_{j-1/2}^{\mathfrak{sl}(2,\mathbb{R})}. \tag{B.33}$$

The irreducible representations in $\mathfrak{sl}(2,\mathbb{R})$ are however not unitary, since this would require $j = -\frac{1}{2}+ik$. Calculating the full commutation relations, one finds that they indeed satisfy the $\mathfrak{osp}(1|2,\mathbb{R})$-algebra Eq 4.83, up to a sign factor in the anticommutators [40];

$$\begin{aligned}
[H,E_\pm] &= \pm iE_\pm, \qquad [E_+,E_-] = 2iH, \qquad [H,F_\pm] = \pm\frac{1}{2}iF_\pm,\\
[E_\pm,F_\mp] &= iF_\pm, \qquad \{F_+,F_-\} = -\frac{1}{2}iH, \qquad \{F_\pm,F_\pm\} = \pm\frac{1}{2}iE_\pm.
\end{aligned} \tag{B.34}$$

This is consistent with the fact that the generators in the Borel-Weil realization are represented by anticommuting Grassmann differential operators, whereas the entries in the fermionic generators of the matrixrepresentation Eq B.9 are real numbers. In accordance with [40], we denote the algebra satisfied by the Borel-Weil generators the *opposite* $\mathfrak{osp}(1|2,\mathbb{R})$-superalgebra.
To account for the opposite algebra, one makes a slight modification in the definition of the sCasimir compared to Eq B.14,

$$\begin{aligned}
\mathcal{Q} &= (iF_-)(iF_+) - (iF_+)(iF_-) + \frac{1}{8}\\
&= \frac{1}{2}(\partial_\vartheta + \vartheta\partial_x)(-\frac{1}{2}x\partial_\vartheta - \frac{1}{2}x\vartheta\partial_x + j\vartheta) - \frac{1}{2}(-\frac{1}{2}x\partial_\vartheta - \frac{1}{2}x\vartheta\partial_x + j\vartheta)(\partial_\vartheta + \vartheta\partial_x) + \frac{1}{8}\\
&= -\frac{1}{4}\cancel{x\partial_x} + \frac{1}{4}\cancel{x\vartheta\partial_\vartheta\partial_x} + \frac{j}{2} - \frac{j}{2}\vartheta\partial_\vartheta - \frac{1}{4}\vartheta\partial_\vartheta - \frac{1}{4}\cancel{x\vartheta\partial_\vartheta\partial_x} + \frac{1}{4}\cancel{x\partial_x} - \frac{1}{4}\cancel{x\vartheta\partial_\vartheta\partial_x} + \frac{1}{4}\cancel{x\vartheta\partial_x\partial_\vartheta} - \frac{j}{2}\vartheta\partial_\vartheta + \frac{1}{8}\\
&= \left(\frac{1}{8}+\frac{j}{2}\right)(1-2\vartheta\partial_\vartheta).
\end{aligned} \tag{B.35}$$

For any supernumber, it is readily seen that $(-)^F = (1-2\vartheta\partial_\vartheta)$ changes sign when acting on a fermionic entry, while acts as the identity on any bosonic entry. This agrees with the general definition of the sCasimir, acting on a spin-j representation Eq B.14.
Due to unitarity of the principal series representation, all bosonic generators $H, E_\pm$ are hermitian with respect to the measure $dxd\vartheta$. Due to $\{F^\pm, F^\pm\} = \pm\frac{1}{2}iE^\pm$, the fermionic generators cannot be hermitian (e.g. $(F^+F^+)^\dagger = \mp i\frac{1}{4}(E^+)^\dagger = \mp i\frac{1}{4}E^+ \neq F^+F^+$). However, the combination of the fermionic generators with their respective Grassmann group variables $\theta_\pm$ are hermitian in the usual sense (the fermionic generators do not square to an anti-hermitian generator anymore; instead $[\theta'^\pm F_\pm, \theta^\pm F^\pm] = \mp\frac{1}{2}i\,\theta'^\pm\theta^\pm\,E^\pm$).

B.1.4 Gauss parametrization of the OSp$(1|2,\mathbb{R})$ supergroup manifold

To proceed, it is useful to introduce the Gauss parametrization of the OSp$(1|2,\mathbb{R})$ group manifold, analogously to the Gauss parametrization of SL$(2,\mathbb{R})$ (c.f. Eq 2.148). In particular, the group manifold is characterized in terms of $\phi, \gamma^+, \gamma^-, \theta^+, \theta^-$:

$$g(\phi, \gamma^-, \gamma^+ \mid \theta^-, \theta^+) = e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+}. \tag{B.36}$$

Using the explicit exponentiations Eq B.26;

$$g(\phi, \gamma^-, \gamma^+ \mid \theta^-, \theta^+) = \left(\begin{array}{cc|c} e^{-\phi} & \gamma^+ e^{-\phi} & e^{-\phi}\theta^+ \\ \gamma^- e^{-\phi} & e^{\phi} + \gamma^-\gamma^+ e^{-\phi} - \theta^-\theta^+ & \gamma^- e^{-\phi}\theta^+ - \theta^- \\ \hline e^{-\phi}\theta^- & \gamma^+ e^{-\phi}\theta^- + \theta^+ & 1 + e^{-\phi}\theta^-\theta^+ \end{array}\right). \tag{B.37}$$

One can calculate the Haar measure on the Gauss manifold using similar reasoning of section 2.6.1. In particular, one considers the metric on OSp$(1|2,\mathbb{R})$ (c.f. Eq 2.149):

$$ds^2 = \frac{1}{2}\text{STr}((g^{-1}dg)^2) = g_{ij} dx^i dx^j. \tag{B.38}$$

This leads to the natural volume form on the supergroup manifold equipped with the metric g_{ij}: $dg = \sqrt{\text{sdet}(g)} dx^1 \wedge \cdots \wedge dx^n$. n is the dimension of the $\mathfrak{osp}(1|2,\mathbb{R})$-algebra. After a lengthy, but otherwise straightforward calculation (preferably on Mathematica), we verify [40]:

$$\begin{aligned} g^{-1}dg = &\left(2d\phi - 2\gamma^+ e^{-2\phi} d\gamma^- - 2\left(e^{-\phi}\theta^+ + \gamma^+ e^{-2\phi}\theta^-\right) d\theta^-\right) iH \qquad (B.39)\\ &+ \left(2\gamma^+ d\phi - \gamma^{+2} e^{-2\phi} d\gamma^- + d\gamma^+ - (\gamma^{+2} e^{-2\phi}\theta^- + 2\gamma^+ e^{-\phi}\theta^+) d\theta^- - \theta^+ d\theta^+\right) iE_+ \\ &+ \left(e^{-2\phi} d\gamma^- + e^{-2\phi}\theta^- d\theta^-\right) iE_- \\ &+ \left(2\theta^+ d\phi - 2\gamma^+ e^{-2\phi}\theta^+ d\gamma^- + 2(\gamma^+ e^{-\phi} + 2\gamma^+\theta^-\theta^+) d\theta^- + 2d\theta^+\right) iF_+ \\ &+ \left(-2e^{-2\phi}\theta^+ d\gamma^- + 2(e^{-\phi} + e^{-2\phi}\theta^-\theta^+) d\theta^-\right) iF_-. \qquad (B.40) \end{aligned}$$

One writes this in terms of $g^{-1}dg = \sum_{ij}(iJ_i) I^i{}_j dx^j$ for a transformation matrix

$$I^i{}_j = \left(\begin{array}{ccc|cc} 2 & -2\gamma^+ e^{-2\phi} & 0 & -2\left(e^{-\phi}\theta^+ + \gamma^+ e^{-2\phi}\theta^-\right) & 0 \\ 2\gamma^+ & -\gamma^{+2} e^{-2\phi} & 1 & -(\gamma^{+2} e^{-2\phi}\theta^- + 2\gamma^+ e^{-\phi}\theta^+) & -\theta^+ \\ 0 & e^{-2\phi} & 0 & e^{-2\phi}\theta^- & 0 \\ \hline 2\theta^+ & -2\gamma^+ e^{-2\phi}\theta^+ & 0 & 2\gamma^+ e^{-\phi} + 2\gamma^+\theta^-\theta^+ & 2 \\ 0 & -2e^{-2\phi}\theta^+ & 0 & 2e^{-\phi} + 2e^{-2\phi}\theta^-\theta^+ & 0 \end{array}\right) \tag{B.41}$$

, where the rows label the generators in the ordening $(iH, iE_+, iE_-, iF_+, iF_-)$, while the columns label the group parameters in the ordening $(\phi, \gamma^-, \gamma^+, \theta^-, \theta^+)$. Using the definition of the Cartan-Killing metric, the metric on the supergroup manifold is:

$$g_{ij}dx^i dx^j = \frac{1}{2}\mathrm{STr}(g^{-1}dg)^2) = \frac{1}{2}I^i{}_j I^l{}_m \mathrm{STr}(iJ_i iJ_l)dx^j dx^m = \frac{1}{4}I^i{}_j I^l{}_m \kappa_{il} dx^j dx^m \tag{B.42}$$

, where κ_{ij} is the Cartan-Killing metric defined in Eq B.12 with $\mathrm{sdet}(\kappa_{ij}) = 4$. Taking the superdeterminant on both sides leads to $\mathrm{sdet}(I)^2 = \mathrm{sdet}(g)$. The natural volume form on the supergroup manifold is therefore in one-to-one relation with the superdeterminant of the transition matrix I defined above. Using the definition of the Berezinian Eq B.7, we compute:

$$\mathrm{sdet}(I) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{-2e^{-2\phi} - 2e^{-3\phi}\theta_-\theta_+}{-4e^{-\phi} - 4e^{-2\phi}\theta^-\theta^+} = \frac{1}{2}e^{-\phi}$$

, leading to the volume form

$$\boxed{dg = \frac{1}{2}e^{-\phi}[d\phi d\gamma^- d\gamma^+ \mid d\theta^- d\theta^+]}. \tag{B.43}$$

The brackets denote an integration form on the supermanifold.

B.1.5 Mixed parabolic matrix element and the Plancherel measure

Left-parabolic eigenfunction

To deduce the Plancherel measure on $\mathrm{OSp}(1|2,\mathbb{R})$, [40] used a generalization of the orthogonality theorem Eq A.30 to functions defined on the superline $\mathbb{R}^{1|1}$:

$$\int dg\, R^k_{x|\alpha,y|\beta}(g)^*\, R^{k'}_{x'|\alpha',y'|\beta'}(g) \equiv \frac{\delta(k-k')\delta(x-x')\delta(y-y')\delta(\alpha-\alpha')\delta(\beta-\beta')}{\rho(k)}. \tag{B.44}$$

In principle, we can use any suitable basis to obtain the Plancherel measure. For physical applications, I review the results in the mixed parabolic basis.

The parabolic basis diagonalizes the parabolic group elements $iE_\pm, iF_\pm$. To deduce the eigenfunctions of the left group elements, we take inspiration from the harmonic functions on $\mathrm{SL}(2,\mathbb{R})$ (c.f. Eq A.17). The harmonic modes constitute a Fourier basis of the square-integrable functions on $\mathbb{R}$. The analogous normalized[5] super-Fourier modes on $\mathbb{R}^{1|1}$ are defined as: [40]:

$$\langle x, \vartheta \mid \nu_-, \beta\rangle = \frac{1}{\sqrt{2\pi}}e^{i\nu_- x - \beta\vartheta} = \frac{1}{\sqrt{2\pi}}(1-\beta\vartheta)e^{i\nu_- x} \tag{B.47}$$

[5]These modes are normalized in the sense that

$$\langle \nu'_-, \beta' \mid \nu_-, \beta\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} dx e^{i(\nu_- - \nu'_-)x}\int d\vartheta\vartheta\,(\beta-\beta') = \delta(\nu_- - \nu'_-)\delta(\beta-\beta') \tag{B.45}$$

, where we needed that β is a purely imaginary Grassmann number. Using the same reasoning, we see that they form a complete set of states in the sense that:

$$\int_{\mathbb{R}} d\nu_- \int d\alpha\, |\nu_-, \alpha\rangle\langle\nu_-, \alpha| = \mathbf{1} \tag{B.46}$$

, for any imaginary Grassmann number $\beta^* \equiv -\beta$. We readily see that these are eigenfunctions of the commuting operators $iE_- = \partial_x$ with eigenvalue $i\nu_-$, and ∂_ϑ with eigenvalue β. The latter follows from

$$\partial_\vartheta \langle x, \vartheta \mid \nu_-, \beta\rangle = \frac{\beta}{\sqrt{2\pi}} e^{i\nu_- x} = \beta \langle x, \vartheta \mid \nu_-, \beta\rangle .$$

Although these vectors diagonalize iE_-, they are not eigenfunctions of the fermionic operator iF_-, since one can prove that there exists only one (in this case ∂_ϑ) simultaneously diagonalizable fermionic operator. On the other hand, this might not be a problem, since the basis does not necessarily need to diagonalize all generators in the Plancherel decomposition. One can calculate the adjoint action of the left group elements $e^{2\theta^- iF_-}e^{\gamma^- iE_-}$ on $\langle \nu_-, \beta|$ to be:

$$\langle \nu_-, \beta \mid e^{2\theta^- iF_-} e^{\gamma^- iE_-} \mid x, \vartheta\rangle = \frac{1}{\sqrt{2\pi}} e^{i\nu_-\gamma^-}(1 + \theta^-\beta + (\beta + i\nu_-\theta^-)\vartheta)e^{-i\nu_- x} \tag{B.48}$$

, where ν_- is the eigenvalue under E_-. This follows from the adjoint action of $(iF_-)^\dagger = -\frac{1}{2}(\partial_\vartheta + \vartheta\partial_x)$ on the bosonic eigenmode $\langle x, \vartheta \mid \nu_-, \beta\rangle$ (with *fermionic* top component!), whereby

$$\int dx d\vartheta \langle x, \vartheta \mid \nu_-, \beta\rangle^* (\partial_\vartheta + \vartheta\partial_x) \langle x, \vartheta \mid \nu_-, \beta\rangle = -\int dx d\vartheta ((\partial_\vartheta + \vartheta\partial_x) \langle x, \vartheta \mid \nu_-, \beta\rangle)^* \langle x, \vartheta \mid \nu_-, \beta\rangle .$$

Since the representation is unitary, the generator iE_- is anti-hermitian (E_- is hermitian) in the usual sense: $(iE_-)^\dagger = -iE_-$. Using the Borel-Weil representation, the action of $(iF_-)^\dagger$ and $(iE)^\dagger$ on $f(x, \vartheta)$ is

$$((e^{2\theta^- iF_-})^\dagger \cdot f)(x, \vartheta) = f(x - \theta^-\vartheta, \vartheta - \theta^-) \tag{B.49}$$

$$((e^{\gamma^- iE_-})^\dagger \cdot f)(x, \vartheta) = f(x - \gamma^-, \vartheta). \tag{B.50}$$

Acting on $\langle x, \vartheta \mid \nu_-, \beta\rangle$, we have:

$$\langle x, \vartheta \mid \nu_-, \beta\rangle = \frac{1}{\sqrt{2\pi}}(1 - \beta\vartheta)e^{i\nu_- x} \xrightarrow{e^{2\theta^-(iF_-)^\dagger}} \frac{1}{\sqrt{2\pi}}\left(1 - \beta\vartheta + \beta\theta^-\right) e^{i\nu_-(x - \theta^-\vartheta)} \tag{B.51}$$

$$\xrightarrow{e^{-\gamma^- iE_-}} \frac{1}{\sqrt{2\pi}}\left(1 - \beta\vartheta + \beta\theta^-\right) e^{i\nu_-(x - \gamma^- - \theta^-\vartheta)} \tag{B.52}$$

$$= \frac{1}{\sqrt{2\pi}}\left(1 + \beta\theta^- - (\beta + i\nu_-\theta^-)\vartheta\right) e^{i\nu_- x} e^{-i\nu_-\gamma^-}. \tag{B.53}$$

Using that β is a purely imaginary Grassmann number and that the fermionic order is preserved under complex conjugation, the complex conjugate is readily verified to reproduce Eq B.48:

$$\begin{aligned}\langle \nu_-, \beta \mid e^{2\theta^- iF_-} e^{\gamma^- iE_-} \mid x, \vartheta\rangle &= \langle x, \vartheta \mid e^{-\gamma^- iE_-} e^{2\theta^-(iF_-)^\dagger} \mid \nu_-, \beta\rangle^* \\ &= \frac{1}{\sqrt{2\pi}} e^{i\nu_-\gamma^-}\left(1 - \beta\theta^- + (\beta + i\nu_-\theta^-)\vartheta\right) e^{-i\nu_- x}.\end{aligned}$$

Right-parabolic eigenfunction

The generalization of the right eigenvector Eq A.19 to the superline is given by the super-Fourier-inverse [40]:

$$\langle x,\vartheta\,|,\nu_+,\alpha\rangle = \frac{1}{\sqrt{2\pi}}\,|x|^{2j}e^{i\nu_+/x}e^{\alpha\vartheta/x} = \frac{1}{\sqrt{2\pi}}\left(1+\frac{\alpha\vartheta}{x}\right)\frac{|x|^{ik}}{\sqrt{x}}e^{i\nu_+/x} \tag{B.54}$$

, using $j=-\frac{1}{4}+\frac{ik}{2}$ and $\alpha^*=-\alpha$. These are eigenmodes of both iE_+ and $x\partial_\vartheta$. We readily see that the latter has eigenvalue $-\alpha$. To realize the former, we consider the Borel-Weil action[6] of the matrix element $e^{\gamma^+ iE_+}$ on $\langle x,\vartheta\,|,\nu_+,\alpha\rangle$;

$$\langle x,\vartheta\mid e^{\gamma^+ iE_+}\mid\nu_+,\alpha\rangle = \frac{1}{\sqrt{2\pi}}e^{i\nu_+\gamma^+}\left(1+\frac{\alpha\vartheta}{x}\right)\frac{|x|^{ik}}{\sqrt{x}}e^{i\nu_+/x} \tag{B.55}$$

, with eigenvalue γ^+ under E_+. It is however not an eigenvector under iF_+, as can be readily checked. The composite action is:

$$\begin{aligned}
\langle x,\vartheta\mid\nu_+,\alpha\rangle \xrightarrow{e^{2\theta^+ iF_+}} & \frac{1}{\sqrt{2\pi}}\left(1+\frac{\alpha(\vartheta-\theta^+x)}{x}\right)\frac{|x|^{ik}}{\sqrt{x}}e^{i\frac{\nu_+}{x}(1+\theta^+\vartheta)}\\
=&\frac{1}{\sqrt{2\pi}}\left(1+\theta^+\alpha+\frac{\alpha\vartheta}{x}+\frac{i\nu_+}{x}\theta^+\vartheta\right)\frac{|x|^{ik}}{\sqrt{x}}e^{i\nu_+/x}\\
=&\frac{1}{\sqrt{2\pi}}\left(1+\theta^+\alpha+\frac{(\alpha+i\nu_+\theta^+)\vartheta}{x}\right)\frac{|x|^{ik}}{\sqrt{x}}e^{i\nu_+/x} &\text{(B.56)}\\
\xrightarrow{e^{\gamma^+ iE_+}}&\frac{1}{\sqrt{2\pi}}e^{i\nu_+\gamma^+}\left(1+\theta^+\alpha+\frac{(\alpha+i\nu_+\theta^+)\vartheta}{x}\right)\frac{|x|^{ik}}{\sqrt{x}}e^{i\nu_+/x}. &\text{(B.57)}
\end{aligned}$$

Mixed-parabolic matrix element

The Borel-Weil action of the hyperbolic element $e^{2\phi iH}$ on $f(x,\vartheta)$ is:

$$(e^{2\phi iH}\circ f)(x,\vartheta) = e^{-(1/2-ik)\phi}f\left(e^{-2\phi}x,e^{-\phi}\vartheta\right)$$

[6] In general, the action on any function $f(x,\vartheta)$ is (B.27):

$$(e^{\gamma^+ iE_+}\cdot f)(x,\vartheta) = \frac{|\gamma^+x+1|^{2j}}{\mathrm{sgn}(\gamma^+x+1)^{1/2}}f\left(\frac{x}{\gamma^+x+1},\frac{\vartheta}{\gamma^+x+1}\right) = \frac{|\gamma^+x+1|^{ik}}{(\gamma^+x+1)^{1/2}}f\left(\frac{x}{\gamma^+x+1},\frac{\vartheta}{\gamma^+x+1}\right)$$

$$(e^{2\theta^+ iF_+}\cdot f)(x,\vartheta) = |1+\theta^+\vartheta|^{-1/2+ik}f\left(\frac{x}{1+\theta^+\vartheta},\frac{\vartheta-\theta^+x}{1+\theta^+\vartheta}\right).$$

This allows us to calculate a general Gauss-parameterized matrix element in the mixed parabolic basis, by inserting the identity $\int dx d\vartheta \, |x,\vartheta\rangle\langle x,\vartheta| = \mathbf{1}$:

$$\begin{aligned}
&\langle \nu_-,\beta \mid e^{2\theta^- iF_-}e^{\gamma^- iE_-}e^{2\phi iH}e^{\gamma^+ iE_+}e^{2\theta^+ iF_+} \mid \nu_+,\alpha\rangle \\
&= \int_{-\infty}^{\infty} dx d\vartheta \, \langle \nu_-,\beta \mid e^{2\theta^- iF_-}e^{\gamma^- iE_-} \mid x,\vartheta\rangle \, e^{2\phi iH} \, \langle x,\vartheta \mid e^{\gamma^+ iE_+}e^{2\theta^+ iF_+} \mid \nu_+,\alpha\rangle \\
&= e^{-(1/2-ik)\phi}\int_{-\infty}^{\infty} dx d\vartheta \, \langle \nu_-,\beta \mid e^{2\theta^- iF_-}e^{\gamma^- iE_-} \mid x,\vartheta\rangle\langle e^{-2\phi}x, e^{-\phi}\vartheta \mid e^{\gamma^+ iE_+}e^{2\theta^+ iF_+} \mid \nu_+,\alpha\rangle \\
&= e^{(1/2+ik)\phi}\int_{-\infty}^{\infty} dx d\vartheta \, \langle \nu_-,\beta \mid e^{2\theta^- iF_-}e^{\gamma^- iE_-} \mid e^{\phi}x,\vartheta\rangle\langle e^{-\phi}x, e^{-\phi}\vartheta \mid e^{\gamma^+ iE_+}e^{2\theta^+ iF_+} \mid \nu_+,\alpha\rangle
\end{aligned}$$

, where in the last line we shifted $x \to e^{\phi}x$. Plugging in the explicit wavefunctions determined above yields:

$$= \frac{1}{2\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}\int_{-\infty}^{\infty} dx \left((\beta + i\nu_-\theta^-)(1+\theta^+\alpha) + (1-\beta\theta^-)\frac{(\alpha + i\nu_+\theta^+)}{x}\right)\frac{|x|^{ik}}{\sqrt{x}}e^{ie^{\phi}\nu_+/x}e^{-i\nu_- e^{\phi}x} \tag{B.58}$$

, where we immediately integrated out the fermionic superpartner ϑ, leaving only the coefficients of the terms linear in ϑ. Next, we split the integral into the positive and negative real axis, and make use of the Bessel integral representation $(\nu_+,\nu_- > 0)$ [40]

$$\int_0^{\infty} dx \; x^{2j-1}e^{\pm(i\nu_- e^{\phi}x - i\nu_+ e^{\phi}/x)} = 2e^{\pm(i\pi j)}\left(\frac{\nu_+}{\nu_-}\right)^j K_{2j}(2e^{\phi}\sqrt{\nu_-\nu_+}). \tag{B.59}$$

Considering the first term, we have

$$\frac{1}{2\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}(\beta + i\nu_-\theta^-)(1+\theta^+\alpha)\left(\int_0^{\infty} dx \frac{x^{ik-1/2}}{i}e^{-ie^{\phi}\nu_+/x}e^{+i\nu_- e^{\phi}x} + \int_0^{\infty} dx \; x^{ik-1/2}e^{ie^{\phi}\nu_+/x}e^{-i\nu_- e^{\phi}x}\right)$$

, whose terms are evaluated from the Bessel integral representation with spin $j = \frac{ik}{2} + \frac{1}{4}$

$$\begin{aligned}
&= \frac{1}{2\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}(\beta + i\nu_-\theta^-)(1+\theta^+\alpha)\left(\frac{\nu_+}{\nu_-}\right)^{\frac{ik}{2}+\frac{1}{4}} K_{ik+\frac{1}{2}}(2e^{\phi}\sqrt{\nu_-\nu_+})\left(2e^{\frac{i\pi}{2}(ik-\frac{1}{2})} + 2e^{-i\frac{\pi}{2}(ik+\frac{1}{2})}\right) \\
&= \frac{2}{\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}(\beta + i\nu_-\theta^-)(1+\theta^+\alpha)\left(\frac{\nu_+}{\nu_-}\right)^{\frac{ik}{2}+\frac{1}{4}} K_{ik+\frac{1}{2}}(2e^{\phi}\sqrt{\nu_-\nu_+})e^{-\frac{i\pi}{4}}\cosh\left(\frac{\pi k}{2}\right).
\end{aligned}$$

Considering the second term in Eq B.58, we evaluate it in the Bessel integral representation with spin $j = \frac{ik}{2} - \frac{1}{4}$ due to the additional x in the denominator;

$$\begin{aligned}
&\frac{1}{2\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}(1-\beta\theta^-)(\alpha + i\nu_+\theta^+)\left(\int_0^{\infty} dx \frac{x^{ik-3/2}}{-i}e^{-ie^{\phi}\nu_+/x}e^{+i\nu_- e^{\phi}x} + \int_0^{\infty} dx \; x^{ik-3/2}e^{ie^{\phi}\nu_+/x}e^{-i\nu_- e^{\phi}x}\right) \\
&= \frac{1}{\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}(1-\beta\theta^-)(\alpha + i\nu_+\theta^+)\left(\frac{\nu_+}{\nu_-}\right)^{\frac{ik}{2}-\frac{1}{4}} K_{ik-\frac{1}{2}}(2e^{\phi}\sqrt{\nu_+\nu_-})\left(e^{\frac{i\pi}{2}(ik+\frac{1}{2})} + e^{\frac{i\pi}{2}(-ik+\frac{1}{2})}\right) \\
&= \frac{2}{\pi}e^{i\nu_+\gamma^+}e^{i\nu_-\gamma^-}e^{\phi}(1-\beta\theta^-)(\alpha + i\nu_+\theta^+)\left(\frac{\nu_+}{\nu_-}\right)^{\frac{ik}{2}-\frac{1}{4}} K_{ik-\frac{1}{2}}(2e^{\phi}\sqrt{\nu_+\nu_-})e^{i\pi/4}\cosh\left(\frac{\pi k}{2}\right).
\end{aligned}$$

Summarized, the total mixed-parabolic matrix element is given by:

$$\langle \nu_-, \beta \mid g \mid \lambda_+, \alpha \rangle \tag{B.60}$$

$$= \frac{2}{\pi} e^{i\nu_+\gamma^+} e^{i\nu_-\gamma^-} e^{\phi} \cosh\left(\frac{\pi k}{2}\right) \Bigg((1-\beta\theta^-)(\alpha + i\nu_+\theta^+) \left(\frac{\nu_+}{\nu_-}\right)^{\frac{ik}{2}-\frac{1}{4}} e^{i\pi/4} K_{ik-\frac{1}{2}}(2e^{\phi}\sqrt{\nu_+\nu_-}) + (\beta + i\nu_-\theta^-)(1+\theta^+\alpha) \left(\frac{\nu_+}{\nu_-}\right)^{\frac{ik}{2}+\frac{1}{4}} e^{-\frac{i\pi}{4}} K_{ik+\frac{1}{2}}(2e^{\phi}\sqrt{\nu_-\nu_+}) \Bigg). \tag{B.61}$$

The Plancherel measure is deduced from the orthogonality relation

$$\int dg \ \langle \nu_-, \beta \mid g \mid \nu_+, \alpha \rangle^*_{k_1} \langle \nu'_-, \beta' \mid g \mid \nu'_+, \alpha' \rangle_{k_1} = \frac{\delta(k_1-k_2)\delta(\alpha-\alpha')\delta(\beta-\beta')\delta(\nu_- - \nu'_-)\delta(\nu_+ - \nu'_+)}{\rho(k)} \tag{B.62}$$

, with repspect to the Haar measure Eq B.43. Using the analytically continued orthogonality identities of the Bessel functions [40]

$$\int_0^\infty dx \ \left(K_{\frac{1}{2}+ik}(x)K_{\frac{1}{2}+ik'}(x) + K_{\frac{1}{2}-ik}(x)K_{\frac{1}{2}-ik'}(x)\right) = \frac{\pi^2}{\cosh \pi k}\delta(k+k') \tag{B.63}$$

$$\int_0^\infty dx \ \left(K_{\frac{1}{2}+ik}(x)K_{\frac{1}{2}-ik'}(x) + K_{\frac{1}{2}-ik}(x)K_{\frac{1}{2}+ik'}(x)\right) = \frac{\pi^2}{\cosh \pi k}\delta(k-k') \tag{B.64}$$

, it is a straightforward but rather lengthy calculation to deduce the correct Plancherel measure. The details can be found in [40], leading to:

$$\rho(k) = \frac{1}{4\pi^2}\frac{\cosh(\pi k)}{\cosh^2\left(\frac{\pi k}{2}\right)} = \frac{1}{2\pi^2}\frac{\cosh(\pi k)}{1+\cosh(\pi k)}. \tag{B.65}$$

Reminiscent to SL$(2,\mathbb{R})$, this is not the end of the story since the full OSp$(1|2,\mathbb{R})$-supergroup manifold is actually covered by eight disconnected patches. Using similar arguments to [24], [40] finds that each patch has the same weight, and summing over the different patches leads simply to a factor eight in the total inner product. The final Plancherel measure on OSp$(1|2,\mathbb{R})$ is therefore:

$$\boxed{\rho(k) = \frac{1}{16\pi^2}\frac{\cosh(\pi k)}{1+\cosh(\pi k)}}. \tag{B.66}$$

B.2 Representation theory of OSp$^+(1|2,\mathbb{R})$

In [40], it was argued that the relevant representation theory for gravity is in fact the subsemisupergroup OSp$^+(1|2,\mathbb{R})$, analogous to the story for bosonic JT quantum gravity.
The subsemisupergroup in the defining representation consists of all OSp$(1|2,\mathbb{R})$-matrices (Eq B.1) whose bosonic entries take only positive values: $a, b, c, d > 0$, with no further constraints on the fermionic Grassmann entries, since the positivity of a supernumber is determined entirely by its body (see footnote 2). This

property is preserved under group multiplication $g_1 \cdot g_2$, for which the explicit composition is written in Eq B.22. We see that each entry remains positive if the bosonic entries of both g_1 and g_2 are constrained to positive values.
This time however, the entire (semi)supergroup manifold is covered by a single patch in the Gauss decomposition $g(\phi, \gamma^-, \gamma^+ \mid \theta^-, \theta^+) = e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+}$, where we only need to constrain $\gamma^+,\ \gamma^- > 0$:

$$g(\phi, \gamma^-, \gamma^+ \mid \theta^-, \theta^+) = \left(\begin{array}{cc|c} e^{-\phi} & \gamma^+ e^{-\phi} & e^{-\phi}\theta^+ \\ \gamma^- e^{-\phi} & e^{\phi} + \gamma^-\gamma^+ e^{-\phi} - \theta^-\theta^+ & \gamma^- e^{-\phi}\theta^+ - \theta^- \\ \hline e^{-\phi}\theta^- & \gamma^+ e^{-\phi}\theta^- + \theta^+ & 1 + e^{-\phi}\theta^-\theta^+ \end{array} \right). \tag{B.67}$$

One constructs the principal series representation as the projective action of the subsemisupergroup on the target space of square integrable functions $L^2(\mathbb{R}^{+1|1})$ on the positive superline $\mathbb{R}^{+1|1} = \{(x|\vartheta) : \ x > 0\}$. Again, since positivity of a supernumber is determined entirely by its body, positivity is preserved under the projective action of an element of OSp$^+(1|2,\mathbb{R})$, which maps the bosonic coordinate x to $\frac{ax+c+\beta\vartheta}{bx+d+\delta\vartheta}$.
More concretely, the principal series action of a group element $g \in$ OSp$^+(1|2,\mathbb{R})$ on a square integrable function $f \in L^2(\mathbb{R}^{1|1})$ is defined as Eq B.21, without the additional sign factors and absolute values:

$$\boxed{\langle x \mid g \mid f\rangle = (g \cdot f)(x, \vartheta) = (bx + d + \delta\vartheta)^{2j}\ f\left(\frac{ax + c + \beta\vartheta}{bx + d + \delta\vartheta}, -\frac{\alpha x + \gamma - e\vartheta}{bx + d + \delta\vartheta}\right)}. \tag{B.68}$$

This defines a spin-j representation of the $\mathfrak{osp}(1|2,\mathbb{R})$ algebra, whose infinitesimal action is given by the Borel-Weil realization of the opposite superalgebra Eq B.32 in the anticommutators. This is again consistent with the fact that the fermionic first order differential operators are represented by Grassmann operators, rather than the defining bosonic matrices. The principal continuous series representation on OSp$^+(1|2,\mathbb{R})$ is both unitary and irreducible [40]. In particular, one again constrains the spin label to:

$$j = -\frac{1}{4} + \frac{ik}{2}, \qquad \text{with } k \in \mathbb{R}. \tag{B.69}$$

Within the subsemisupergroup, [40] motivates that the only relevant irreducible representations in the Plancherel decomposition are in fact the principal continuous series representation, upon considering the Casimir eigenvalue problem in the relevant positive subsector of the entire supergroup manifold. It is noted that within this subsector only the principal series wavefunctions appear in the harmonic analysis. In particular, the discrete series representations do not appear in the Casimir eigenfunctions for the subsemigroup.

B.2.1 Gravitational matrix elements in the mixed parabolic basis

Left gravitational eigenstates

Having the gravitational coset boundary conditions in mind, one calculates the explicit matrix element in a mixed parabolic basis. This allows us to deduce directly the Plancherel measure on $\mathrm{OSp}^+(1|2,\mathbb{R})$. As opposed to the state vectors of $\mathrm{OSp}(1|2,\mathbb{R})$, one considers eigenfunctions of iE_+ and iE_- with a bosonic top component. This allows to streamline the diagonalization procedure, as demonstrated bellow. For these modified eigenstates $|\nu_+,\epsilon_+\rangle$, $|\nu_-,\epsilon_-\rangle$, we consider the matrix element

$$\begin{aligned}\langle \nu_-,\epsilon_- \mid g \mid \nu_+,\epsilon_+\rangle &= \langle \nu_-,\epsilon_- \mid e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+} \mid \nu_+,\epsilon_+\rangle \\ &= \int_0^\infty dx d\vartheta \; \langle \nu_-,\epsilon_- \mid x,\vartheta\rangle e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+} \langle x,\vartheta \mid \nu_+,\epsilon_+\rangle .\end{aligned}$$

In particular, the left parabolic eigenfunction of iE_- are given by:

$$\boxed{\langle x,\vartheta \mid \nu_-,\epsilon_-\rangle = \frac{1}{\sqrt{2\pi}}\, e^{-\nu_- x + i\epsilon_-\sqrt{\nu_-}\vartheta} = \frac{1}{\sqrt{2\pi}}(e^{-\nu_- x} + i\epsilon_-\sqrt{\nu_-}\vartheta e^{-\nu_- x})} \tag{B.70}$$

, where $\epsilon_- = \{-1,1\}$ is a $\mathbb{Z}_2$-phase factor for consistency. Acting with $iE_- = \partial_x$ demonstrates that the eigenvalue under E_- is $i\nu_-$ (eigenvalue of $iE_- = -\nu_-$). We are hence working with exponentially damped eigenmodes on $\mathbb{R}^{+1|1}$ with imaginary eigenvalue under E_-, instead of the harmonically oscillating modes on $\mathbb{R}^{1|1}$ with real eigenvalue under E_-. This should of course be compared to the restriction to the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ (c.f. Eq A.48) in the bosonic case. Since the combination (iE_-) is anti-hermitian, the adjoint action is:

$$\langle \nu_-,\epsilon_- \mid x,\vartheta\rangle (iE_-) = ((iE_-)^\dagger \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle)^\dagger = \nu_- \langle \nu_-,\epsilon_- \mid x,\vartheta\rangle . \tag{B.71}$$

The generator $iF_- = \frac{1}{2}(\partial_\vartheta + \vartheta\partial_x)$ is not simply anti-hermitian with respect to the measure $dxd\vartheta$. Instead, partially integrating on a square integrable function with a bosonic top and bottom component yields the adjoint action $(\partial_\vartheta)^\dagger = \partial_\vartheta$ and $(\vartheta\partial_x)^\dagger = -\vartheta\partial_x$. The adjoint action is thus characterized by[7] $(iF_-)^\dagger = \frac{1}{2}(\partial_\vartheta - \vartheta\partial_x)$. Its action on $\langle x,\vartheta \mid \nu_-,\epsilon_-\rangle$ is readily given by:

$$\begin{aligned}(iF_-)^\dagger \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle &= \frac{1}{2}(\partial_\vartheta - \vartheta\partial_x)\frac{1}{\sqrt{2\pi}}(e^{-\nu_- x} + i\epsilon_-\sqrt{\nu_-}\vartheta e^{-\nu_- x}) \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{i\epsilon_-\sqrt{\nu_-}}{2} e^{-\nu_- x} + \frac{\vartheta}{2}\nu_- e^{-\nu_- x}\right) = \frac{i\epsilon_-\sqrt{\nu_-}}{2}\frac{1}{\sqrt{2\pi}}\left(e^{-\nu_- x} - i\epsilon_-\sqrt{\nu_-}\vartheta e^{-\nu_- x}\right).\end{aligned}$$

This leads exactly to the adjoint action of $(iF_-)^\dagger$ on $\langle x,\vartheta \mid \nu_-,\epsilon_-\rangle$;

$$(iF_-)^\dagger \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle = \frac{i\epsilon_-\sqrt{\nu_-}}{2}\langle x,\vartheta \mid \nu_-,-\epsilon_-\rangle . \tag{B.72}$$

As argued before, the states $\langle x,\vartheta \mid \nu_-,\epsilon_-\rangle$ cannot be simultaneous eigenvectors of both iE_- and iF_-. Instead, we see that the action of $(iF_-)^\dagger$ maps the states $\langle x,\vartheta \mid \nu_-,\epsilon_-\rangle$ to its opposite $\langle x,\vartheta \mid \nu_-,-\epsilon_-\rangle$ under $\mathbb{Z}_2$.

[7] This should be contrasted with the adjoint action on functions with fermionic top component discussed in the previous section.

The mutual "eigenvalue" under $(iE_-)^\dagger$ and $(iF_-)^\dagger$ indeed constrains the value of the bosonic top component ϵ_-. Indeed, due to the anti-commutation relation $\{F_-, F_-\} = -\frac{iE_-}{2}$, the fermionic generator squares to the anti-hermitian generator iE_-: $(iF_-)^\dagger(iF_-)^\dagger = -\frac{iE_-}{4}$. Acting on $\langle x, \vartheta \mid \nu_-, \epsilon_-\rangle$:

$$
\begin{aligned}
iE_- \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle &= -4(iF_-)^\dagger(iF_-)^\dagger \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle = -4\frac{i\epsilon_-\sqrt{\nu_-}}{2}(iF_-)^\dagger \langle x,\vartheta \mid \nu_-,-\epsilon_-\rangle \\
&= -\epsilon_-^2\, \nu_- \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle .
\end{aligned}
$$

On the other hand, the eigenvalue under iE_- was established to be $iE_- \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle = -\nu_- \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle$. This therefore indeed restricts the bosonic parameter to $\mathbb{Z}_2$-sign factor $\epsilon_\pm \in \{-1,+1\}$.
On the other hand, the fermionic generator (iF_-) is always accompanied by a Grassmann variable θ^-. Transmuting θ^- to the left, then taking the hermitian conjugate yields a flip under $\mathbb{Z}_2$ in the order-preserving complex conjugation convention:

$$
\begin{aligned}
\langle \nu_-,\epsilon_- \mid x;\vartheta\rangle \theta^-(iF_-) &= (\theta^-(iF_-)^\dagger \langle x,\vartheta \mid \nu_-,-\epsilon_-\rangle)^\dagger = \left(-\frac{i\epsilon_-\sqrt{\nu_-}}{2}\theta^- \langle x,\vartheta \mid \nu_-,\epsilon_-\rangle\right)^\dagger \\
&= \frac{i\epsilon_-\sqrt{\nu_-}}{2}\theta^- \langle \nu_-,\epsilon_- \mid x,\vartheta\rangle .
\end{aligned} \tag{B.73}
$$

Right gravitational eigenstates

The right parabolic eigenstates are given by:

$$
\boxed{\langle x,\vartheta \mid \nu_+,\epsilon_+\rangle = \frac{1}{\sqrt{2\pi}} x^{2j} e^{-\nu_+/x - i\epsilon_+\sqrt{\nu_+}\vartheta/x} = \frac{1}{\sqrt{2\pi}}(x^{2j}e^{-\nu_+/x} - i\epsilon_+\sqrt{\nu_+}\vartheta x^{2j-1}e^{-\nu_+/x})}. \tag{B.74}
$$

For consistency, we again take ϵ_+ to be a $\mathbb{Z}_2$-valued sign factor $\epsilon_+ \in \{-1,1\}$. Using similar reasoning to Eq B.55, the Borel-Weil action of $e^{\gamma^+ iE_+}$ directly leads to the eigenvalue under E_+ of $E_+ = i\nu_+$. On the other hand, we find the action of (iF_+) by explicitly calculating

$$
\begin{aligned}
iF_+ \langle x,\vartheta \mid \nu_+,\epsilon_+\rangle &= \frac{1}{\sqrt{2\pi}}(-\frac{1}{2}x\partial_\vartheta - \frac{1}{2}x\vartheta\partial_x + j\vartheta)(x^{2j}e^{-\nu_+/x} - i\epsilon_+\sqrt{\nu_+}\vartheta x^{2j-1}e^{-\nu_+/x}) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{i\epsilon_+\sqrt{\nu_+}}{2}x^{2j}e^{-\nu_+/x} - \frac{2j}{2}\cancel{x^{2j}\vartheta e^{-\nu_+/x}} - \frac{\nu_+}{2}x^{2j-1}\vartheta e^{-\nu_+/x} + \cancel{j\vartheta x^{2j}e^{-\nu_+/x}}\right) \\
&= \frac{1}{\sqrt{2\pi}}\frac{i\epsilon_+\sqrt{\nu_+}}{2}\left(x^{2j}e^{-\nu_+/x} + i\epsilon_+\sqrt{\nu_+}\vartheta x^{2j-1}e^{-\nu_+/x}\right) \\
&= \frac{i\epsilon_+\sqrt{\nu_+}}{2}\langle x,\vartheta \mid \nu_+,-\epsilon_+\rangle .
\end{aligned}
$$

Again, we see that the action of iF_+ transforms $\langle x,\vartheta \mid \nu_+,\epsilon_+\rangle$ into its opposite $\langle x,\vartheta \mid \nu_+,-\epsilon_+\rangle$ under $\mathbb{Z}_2$. Due to the algebra relation $\{F_+,F_+\} = \frac{iE_+}{2}$, the parameter ν_+ is also restricted to a $\mathbb{Z}_2$-phase factor $\epsilon_+ \in \{-1,1\}$. Accompanied by the Grassmann parameter θ^+, the "eigenstate" $\langle x,\vartheta \mid \nu_+,-\epsilon_+\rangle$ transforms back when transmuting θ^+ to the right:

$$
\theta^+(iF_+)\langle x,\vartheta \mid \nu_+,\epsilon_+\rangle = \frac{i\epsilon_+\sqrt{\nu_+}}{2}\langle x,\vartheta \mid \nu_+,\epsilon_+\rangle\theta^+ . \tag{B.75}
$$

Using the (adjoint) action of the left- and the right-parabolic group elements on respectively $\langle \nu_-, \epsilon_- \mid x, \vartheta\rangle$ and $\langle x, \vartheta \mid \nu_+, \epsilon_+\rangle$, the mixed parabolic matrix element is given entirely by the action of the hyperbolic group element:

$$\begin{aligned} \langle \nu_-, \epsilon_- \mid g \mid \nu_+, \epsilon_+\rangle &= \langle \nu_-, \epsilon_- \mid e^{2\theta^- iF_-} e^{\gamma^- iE_-} e^{2\phi iH} e^{\gamma^+ iE_+} e^{\gamma^+ iE_+} e^{2\theta^+ iF_+} \mid \nu_+, \epsilon_+\rangle \\ = \left(1 + i\epsilon_-\sqrt{\nu_-}\theta^- + i\epsilon_+\sqrt{\nu_+}\theta^+ - \epsilon_-\epsilon_+\sqrt{\nu_-\nu_+}\theta^-\theta^+\right) e^{\gamma^-\nu_- - \gamma^+\nu_+} \\ \int dx d\vartheta \ \langle \nu_-, \epsilon_- \mid x, \vartheta\rangle e^{2\phi iH} \langle x, \vartheta \mid \nu_+, \epsilon_+\rangle . \end{aligned} \tag{B.76}$$

Thereby only the hyperbolic group parameter $e^{2\phi iH}$ is relevant in the integral. Resembling the nomenclature of SL$^{(+)}(2,\mathbb{R})$, we call the eigenvectors Eqs B.70 and B.74 the left and right *Whittaker vector* respectively. The matrix element inside the integral $\langle \nu_-, \epsilon_- \mid e^{2\phi iH} \mid \nu_+, \epsilon_+\rangle$ is called the *Whittaker function*. We readily compute it by integrating over ϑ, and by using the integral identity [40]:

$$\int_0^\infty dx \ x^{2j-1} e^{-\nu_- x - \nu_+/x} = 2\left(\frac{\nu_+}{\nu_-}\right)^j K_{2j}(2\sqrt{\nu_-\nu_+}). \tag{B.77}$$

We first realize the action of $e^{2\phi iH}$ on $\langle x, \vartheta \mid \nu_+, \epsilon_+\rangle$ by virtue of the Borel-Weil realization Eq B.68 $\langle x, \vartheta \mid e^{2\phi iH} \mid \nu_+, \epsilon_+\rangle = e^{2j\phi}\langle e^{-2\phi}x, e^{-\phi}\vartheta \mid \nu_+, \epsilon_+\rangle$, leading to:

$$\begin{aligned} \langle \nu_-, \epsilon_- \mid e^{2\phi iH} \mid \nu_+, \epsilon_+\rangle &= \frac{1}{2\pi}\int_0^\infty dx \int d\vartheta (e^{-\nu_- x} - i\epsilon_-\sqrt{\nu_-}\vartheta e^{-\nu_- x}) \\ &\quad \times e^{2j\phi}\left(e^{-4j\phi}x^{2j}e^{-\nu_+ e^{2\phi}/x} - i\epsilon_+\sqrt{\nu_+}e^{-\phi}\vartheta e^{-2(2j-1)\phi}x^{2j-1}e^{-\nu_+ e^{2\phi}/x}\right) \\ &= \frac{1}{2\pi}\int_0^\infty dx \ \left(-i\epsilon_-\sqrt{\nu_-}e^{-2j\phi}x^{2j}e^{-\nu_+ e^{2\phi}/x - \nu_- x} - i\epsilon_+\sqrt{\nu_+}e^{-(2j-1)\phi}x^{2j-1}e^{-\nu_+ e^{2\phi}/x - \nu_- x}\right) \\ &= \frac{1}{\pi i}\left(\epsilon_-\sqrt{\nu_-}e^{-2j\phi}\left(e^{2\phi}\frac{\nu_+}{\nu_-}\right)^{j+1/2} K_{2j+1}(2e^\phi\sqrt{\nu_-\nu_+}) + \epsilon_+\sqrt{\nu_+}e^{-(2j-1)\phi}\left(e^{2\phi}\frac{\nu_+}{\nu_-}\right)^j K_{2j}(2e^\phi\sqrt{\nu_-\nu_+})\right) \\ &= \frac{1}{\pi i}\frac{\nu_+^{j+1/2}}{\nu_-^j}e^\phi\left(\epsilon_- K_{2j+1}(2e^\phi\sqrt{\nu_-\nu_+}) + \epsilon_+ K_{2j}(2e^\phi\sqrt{\nu_-\nu_+})\right) \end{aligned}$$

, which combined with the prefactor proportional to the parabolic group elements, leads to the total mixed parabolic matrix element ($j = -\frac{1}{4} + \frac{ik}{2}$):

$$\begin{aligned} \langle \nu_-, \epsilon_- \mid g \mid \nu_+, \epsilon_+\rangle &= \left(1 + i\epsilon_-\sqrt{\nu_-}\theta^- + i\epsilon_+\sqrt{\nu_+}\theta^+ - \epsilon_-\epsilon_+\sqrt{\nu_-\nu_+}\theta^-\theta^+\right) e^{\gamma^-\nu_- - \gamma^+\nu_+} \\ &\quad \times \frac{1}{\pi i}\frac{(\nu_+)^{\frac{1}{4}+\frac{ik}{2}}}{(\nu_-)^{-\frac{1}{4}+\frac{ik}{2}}}e^\phi\left(\epsilon_- K_{\frac{1}{2}+ik}(2e^\phi\sqrt{\nu_-\nu_+}) + \epsilon_+ K_{\frac{1}{2}-ik}(2e^\phi\sqrt{\nu_-\nu_+})\right). \end{aligned} \tag{B.78}$$

The symmetry of the Bessel function $K_{ik-\frac{1}{2}}(x) = K_{\frac{1}{2}-ik}(x)$ was used to obtain the second term [40].

B.2.2 Plancherel measure on OSp$^+(1|2,\mathbb{R})$

For future purposes, we can basically neglect the parabolic prefactor in front of the Whittaker function, since the integral over the supergroup parameters $\theta^\pm$ yields an irrelevant overall constant. The resulting matrix element is simply the Whittaker function, and depends only on the hyperbolic group element:

$$\begin{aligned}R^k_{\nu_+,\epsilon_+;\nu_-,\epsilon_-}(\phi) =&\langle \nu_-,\epsilon_- \mid e^{2\phi i H} \mid \nu_+,\epsilon_+\rangle \\ =&\frac{1}{\pi i}\frac{(\nu_+)^{\frac{1}{4}+\frac{ik}{2}}}{(\nu_-)^{-\frac{1}{4}+\frac{ik}{2}}} e^{\phi}\left(\epsilon_- K_{\frac{1}{2}+ik}(2e^\phi\sqrt{\nu_-\nu_+}) + \epsilon_+ K_{\frac{1}{2}-ik}(2e^\phi\sqrt{\nu_-\nu_+})\right).\end{aligned} \tag{B.79}$$

To derive the Plancherel measure, we consider the group integral over the hyperbolic parameter ϕ with respect to the Haar measure Eq B.43;

$$\int_{-\infty}^{+\infty}\left(\frac{1}{2}e^{-\phi}\right) R^k_{\nu_+,\epsilon_+;\nu_-,\epsilon_-}(\phi)^* R^k_{\nu_+,\epsilon_+;\nu_-,\epsilon_-}(\phi) \equiv \frac{\delta(k-k')}{\rho(k)} \tag{B.80}$$

, for which one uses the integral identity [40]

$$\begin{aligned}&\int_{-\infty}^{\infty} d\phi \left(\frac{e^\phi}{2}\right)\left(K_{\frac{1}{2}-ik}(2\sqrt{\nu_-\nu_+}e^\phi) + \epsilon_-\epsilon_+ K_{\frac{1}{2}+ik}(2\sqrt{\nu_-\nu_+}e^\phi)\right) \\ &\qquad \times \left(K_{\frac{1}{2}+ik'}(2\sqrt{\nu_-\nu_+}e^\phi) + \epsilon_-\epsilon_+ K_{\frac{1}{2}-ik'}(2\sqrt{\nu_-\nu_+}e^\phi)\right) = \frac{\pi^2\delta(k-k')}{4\sqrt{\nu_-\nu_+}\,\cosh(\pi k)}.\end{aligned} \tag{B.81}$$

This holds for any sign of $\epsilon_-\epsilon_+$. Inserting the prefactor of the Whitakker function Eq B.79, one directly obtains up to normalization, the proper Plancherel measure:

$$\boxed{\rho(k) = \cosh(\pi k)}. \tag{B.82}$$

The evaluation is significantly simplified compared to the discussion of OSp$(1|2,\mathbb{R})$, since we only cover the positive superline and tberefore do not need to split the integral representation of the Bessel function along positive and negative values. The supergroup manifold is furthermore covered by only a single patch.

Appendix C

Virasoro Coadjoint Orbits and the Symplectic measure

We start from transforming the finite-temperature Schwarzian action for reparameterization modes of the Poincaré time $F = \tan \frac{\pi}{\beta} f(\tau)$ to reparameterization modes of the thermal circle $f(\tau)$ (Eq 1.105):

$$I = -C \int_0^\beta d\tau \, \{F(\tau), \tau\} = -C \int_0^\beta d\tau \left[\{f(\tau), \tau\} + \frac{2\pi^2}{\beta^2} f'(\tau)^2 \right]. \tag{C.1}$$

It is convenient to redefine the reparametrization mode $f(\tau) \to \frac{\beta}{2\pi} f(\frac{2\pi}{\beta}\tau)$ into a 2π periodic mode: $f(\tau + 2\pi) = f(\tau) + 2\pi$. Using that the Schwarzian derivative is invariant under dilations, and further using the chain rule, we may equivalently write:

$$I = -\frac{2\pi C}{\beta} \int_0^{2\pi} d\tau \left\{ \tan \frac{f}{2}, \tau \right\} = -\frac{2\pi C}{\beta} \int_0^{2\pi} d\tau \left[\{f, \tau\} + \frac{1}{2} f'^2 \right] \tag{C.2}$$

$$= \frac{\pi C}{\beta} \int_0^{2\pi} d\tau \left(\frac{f''^2}{f'^2} - f'^2 \right). \tag{C.3}$$

The last definition is a convenient rewriting of the finite-temperature Schwarzian action, obtained by partially integrating the Schwarzian derivative

$$\{f(\tau), \tau\} \equiv \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2 = \left(\frac{(f''(\tau)}{f'(\tau)} \right)' - \frac{1}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2 \simeq -\frac{1}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2.$$

As elaborated before, the integration space is the orbit of a particular constant coadjoint vector under the Virasoro group. Coadjoint orbits are symplectic manifolds, which inherit a natural symplectic form. Therefore, as a first step to understand the integration measure deduced from the symplectic form, we should understand how the integration space is related to the Virasoro algebra.

C.1 Virasoro algebra and coadjoint vectors

The Lie algebra of the group of diffeomorphisms of the thermal circle diff(S_1) (without a central extension) consists of vector fields $V \equiv V(\tau)\frac{\partial}{\partial\tau}$, defined on S_1 where $\tau \in [0, 2\pi]$. It is a general fact that the variation of a vector along a vector field $W = W(\tau)\frac{\partial}{\partial\tau}$ is given by the Lie derivative[1] of V along W;

$$\delta_W V(\tau) = \mathcal{L}_W V(\tau) = [W(\tau)\frac{\partial}{\partial\tau}, V(\tau)\frac{\partial}{\partial\tau}] = (WV' - VW')\frac{\partial}{\partial\tau}.$$

The coadjoint vector[2] is defined as the quadratic differential $\phi(\tau)(d\tau)^2$ [66], which pairs with the adjoint vectors via the inner product $\langle\phi \mid V\rangle = \int_0^{2\pi} d\tau\phi V$. Upon requiring that the inner product is invariant under infinitesimal transformations of $\delta_W V$, the variation of the coadjoint vector is $\delta_W\phi = 2W'\phi + W\phi'$, such that indeed:

$$\delta_W\left(\int d\tau\phi V\right) = \int d\tau\,[(\delta_W V)\phi + V(\delta_W\phi)] = \int d\tau\,\left[WV'\phi - VW'\phi + 2W'V\phi + VW\phi'\right] = 0.$$

This can be seen trivially by partially integrating the last term.

The Virasoro group is the infinite dimensional Lie group obtained as the universal central extension of the group of reparametrizations of the circle diff(S_1). The corresponding Virasoro Lie algebra is isomorphic to the unique central extension of the Witt algebra, and obeys the Lie algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \tag{C.4}$$

, where the eigenvalue of c is the central charge of the central extension. An equivalent, more practical form is obtained by shifting $L_0 \to L_0 + \frac{c}{24}$,

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n}}. \tag{C.5}$$

We imagine the generators to be the complete set of functions $L_m = ie^{im\tau}\frac{\partial}{\partial\tau}$, defined on the thermal circle $\tau \equiv \tau + 2\pi$. The Virasoro algebra is realized[3] on this set of functions with the central extended Lie bracket:

$$[V(\tau)\frac{d}{d\tau}, W(\tau)\frac{d}{d\tau}] = (VW' - WV')\frac{d}{d\tau} + \frac{ic}{48\pi}\int_0^{2\pi} d\tau\,(VW''' - V'''W). \tag{C.6}$$

[1]Consider the variation of $U^\mu(x)$ along a vector field $\xi^\mu(x)$ to $x'^\mu = x^\mu - \xi^\mu(x)$. This results in $\delta_\xi U^\mu \equiv U'^\mu(x) - U^\mu(x) = U'^\mu(x') + \xi^\nu(x)\partial_\nu U'^\mu(x') - U^\mu(x) = \frac{\partial x'^\mu}{\partial x^\nu}U^\nu(x) - U^\mu(x) + \xi^\nu(x)\partial_\nu U'^\mu(x') \simeq \xi^\nu(x)\partial_\nu U^\mu(x) - \partial_\nu\xi^\mu(x)U^\nu(x) \equiv \mathcal{L}_\xi U^\mu(x)$

[2]A more formal definition of the coadjoint representation starts by considering a Lie group G and its Lie algebra $\mathfrak{g}$. The adjoint representation of the action of the Lie algebra is a homomorphism of the Lie algebra on itself: $R_{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ x \mapsto ad_x$, where ad_x is a linear function on the algebra $\mathfrak{g}$, defined by $ad_x(y) = [x, y]$, for any $y \in \mathfrak{g}$. By the Jacobi identity on $\mathfrak{g}$, this map is a homomorphism that preserves the commutation relations $R_{ad}(x) \circ R_{ad}(y) - R_{ad}(y) \circ R_{ad}(x) = R_{ad}([x, y])$. By considering the Cartan-Killing metric on the Lie algebra $\langle g \mid h\rangle$, one defines the coadjoint representation Ad* $: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ as the dual space of the adjoint representation, defined by $\langle ad_x^*\mu \mid y\rangle \equiv \langle\mu \mid -ad_x(y)\rangle = \langle\mu \mid -[x, y]\rangle$, for $x, y \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The coadjoint orbit of $\mu \in \mathfrak{g}^*$ is defined as the orbit $Ad_G^*\mu$ of μ under the action of coadjoint action of G, or as the quotient space G/G_μ, for the isotropy subgroup G_μ of μ with respect to the coadjoint action of G. The coadjoint orbit admits a natural symplectic structure defined by the closed, non-degenerate, G-invariant 2-form $\omega_\nu(ad_x^*\nu, ad_y^*\nu) \equiv \langle\nu \mid [x, y]\rangle$ for $x, y \in \mathfrak{g}$, $\nu \in Ad_G^*(\mu)$.

[3]Explicitly; $[L_m, L_n] = (-in + im)e^{i(n+m)\tau}\frac{\partial}{\partial\tau} + \frac{ic}{48\pi}\int_0^{2\pi} d\tau\left[in^3e^{i(n+m)\tau} - im^3e^{i(n+m)\tau}\right] = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n}$, from the orthogonality $\frac{1}{2\pi}\int_0^{2\pi} e^{i(m+n)\tau}d\tau = \delta_{m+n}$.

A general element of the Virasoro adjoint representation is labeled by a pair (V, a), representing $V(\tau)\frac{\partial}{\partial\tau} - iac$, with V an adjoint vector and a a real number related to the central extension. We can expand this vector into generators of the Virasoro algebra L_m, plus a multiple of the central element. The Lie bracket determines the infinitesimal change in (V, a) under a vector W

$$\delta_W(V, a) = \left(WV' - W'V, \frac{1}{48\pi}\int_0^{2\pi} d\tau\, (W'''V - V'''W)\right). \tag{C.7}$$

The quadratic differentials of coadjoint elements are extended with a dual central element $\tilde{c}$ that acts on the central element of the adjoint vectors c such that $\tilde{c}(c) = 1$. The coadjoint vectors are labeled by pairs (ϕ, b), representing $\phi(\tau)d\tau^2 + ib\tilde{c}$. By analogy to the centerless diff(S_1), we define the inner product between an adjoint vector (V, a) and a coadjoint vector (ϕ, b) by:

$$\langle(\phi, b) \mid (V, a)\rangle \equiv \int_0^{2\pi} d\tau\, (V(\tau)\phi(\tau)) + ab \tag{C.8}$$

We take b to be a constant in the coadjoint pair. Since the variation of V contains a term $WV' - W'V$, we know from the centerless diff(S_1) that the coadjoint variation contains $2W'\phi + W\phi' \subset \delta_W f$. The additional term should compensate the variation in the center of Eq C.7:

$$\int d\tau\, \delta_W\phi V + \frac{1}{48\pi}\int d\tau\, (VW''' - V'''W)b = \int d\tau\, \delta_W\phi V + \frac{1}{24\pi}\int d\tau\, VW'''b \subset \delta_W(\langle(\phi, b) \mid (V, a)\rangle)$$

Therefore, in order for the inner product to be invariant under infinitesimal variation of W, the correct transformation rule is

$$\delta_W\phi(\tau) = 2W'\phi + W\phi' - \frac{W'''b}{24\pi}, \qquad \delta_W b = 0. \tag{C.9}$$

The coadjoint orbit W_β of a coadjoint vector $\beta = (\phi_0(\theta), t)$ consists of all coadjoint vectors into which β can be transformed by the action of the Virasoro group. Since this is a homogeneous space, every γ in W_β can be obtained from β by a reparametrization of the thermal circle $\tau \to f(\tau)$. The orbit of an element β under the Virasoro group may also be defined as the quotient of the Virasoro group and the isotropy subgroup whose infinitesimal action leaves the coadjoint element invariant. In this case, the action of certain reparametrization elements $f(\tau)$ leaves the point β invariant, and corresponds to the stabilizer of β. The action on a coadjoint vector preserves the central element b, which is a constant that characterizes the orbit.
One can show [66] [20] that the integrated form of Eq C.9 transforms the quadratic differential ϕ_0 to:

$$\tilde{\phi}(\tau) \equiv \mathrm{Ad}^*_{f^{-1}}(\phi_0(\tau)) = f'(\tau)^2\phi_0(f(\tau)) - \frac{b}{24\pi}\{f(\tau), \tau\} \tag{C.10}$$

, while leaving b invariant. For infinitesimal transformations $f(\tau) \to \tau + W$, the variation of the Schwarzian derivative to first order is:

$$\{f(\tau), \tau\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \to \frac{W'''}{1 + W'} - \frac{3}{2}\frac{W''^2}{(1 + W')^2} = W''' + \mathcal{O}(W^2)$$

Together with the variation of $f'(\tau)^2\phi_0(f(\tau)) = (1 + W')^2\phi_0(\tau + W) = \phi_0 + 2W'\phi_0(\tau) + W\phi_0' + \mathcal{O}(W^2)$,

this reproduces Eq C.9 to first order in W. From the finite transformation law of a coadjoint vector, we may interpret ϕ_0 as a stress tensor in a 2d CFT. This is the context in which the Schwarzian derivative naturally shows up.
The finite transformation of an initial ϕ_0 defines a coadjoint orbit $\text{Ad}^*_{f^{-1}}(\phi_0(\tau))$ under the Virasoro group f. The corresponding orbit of the adjoint element $(v(\tau), a)$ in the Virasoro algebra is consequently:

$$\text{Ad}_{f^{-1}}(V(\tau), a) = \left(\frac{V(f(\tau))}{f'(\tau)}, a + \frac{1}{24\pi}\int_0^{2\pi} d\tau \, \frac{V(f(\tau))}{f'(\tau)} \{f(\tau), \tau\}\right). \tag{C.11}$$

It is readily seen that this leaves the inner product $\langle(\phi, b) \mid (v, a)\rangle$ invariant:

$$\langle(\phi, b) \mid (V, a)\rangle = \int_0^{2\pi} d\tau \; f'(\tau)(V(f(\tau))\phi(f(\tau)) + ab = \int_0^{2\pi} d\tau \; V(\tau)\phi(\tau) + ab.$$

The finite transformation Eq C.10 represents the coadjoint orbit of the coadjoint vector $\phi_0(\tau)$, obtained by continuously transforming $\phi_0(\tau)$ by varying f. Some of the coadjoint vectors may be invariant under the action for some f-configurations. Note that any generic constant element ϕ_0 is invariant under the $U(1)$ action of $f(\tau) = \tau + \epsilon$ for a constant ϵ; $\text{Ad}^*_{f^{-1}}(\phi_0) = \phi_0$. Therefore, the natural coadjoint orbit of the latter is the quotient space $\text{diff}(S_1)/U(1)$.
In the case of interest for the Schwarzian boundary theory, we choose a particular identity coadjoint vector $\phi_0 = -\frac{b}{48\pi}$. The coadjoint orbit, obtained by continuously varying f, is the finite-temperature Schwarzian derivative:

$$\text{Ad}^*_{f^{-1}}(-\frac{b}{48\pi}) = -\frac{b}{24\pi}\left[\{f(\tau), \tau\} + \frac{f'(\tau)^2}{2}\right] = -\frac{b}{24\pi}\left\{\tan\frac{f(\tau)}{2}, \tau\right\}. \tag{C.12}$$

In this case, the stabilizer corresponds to the projective $\text{SL}(2, \mathbb{R})$ transformations on $F(\tau) \equiv \tan\frac{f(\tau)}{2}$, that leave the Schwarzian derivative invariant:

$$-\frac{b}{24\pi}\left\{\frac{aF + b}{cF + d}, \tau\right\} = -\frac{b}{24\pi}\{F(\tau), \tau\}.$$

Therefore, we come to the conclusion that the $\text{diff}(S_1)/\text{SL}(2, \mathbb{R})$ integration space of the Schwarzian path integral corresponds to the coadjoint orbit of a particular identity element under the action of the Virasoro group. The Hamiltonian corresponds to time translations $\tau \to \tau + \epsilon$, generated by L_0. The latter corresponds to a constant adjoint vector. The associated group action is obtained by pairing the coadjoint orbit $\text{Ad}^*_{f^{-1}}$ with this constant vector. This is just the Schwarzian action:

$$I \;=\; \langle \text{Ad}^*_{f^{-1}}, c^{te}\rangle \propto \int_0^{2\pi} d\tau \left\{\tan\frac{f}{2}, \tau\right\}. \tag{C.13}$$

C.2 Symplectic structure of coadjoint orbits

A symplectic manifold (M, ω) is a manifold M, equipped with a non-degenerate closed two-form ω, which is called the symplectic form [67]. In local coordinates, we can write $\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu$. The factor $\frac{1}{2}$ prevents overcounting in the antisymmetric indices. This defines an antisymmetric matrix $\omega_{\mu\nu}$. The condition that this

is additionally closed requires the exterior derivative to vanish $d\omega = 0$. By non-degeneracy, there exists an inverse matrix $\omega^{\mu\rho}\omega_{\rho\nu} = \delta^{\nu}_{\mu}$. Note that this requires the manifold to be even dimensional, since it must contain an invertible antisymmetric matrix. This follows from taking the determinant on both sides of this identity.
As an example of a symplectic manifold, consider the cotangent bundle T^*M on a manifold M. This requires the introduction of base coordinates x^i and fibre coordinates p_i, for which one can introduce the one-form $\theta \equiv p_i dx^i$. The symplectic form on this manifold is constructed by taking the exterior derivative on this one-form $\omega \equiv d\theta = dp_i \wedge dx^i$. This has a nice interpretation in classical mechanics, where x^i are the configurational coordinates on a manifold, and p_i its momenta.
Every differentiable function $H : M \to \mathbb{R}$ determines a vector field $X^{\mu}_{H} = \omega^{\mu\nu}\partial_\nu H$ that generates a symplectomorphism in the sense of:

$$\mathcal{L}_{X_H}\omega = 0.$$

Since it is often awkward to work with the inverse symplectic matrix, we can write this equivalently in terms of an interior derivative: $X^{\mu}_{H} = \omega^{\mu\nu}\partial_\nu H \quad \leftrightarrow \quad i_{X_H}\omega = dH$. This defines a vector field X_H under which the symplectic form is invariant;

$$\begin{aligned}\mathcal{L}_{X_H}\omega &\equiv i_{X_H}d\omega + d(i_{X_H}\omega) \\ &= d^2 H = 0\end{aligned}$$

In the first identity, I used the equivalent definition of the Lie derivative [59], and the closure property of the symplectic form. Furthermore, $d^2 \equiv 0$ by commutation of the derivatives. Note that converse is also true (locally). Any generator of a symplectomorphism comes with a Hamiltonian vector field. This follows from the Poincaré lemma [59] that states that any closed form is locally exact. We call H the Hamiltonian and X_H a Hamiltonian vector field. Canonical transformations in classical mechanics are the symplectomorphisms of $\omega \equiv d\theta = dp_i \wedge dx^i$. This defines the Poisson bracket on the symplectic manifold:

$$\{f, g\} \equiv \frac{\partial f}{\partial x^\mu}\omega^{\mu\nu}(x)\frac{\partial g}{\partial x^\nu} \quad \to \quad \mathcal{L}_{X_H}(g) = \{g, H\}.$$

Any coadjoint orbit of the Virasoro group is a symplectic manifold that carries a natural symplectic structure, defined by the two-form (see footnote 2 for a more precise definition):

$$\omega(u, v) = \langle(\tilde{\phi}(\tau), c) \mid [(u, 0), (v, 0)]\rangle \tag{C.14}$$

, where $(\tilde{\phi}(\tau), b)$ is an element in the coadjoint orbit of $\phi_0 = -\frac{b}{48\pi}$, and u, v are elements of the adjoint representation. This is obviously antisymmetric, f-invariant (by construction of the inner product), and will be shown to be closed $d\omega = 0$. Using the explicit commutation relation Eq C.6, we obtain:

$$[(u, 0), (v, 0)] = \left(uv' - vu', -\frac{1}{48\pi}\int_0^{2\pi} d\tau \, (uv''' - vu''')\right)$$

The commutation between two centerless adjoint vectors yields an anomalous central element. Working out the inner product with the explicit form of an element in the coadjoint orbit of ϕ_0 (Eq C.10) yields:

$$
\begin{aligned}
\omega(u,v) &= \int_0^{2\pi} d\tau\,(uv' - vu')\left(-\frac{b}{48\pi}f'(\tau)^2 - \frac{b}{24\pi}\{f(\tau),\tau\}\right) - \frac{b}{48\pi}\int_0^{2\pi} d\tau\,(uv''' - vu''') \\
&\simeq -\frac{b}{24\pi}\int_0^{2\pi} d\tau\,(uv' - vu')\left(\{f(\tau),\tau\} + \frac{1}{2}f'(\tau)^2\right) + \frac{b}{48\pi}\int_0^{2\pi} d\tau\,(u'v'' - v'u'') \\
&= \frac{b}{48\pi}\int_0^{2\pi} d\tau\,\left[(u'v'' - v'u'') - 2(uv' - vu')\left\{\tan\frac{f(\tau)}{2},\tau\right\}\right].
\end{aligned}
$$

Elements in the adjoint vector space correspond to infinitesimal changes of the inverse group elements f^{-1}. Since they parameterize the tangent space to the coadjoint orbits, we can likewise treat them as differentials $u(\tau) = v(\tau) = d(f^{-1}) = \frac{df}{f'}$. A more natural way to write the symplectic form is by interpreting these differentials as one-forms, and by using the antisymmetric wedge product. Up to a constant prefactor, this is:

$$
\omega = \int_0^{2\pi} d\tau\,\left[\left(\frac{df(\tau)}{f'(\tau)}\right)' \wedge \left(\frac{df(\tau)}{f'(\tau)}\right)'' - 2\left\{\tan\frac{f(\tau)}{2},\tau\right\}\left(\frac{df(\tau)}{f'(\tau)}\right) \wedge \left(\frac{df(\tau)}{f'(\tau)}\right)'\right]. \tag{C.15}
$$

A lengthy, but otherwise straightforward calculation shows that this can be also be written as [20]:

$$
\omega = \int_0^{2\pi} d\tau\,\left[\frac{df'(\tau)\wedge df''(\tau)}{f'(\tau)^2} - df(\tau)\wedge df'(\tau)\right] \tag{C.16}
$$

Note that d is the abstract exterior derivative that acts only on the fields f, and not on the coordinate τ. Therefore, it commutes with ∂_τ: $df' = \partial_\tau df$. From the antisymmetry of the wedge product between one-forms, we can neglect the explicit notation of $\wedge$, and view the coordinates df as fermionic partners of f, satisfying the Grassmann algebra. Using the commutation between the exterior derivative d and ∂_τ, and $(df')^2 = 0$, the symplectic form is readily

$$
\omega = \int_0^{2\pi} d\tau\,\left[\left(\frac{df'}{f'}\right)\partial_\tau\left(\frac{df'}{f'}\right) - df\,\partial_\tau df\right]. \tag{C.17}
$$

Since we can write $\left(\frac{df'}{f'}\right) = d\log f'$, the above notation of the symplectic measure is indeed manifestly closed $d\omega \equiv 0$.

The left action of the Virasoro group on f by an infinitesimal diff(S_1) transformation $\delta\tau = \alpha(\tau)$ acts by the Lie bracket $\delta f(\tau) = \mathcal{L}_V f = \alpha(\tau)f'(\tau)$. For $\alpha = 1$, this transformation is associated with a time translation symmetry $\delta f = f'$. In this case, this is a Killing vector field $V = f'$, corresponding to transformations of $f \to f + \delta f$. This in turn generates an associated Hamiltonian from the general definition $i_V\omega = dH$. We can prove that this Hamiltonian is associated with the Schwarzian derivative by explicitly computing the interior derivative[4] on the two-form ω. We replace one copy of df by $\partial_\tau f$ while taking into account the anticommutativity of between i_V and df:

$$
i_V\omega = \int_0^{2\pi} d\tau\,\left[\left(\frac{\partial_\tau f'}{f'}\right)\partial_\tau\left(\frac{df'}{f'}\right) - \left(\frac{df'}{f'}\right)\partial_\tau\left(\frac{\partial_\tau f'}{f'}\right) - \partial_\tau f\,\partial_\tau df + df\,\partial_\tau^2 f\right].
$$

[4]The interior derivative maps a p form into a $(p-1)$-form by replacing one copy of dx^μ by V^μ; $i_V\omega^{(p)} = \frac{1}{(p-1)!}\omega_{\mu\nu\ldots\lambda}V^\mu dx^\nu \wedge dx^\lambda$

Partially integrating yields:

$$i_V\omega = \int_0^{2\pi} d\tau \left[2\left(\frac{\partial_\tau f'}{f'}\right)\partial_\tau\left(\frac{df'}{f'}\right) - 2\partial_\tau f\partial_\tau df\right] = \int_0^{2\pi} d\tau \left[2\left(\frac{f''}{f'}\right)\partial_\tau\left(\frac{df'}{f'}\right) - 2f'df'\right].$$

Using that $\partial_\tau\left(\frac{df'}{f'}\right) = \frac{df''}{f'} - \frac{df'}{(f')^2}f'' = d\left(\frac{f''}{f'}\right)$, we readily note that the Hamiltonian for which $i_V\omega = dH$, is given by:

$$H = \int_0^{2\pi} d\tau \left[\left(\frac{f''}{f'}\right)^2 - (f')^2\right] \tag{C.18}$$

Comparing this with Eq C.3, we deduce that the Hamiltonian associated to time translations $\tau \to \tau + \epsilon$, with the left action on $f(\tau) \to f(\tau) + \epsilon f'(\tau)$ is associated to the Schwarzian action, up to a factor of $\frac{\pi C}{\beta}$:

$$\boxed{H = -2\int_0^{2\pi} d\tau \left\{\tan\frac{f}{2}, \tau\right\}} \tag{C.19}$$

In order for ω to be a proper symplectic measure, it needs to be non-degenerate. However, expanding the reparametrization mode into its Fourier components $f(\tau) = \tau + \sum_n e^{in\tau}u_n$, we see that the two-form Eq C.17 vanishes for certain n. Indeed, at[5] $u_n = 0$;

$$\begin{aligned}
\omega &= \int_0^{2\pi} \left[\left(\frac{df'}{f'}\right)\partial_\tau\left(\frac{df'}{f'}\right) - df\partial_\tau df\right] \\
&= \int_0^{2\pi} \left[\left(\sum_n ine^{in\tau}du_n\right)\partial_\tau\left(\sum_m ime^{im\tau}du_m\right) - \sum_n du_n e^{in\tau}\sum_m ime^{im\tau}du_m\right] \\
&= \sum_{nm}\left[\int_0^{2\pi}\left(-inm^2e^{i(n+m)\tau}du_ndu_m - idu_ndu_m me^{i(n+m)\tau}\right)\right] \\
&= -2\pi i\sum_n (n^3 - n)du_ndu_{-n}.
\end{aligned} \tag{C.20}$$

Therefore, the measure vanishes for $n = -1, 0, +1$. The action associated with these zero-modes spans precisely the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in Eq C.5. Therefore, the two-form ω is non-degenerate on the restriction of the integration space to the right quotient space diff$(S_1)/SL(2,\mathbb{R})$. Here, we modded out over the right action $f(\tau) \to g(f(\tau))$, which acts $\delta f = g(\tau) = \sum_n g_n e^{in\tau}$. The quotient diff$(S_1)/SL(2,\mathbb{R})$ still admits an invariant action under diff(S_1), which descends from the left action of diff(S_1) on itself; $f(\tau) \to f(g(\tau))$, and acts as $\delta f(\tau) = g(\tau)\partial_\tau f$. Therefore, ω is still diff(S_1) invariant, and takes the same form everywhere.
Any $2n$-dimensional symplectic manifold admits a natural integration volume form, obtained by exponentiating the two-form ω by n: $dV = \frac{1}{n!}\omega^n$, where ω^n is non-zero from the non-degeneracy of the symplectic form.

[5]Since ω is diff(S_1) invariant, the same conclusion is true everywhere.

The Pfaffian is defined in this context by[6]

$$dV = \frac{1}{n!}\omega^n = \frac{1}{n!}\frac{1}{2^n}\omega_{i_1 i_2}\dots\omega_{i_{2n-1}i_{2n}}\epsilon_{i_1 i_2 \dots i_{2n-1}i_{2n}} dx^0 \wedge dx^1 \wedge \dots \wedge dx^{2n-1}$$
$$\equiv \text{Pf}(\omega)\, dx^0 \wedge dx^1 \wedge \dots \wedge dx^{2n-1}. \tag{C.21}$$

This is related to the determinant of the $2n \times 2n$-antisymmetric matrix $\omega_{\mu\nu}$ [110];

$$\text{Pf}(\omega) = \sqrt{\det(\omega)}. \tag{C.22}$$

The Pfaffian $\text{Pf}(\omega)$ is related to a Gaussian integral over $2n$ Grassmann variables[7] θ_i;

$$\int d^{2n}\theta e^{\frac{1}{2}\theta_i\omega_{ij}\theta_j} = \int d^{2n}\theta \frac{1}{2^n n!}\omega_{i_1 i_2}\dots\omega_{i_{2n-1}i_{2n}}\theta_{i_1}\theta_{i_2}\dots\theta_{i_{2n-1}}\theta_{i_{2n}} = \text{Pf}(\omega) \tag{C.23}$$

, where I used the unique Taylor expansion of the Grassmann exponential, and used that only the terms linear in the distinct Grassmann fields survive. The generalization to a continuous function $\omega(f,g)$ is trivial by defining Grassmann functions $\theta(\tau)$ and performing a Gaussian path integral instead.
Rewriting the volume form as a fermionic Gaussian integral allows us to define Grassmann variables $\eta = \frac{df(\tau)}{f'(\tau)}$, and to write down the volume form associated with Eq C.15 as:

$$dV = \prod_\tau \eta(\tau) \int \mathcal{D}\eta\, \eta(0)\eta'(0)\eta''(0) \exp\left[\frac{1}{2}\int d\tau \left(\eta'\eta'' - 2\left\{\tan\frac{f}{2},\tau\right\}\right)\eta\eta'\right]. \tag{C.24}$$

The infinite product in front of the integral is due to the change of fermionic variables from df to ψ. The factors $\eta(0)\eta'(0)\eta''(0)$ are introduced to gauge fix the $\text{SL}(2,\mathbb{R})$ zero-modes, and to make the path integral non-trivial over a periodic time interval [20]. It looks like this measure is non local in $f(\tau)$. Remarkably, the path integral turns out to be independent of $f(\tau)$, as shown in [20]. They evaluated the path integral by canonically quantizing the fields, and representing it as a trace over a Hamiltonian, along the lines of Eq D.26. This allows to solve the Schrödinger equation associated with the induced Hamiltonian $H(\tau)$, and to compute the periodic operator $U(2\pi) = \mathcal{P}e^{-\int_0^{2\pi} H(\tau)d\tau}$. The latter turns out to be $U(2\pi) = -1$. The fact that the operator that describes time evolution around the circle is independent of $f(\tau)$ makes the path integral in Eq C.24 independent of the reparametrization mode $f(\tau)$. This basic fact allows us to write the measure completely in terms of the local variables $f(\tau)$;

$$dV = \prod_\tau \eta(\tau) = \prod_\tau \frac{df(\tau)}{f'(\tau)}. \tag{C.25}$$

C.3 One-loop exactness of the Schwarzian theory

We have argued that the integration space $\text{diff}(S_1)/SL(2,\mathbb{R})$ is a symplectic manifold. The $U(1)$ time translations $\tau \to \tau + \alpha$ with $\delta\tau = \alpha$ act on the reparametrization elements as $f(\tau) \to f(\tau) + \delta f(\tau) = f(\tau) + \alpha f'(\tau)$,

[6] The additional factor $\frac{1}{n!}$ prevents overcounting in the n-th exponentiated wedge product.

[7] When introducing n independent "complex conjugates" $\overline{\theta}_i$, one can relate this to the determinant of an $n \times n$ matrix A_{ij} instead; $\int d^n\theta d^n\overline{\theta} e^{\overline{\theta}_i A_{ij}\theta_j} = \det A$.

with an associated Killing vector field $V = f'$. The Hamiltonian of this transformation ($i_V\omega = dH$) was argued to correspond precisely the Schwarzian action, up to a constant factor Eq C.19. Furthermore, the classical solutions $f(\tau) = \tau$ are $U(1)$ invariant, since we consider the quotient space under $\mathrm{SL}(2,\mathbb{R})$, whose action acts as a gauge redundancy under constant shifts $\tau \to \tau + c$. Under this gauge redundancy, we only consider equivalence classes of the subgroup $U(1)$. These basic facts allow us to utilize the Duistermaat-Heckman [68] theorem on supersymmetric localization.

If we wish to evaluate the integral over a symplectic manifold (x^n) on which the action acts as the generator of a $U(1)$ symmetry $x^n \to x^n + v^n$ with $v^n = -\omega^{nm}\partial_m H$, we can introduce Grassmann variables ψ_n and write the associated Pfaffian of the symplectic measure as a Gaussian integral. It turns out that the integral

$$\int \mathcal{D}x \mathrm{Pf}(\omega)\, e^H = \int \mathcal{D}x\mathcal{D}\psi \exp\left[H + \frac{1}{2}\psi_m\omega_{mn}\psi_n\right] \tag{C.26}$$

is one loop exact around the classical saddle point. For this, we consider a supersymmetry transformation:

$$Qx^n = \psi_n, \qquad Q\psi_n = v^n = \omega^{nm}\partial_m H \tag{C.27}$$

, such that $Q^2(x^n) = v^n$ is the generator of the $U(1)$ symmetry associated with H. It follows that the total action is Q-invariant using $d\omega = 0$;

$$\begin{aligned} Q(\omega + H) &= \frac{1}{2}Q(\psi_m\omega_{mn}\psi_n) + (Qx^n)\partial_n H = \frac{1}{2}v^m\omega_{mn}\psi_n - \frac{1}{2}\psi_m\omega_{mn}v^n + \psi_n\partial_n H \\ &= -\frac{1}{2}\partial_a H\omega^{am}\omega_{mn}\psi_n + \frac{1}{2}\psi_m\omega_{mn}\partial_a H\omega^{an} + \psi_n\partial_n H = 0. \end{aligned}$$

It also follows that we can add a Q-exact $U(1)$-invariant term to the action sQV with $Q^2V = 0$ without changing the integral, where s is a continuous parameter;

$$\frac{d}{ds}\int Z = \int \mathcal{D}x\mathcal{D}\psi\; QV\; \exp\left[H + \frac{1}{2}\psi_m\omega_{mn}\psi_n + sQV\right] \tag{C.28}$$

$$= \int \mathcal{D}x\mathcal{D}\psi\; Q\left(V\; \exp\left[H + \frac{1}{2}\psi_m\omega_{mn}\psi_n + sQV\right]\right) \tag{C.29}$$

Assuming that the Q-symmetry is not anomalous, and the volume element is invariant, the last identity is zero, and the path integral is independent of s [111].

We are in this situation, since the action in the path integral Eq 1.133 can be written in the form of Eq C.26 under a redefinition of ψ. For the case at hand, we also rewrite $f(\tau) = \tau + \left(\frac{\beta}{2\pi C}\right)^{1/2}\epsilon$, and take the fermionic symmetry to be [20];

$$Q\epsilon = \psi, \qquad Q\psi = \epsilon'. \tag{C.30}$$

This generates a $U(1)$ symmetry corresponding to $\frac{d}{d\tau}$. We add the term Q-exact term $sQV = s\int d\tau(\epsilon'^2 + \psi'\psi)$ to the path integral, which is found to be additionally $U(1)$-invariant;

$$Q^2V = \int d\tau(2\epsilon'\psi' + \epsilon''\psi - \psi'\epsilon') \simeq \int d\tau(2\epsilon'\psi' - \epsilon'\psi' - \psi'\epsilon') = 0.$$

Although this changes the integrand, it does not change the integral, as elaborated before. Since term sQV has the form of one-loop propagators, we can consider $s \to \infty$. By the above Duitsermaat-Heckman theorem, the value of the integral is independent of s. On the classical saddles of sQV, this term vanishes, and we retain the original classical saddle point value. On the other hand, this renders the two-point propagators arbitrary small, and suppresses higher order corrections to the one-loop determinant. Therefore, the result Eq 1.138 is the entire answer to all orders.

Appendix D

Classical laws of Black Hole thermodynamics

"Black holes are where God divided by zero."

Einstein, Albert

This appendix will cover the basics of the classical laws of hole thermodynamics in 3+1d. Black holes are essentially the simplest solutions of General Relativity. One might disagree on a conceptual basis, but in contrast to other astrophysical objects, they are just vacuum solutions described by only a few parameters (their mass, charge, and angular momentum for a Kerr-Newman BH). The no-hair theorem then essentially implies that any black hole, regardless of how complicated the initial forming conditions might have been, is just described by these parameters. The process of hairloss is described by quasinormal ringing. Therefore, the system will equilibrate to a state that is described unambiguously by a finite set of parameters. The classical laws that follow from Einstein's equations indeed imply black holes to obey laws that look familiar to the laws of classical thermodynamics. Historically, these laws were discovered before Hawking's result on black hole evaporation [7]. Initially, the thermodynamics that seemed to describe black holes was thought to be an accidental similarity. The temperature and entropy could, moreover, only be identified up to a proportionality constant. Hawking's result, however, proved these laws to be more than a formal coincidence. Instead black holes were identified to be systems in thermal equilibrium with a heat bath at infinity with a definite temperature. In quantum gravity, the laws of black hole thermodynamics are manifestations of the true UV descriptions of the black hole microstates. In holography, these degrees of freedom can be described in the dual quantum field theory. The first black hole microstate counting was a string theory calculation by Strominger and Vafa [112]. Strominger also proved in [113] that the BTZ black hole entropy in AdS_3 is one-to-one related to the Cardy formula in the dual CFT_2. The Cardy formula counts microstates in statistical mechanics, and by the holographic duality, they must be microstates of quantum gravity in AdS_3 as well. These results proved unambiguously that the thermodynamics from black holes are manifestations of true quantum microstates, and not some "magic" analogy.

D.1 The Reissner-Nordström black hole

Let us first consider the Reissner-Nordström solutions (RN) of the coupled Einstein-Maxwell equations, describing charged black holes. Consider the Einstein-Maxwell (EM) action in 3+1d. This is a specific case of the Einstein-Hilbert action coupled to a Maxwell matter sector [51]

$$S_{EM} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,R - \frac{1}{4}\int d^4x\,\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{D.1}$$

The first term is the Einstein-Hilbert action, and is the simplest local action one can write down that is composed of second order derivatives of the metric. In a variational context, one imposes Dirichlet boundary conditions for the metric. These boundary conditions require at least second order derivatives to obtain non-trivial dynamics. Although non-trivial, it is a standard exercise in general relativity to obtain the Einstein equation from the variational solutions, see e.g. [60][59]. $F_{\mu\nu}$ is the field strength, $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\mu A_\mu = \partial_\mu A_\mu - \partial_\nu A_\mu$, with covariant derivative $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu}A_\alpha$. The Maxwell-term is minimally coupled to the metric: $F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}$. Writing $R = R_{\mu\nu}g^{\mu\nu}$, and using the identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, the variation of the total action leads to the coupled Einstein-Maxwell equations;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \qquad \nabla_\mu F^{\mu\nu} = 0. \tag{D.2}$$

The term proportional to $\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\mu \delta\Gamma^\rho_{\nu\rho}$ vanishes[1] since it is a total derivative. The stress tensor $T_{\mu\nu}$ is defined as the variation of the matter action S_M with respect to the metric

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = -\frac{1}{4}g_{\mu\nu}F^2 + g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}. \tag{D.3}$$

The solution for a black hole with mass M and electric charge Q is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2}, \qquad \text{with} \qquad f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}. \tag{D.4}$$

The electromagnetic field is related to the charge Q as $A_\mu dx^\mu = -\frac{Q}{4\pi r}dt$, or $F_{r0} = \frac{Q}{4\pi r^2}$. In the asymptotic regime $r \to \infty$, $g_{00} \to -(1 - \frac{2GM}{r})$, such that M can indeed be identified with the mass M of the black hole. It is often more convenient to decompose $f(r)$ in terms of its zeros r_- and r_+:

$$f(r) = \frac{1}{r^2}(r - r_+)(r - r_-), \qquad \text{with} \qquad r_\pm = GM \pm \sqrt{G^2M^2 - \frac{GQ^2}{4\pi}}. \tag{D.5}$$

r_+ is the event horizon, while r_- is the Cauchy horizon. A horizon is generally a Cauchy surface with a null normal vector. For constant r slices ($dr = 0$), this implies $g_{00} \to 0$. The coordinate system breaks down at either horizon, while the true geometry and field strength remain smooth. There is a curvature singularity however at $r = 0$. At the event horizon $r = r_+$, the coordinate r becomes timelike, while t becomes spacelike. When $\frac{dr}{dt} < 0$ at the horizon, the natural direction is inwards. However, once we reach the Cauchy horizon, t becomes timelike again, and r spacelike. One can therefore reverse their direction outwards $\frac{dr}{dt} > 0$, and

[1] A priori, this argument only works for a compact spacetime $\partial\Omega = 0$. For non-compact spacetimes, one requires a suitable Gibbons-Hawking-York (GHY) boundary term in order to obtain a well-defined variational problem with a second order derivative action $S \sim \mathcal{O}(\partial^2 g)$

escape from the interior. Once we reach the Cauchy horizon $r = r_-$, the causal properties again reverse, and the natural direction is outwards until they reach $r = r_+$. The causal structure is summarized in the Penrose diagram in figure D.1. One requires $Q^2 \leqslant 4\pi GM^2$ in order to have $r_- \leqslant r_+$. When the Cauchy horizon coincides with the event horizon $r_- = r_+$, or $Q^2 = 4\pi GM^2$, the black hole is extremal, and has a net zero interaction energy. $Q^2 > 4\pi GM^2$ is prohibited from the cosmic censorship principle, since the curvature singularity at $r = 0$ would not be shielded by a zero of g_{00}.

D.2 Laws of black hole thermodynamics

First law: conservation of energy

It was noticed in the 1970s that small changes in the black hole geometry are described by equations closely parallel to the laws of thermodynamics [114][115]. This becomes very transparent for the RN solutions. Since the metric at the horizon $r = r_+$ is $ds^2 = r_+^2 d\Omega_2^2$, the area is

$$A = 4\pi r_+^2 = 4\pi \left(GM + \sqrt{G^2M^2 - \frac{GQ^2}{4\pi}} \right)^2 .$$

Variation of this area term yields:

$$\delta A = 8\pi r_+ \left(\frac{\partial r_+}{\partial M}\delta M + \frac{\partial r_+}{\partial Q}\delta Q \right) = \frac{16\pi G r_+}{r_+ - r_-} \left(r_+ \delta M - \frac{Q}{4\pi}\delta Q \right) \tag{D.6}$$

This is reminiscent of the first law of thermodynamics, where the heat Q transferred to a system equals the change in internal energy $\Delta E = Q$. For quasistatic changes among equilibria, we have $\delta Q = TdS$. In the presence of an electrical potential Φ, the first law becomes

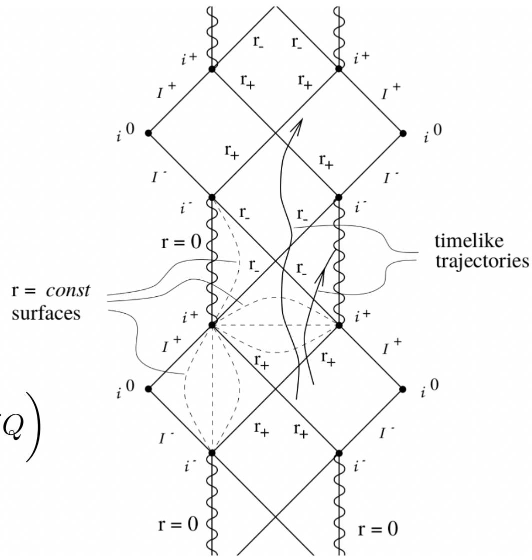


Figure D.1: Penrose-Carter diagram of the Reinssner-Nordström black hole. i_0, $t^\pm$ and $I^\pm$ are the spacelike, past and future timelike, and past and future lightlike infinity respectively. Possible timelike curves, and the curvature singularity at $r = 0$ are indicated. [60]

$$TdS = dE - \Phi dQ \tag{D.7}$$

We can identify the potential at the horizon $\Phi = \frac{Q}{4\pi r_+}$, and the mass M with the energy E. If we further interpret the entropy to be proportional with the area of the horizon $S \sim A$, then the coefficient in front is related to the temperature $T \sim \frac{r_+ - r_-}{r_+^2}$. Of course to identify the entropy and temperature correctly, one should know the exact proportionality factor of at least one of the former quantities. This was only possible after Hawking's calculation of black hole radiation. T is related to the surface gravity of the black hoke, which is defined physically as the acceleration due to gravity near the horizon times a redshift factor. It can be shown that this is constant everywhere on the horizon of a stationary black hole. This makes an analogy with the zeroth law of thermodynamics, where in equilibrium, the temperature is constant.

Second law: entropy never decreases

Hawking showed in 1971 that the black hole horizon can never decrease classically [116]. Therefore, identifying the entropy with the area of the horizon (up to a proportionality factor) is a natural thing to do. The fact that black holes carry entropy is required to salvage the second law of thermodynamics. Indeed, if black holes did not carry any intrinsic entropy, the entropy of the universe could decrease by throwing matter into a black hole. To salvage the second law, one imposes that next to the entropy associated to the matter S_m, the black hole carries entropy themselves S_{BH}, such that in a closed system $\delta(S + S_{BH}) \geqslant 0$. Since only the sum of the entropy should increase, one often talks about the generalized second law of thermodynamics.
In a stronger statement, S_m does not only include classical matter, but also includes gravitons outside the black hole and a vacuum contribution from the quantum fields [117]. The generalized entropy, including this quantum term, is also found to obey the second law of thermodynamics, giving further evidence that it is really an entropy [118]. This result is stronger than the classical area theorem because it also covers phenomena like Hawking radiation, when the area decreases but the generalized entropy increases due to the entropy of Hawking radiation.

D.3 Hawking radiation

To deduce the correct relation between entropy and area via the thermodynamic relations, one first has to know an exact expression of the black hole temperature. This calculation was famously done by Hawking in [7]. By using a quantum field in a fixed black hole background, Hawking was able to predict the existence of a heat bath of thermal radiation in the black hole spacetime. One should stress, however, that this calculation is still semiclassical in nature; one only considers quantum fields in a fixed spacetime, the spacetime itself is not allowed to fluctuate. The calculation relies on the equivalence principle; in a black hole spacetime, a free falling observer is in a locally flat spacetime. Observers hovering at a constant radius near the horizon will need a constant acceleration to overcome the gravitational pull of the black hole. Their natural coordinate system will therefore be the Rindler spacetime. Their vacuum is, of course, not the Minkowksi vacuum state. Minkowksi modes are expanded in Minkowksi plane waves, with their own creation and annihilation operators $a_i^\dagger,\ a_i$ respectively. Each creation/annihilation operator is associated with a specific energy eigenvalue ω_i, such that schematically a scalar field ϕ can be expanded in this basis;

$$\phi = \sum_i (a_i\, e^{-i\omega_i t} + a_i^\dagger\, e^{i\omega_i t}).$$

Sate vectors will be created by applying creation operators, and will be energy eigenvectors of the Hamiltonian. Since the Hamiltonian H_t is the generator of time evolution for Heisenberg operators O: $\frac{i}{\hbar}[H_t, O] = \partial_t O$, a different time coordinate will be associated to different energy eigenmodes, which in turn define the vacuum state via $a_i\,|0\rangle = 0$, for all i. Since Rindler observers have a different choice of time coordinate, their notion of energy is different, and with it, the notion of the vacuum. Since either set of modes forms a complete set, one can indeed show that the Minkowski vacuum corresponds to a thermal spectrum by expressing the Rindler modes in terms of Minkowksi modes via a linear Bogoliubov transformation. This is the Unruh effect of a

Rindler observer. By the equivalence principle, stationary observers near the horizon will detect this thermal spectrum, which can be extrapolated to distant regions by incorporating a redshift factor. The redshifted temperature will asymptotically approximate the Hawking temperature.

A more thorough analysis of Hawking radiation can be found in the excellent lecture notes by Jacobson [119]. By linearly expanding the creation/annihilation modes corresponding to different time coordinates via a Bogoliubov transformation, one has a clear intuiting about the physics governing Hawking radiation. However, this procedure becomes technically difficult rather quickly.
An often useful calculational trick is to express Lorentzian spacetime in a Euclidean manifold by performing a Wick transformation. The derivation of the Hawking temperature is then just a consistency condition on the manifold. This, however, obscures any physical intuition that one has about Hawking temperature.

Most discussions on this subject start with the Schwarzschild geometry. However, to make contact with the previous section, we start again from the RN black hole solution Eq D.4, and ignore the angular dependence which will not be relevant in the following discussion

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + \cancel{r^2 d\Omega^2}. \tag{D.8}$$

The Schwarzschild geometry can be simply obtained by putting $Q = 0$. To obtain a non-zero temperature, consider a sub-extremal black hole $r_+ > r_-$. The outer event horizon $r = r_+$ determines the size of the black hole. To zoom in on the event horizon, introduce $\rho \equiv r - r_+$, and take $\rho \ll r_+$. In this approximation, the metric becomes

$$ds^2 = -\frac{\rho^2(r_+ - r_-)}{r_+^2}dt^2 + \frac{r_+^2}{\rho(r_+ - r_-)}d\rho^2. \tag{D.9}$$

Define now an additional radial coordinate in terms of $dw \equiv \frac{r_+}{\sqrt{\rho(r_+-r_-)}}d\rho$. One can integrate this to the explicit relation $w = 2\frac{r_+\sqrt{\rho}}{\sqrt{r_+-r_-}}$. Inserted in the metric, this becomes;

$$ds^2 = -\frac{w^2}{4}\frac{(r_+ - r_-)^2}{r_+^4}dt^2 + dw^2. \tag{D.10}$$

By defining the temporal Rindler coordinate $\tau \equiv \frac{(r_+-r_-)}{2r_+^2}t$, the manifold near the black hole horizon indeed becomes a patch of the 1+1d Rindler spacetime, added with an angular dependence $Ri_2 \oplus S_2$

$$ds^2 = -w^2 d\tau^2 + dw^2. \tag{D.11}$$

This is equivalent to the Minkowksi space $\mathbb{R}^{1,1}$ $ds^2 = -(dx^0)^2 + (dx^1)^2$ for accelerating observers with acceleration $a = 1/w$;

$$x^1 = w\cosh\tau, \qquad x^0 = w\sinh\tau. \tag{D.12}$$

As such, these coordinates will only cover the right Rindler wedge: $x^1 > 0$, $|x^0| < x^1$. The causal structure of a collapsing star and the near horizon region are displayed in figure D.2. One recognises the early-time geometry to be the Penrose diagram of flat Minkwoski spacetime, while the late-time geometry after the gravitational collapse is described by the Schwarzschild Kruskal coordiantes instead. The near horizon region of a static observer is described by a patch of Rindler spacetime.

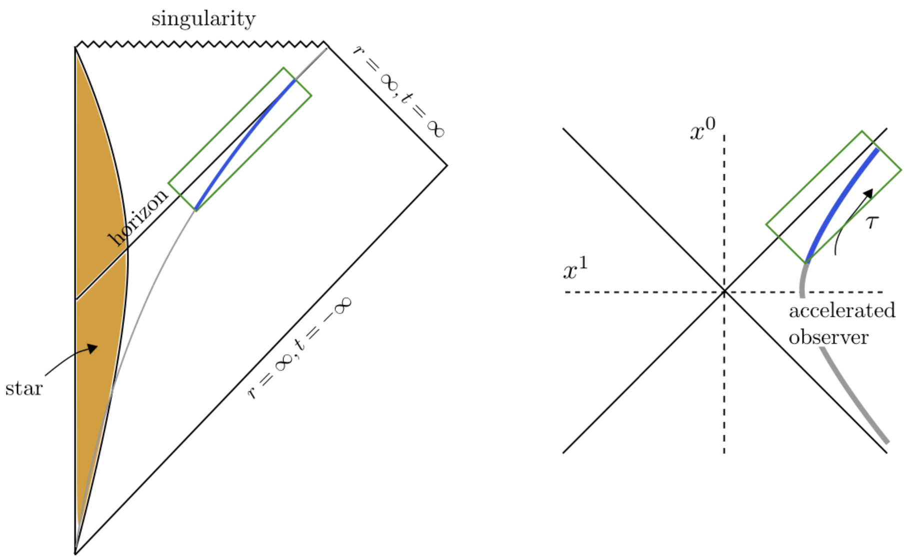


Figure D.2: Left: Penrose diagram of a black hole formed by gravitational collapse. Right: Near horizon region of a constantly accelerating observer, described by a Rindler patch [101].

The Euclidean Black Hole

To derive the Hawking temperature from the near horizon region, we first note that QFT at finite temperature T is periodic in imaginary time with periodicity $t \sim t + i\beta$, where $\beta \equiv 1/T$.

By applying a Wick rotation $t = it_E$ on the black hole spacetime, one finds the Euclidean black hole geometry with Euclidean signature metric $(+,+,+,+)$ at $r \geqslant r_+$:

$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2} dt_E^2 + \frac{r^2}{(r-r_+)(r-r_-)} dr^2 + r^2 d\Omega^2. \tag{D.13}$$

The Wick-rotated time coordinate is periodic and describes a time circle that shrinks down to zero at the horizon. Behind the horizon, $r < r_+$, the geometry does no longer posses a Euclidean signature. Therefore, Euclidean black hole geometries are only well defined outside the horizon. Graphically, the geometry is called the Euclidean cigar, shown in figure D.3

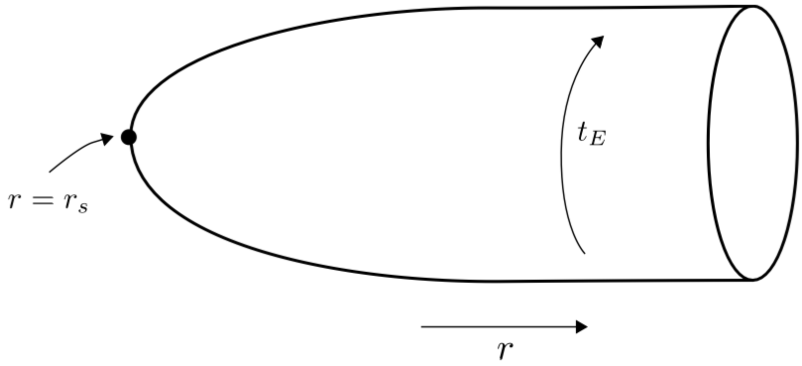


Figure D.3: The Euclidean black hole "cigar" geometry. The geometry is contractible at the horizon $r = r_s$ [101].

In general, any geometry with a contractible time circle describes a black hole manifold. The origin is a smooth point and corresponds to the Euclidean horizon.

By applying a Wick rotation in the near horizon region $x_E^0 = it$, $\tau = i\theta$, the Minkowski manifold $ds^2 = -(dx^0)^2 + (dx^1)^2 \rightarrow ds_E^2 = (dx_E^0)^2 + (dx^1)^2$ is identified with the flat Euclidean plane. The Rindler coordinate transformations Eq D.12 become the standard polar coordinate transformations

$$x_E^0 = w \sin\theta, \qquad x^1 = w \cos\theta$$

that describe the Euclidean plane

$$ds_E^2 = (dx_E^0)^2 + (dx^1)^2 = w^2 d\theta^2 + dw^2. \tag{D.14}$$

Since the black hole geometry is non-singular near the horizon, the plane should not contain a conical singularity. To avoid a conical singularity, the phase angle θ should be periodic in 2π. At a distance w near the horizon, the proper time τ of a Rindler observer is related to $d\tau = w\ d\theta$. Since the periodic Euclidean time coordinate is related to the inverse temperature, we can identify one rotation of θ in 2π with a translation of τ in $\beta = 1/T$, or:

$$T_{proper} = \frac{1}{2\pi w} = \frac{a}{2\pi} \tag{D.15}$$

, where in the second line, the Rindler distance was related to the acceleration $w = 1/a$. This is the proper temperature felt by an observer, and diverges near the horizon $\rho = 0$. This factor tracks the redshifting of photons as they climb the gravitational potential [101]. The Unruh temperature associated to the Rindler time θ undoes the redshift factor

$$T_U = \frac{1}{2\pi}. \tag{D.16}$$

In Eq D.11, the temporal coordinate is identified with the phase angle by $\theta \equiv \frac{(r_+ - r_-)}{2r_+^2} t_E$. A rotation in $\theta :\ 0 \rightarrow 2\pi$ should corresponds with a translation in $t :\ 0 \rightarrow \beta = 1/T$. This identification allows one to deduce directly the Hawking temperature T_H of the RN black hole $T = 1/\beta$;

$$\begin{aligned} 2\pi &\equiv \frac{(r_+ - r_-)}{2r_+^2}\beta \\ \Longleftrightarrow \qquad T_H &= \frac{(r_+ - r_-)}{4\pi r_+^2}. \end{aligned} \tag{D.17}$$

Note that this is not a "derivation" of the Hawking temperature; rather it is a necessary and sufficient condition for the Hawking temperature that needs to be obeyed in order to yield a regular solution to the Einstein equations. The physical intuition is that the temperature needs to be $T = T_H = 1/\beta$ to define a canonical ensemble in thermal equilibrium with a heath bath at infinity. At $T \neq T_H$, the black hole is not in equilibrium with the heat bath, which translates into the development of a conical singularity.
One notes that extremal black holes $r_- = r_+$ have zero temperature, which is often the defining feature of extremal black holes in practice. For the Schwarschild solution with $Q = 0$, one finds directly from Eqs D.5

$$T_H = \frac{1}{8\pi GM}. \tag{D.18}$$

The precise identification of the Hawking temperature allows to derive the thermal entropy from Eq D.6. When identifying $E = M$, the latter can be rewritten in the form of an exact thermodynamic relation $TdS = dM - \Phi dQ$

$$\begin{aligned} \frac{1}{4\pi}\left(\frac{r_+ - r_-}{r_+^2}\right) d\left(\frac{A}{4G_N}\right) &= dM - \frac{Q}{4\pi r_+}dQ \\ \Longleftrightarrow \qquad T\, d\left(\frac{A}{4G_N}\right) &= dM - \Phi dQ. \end{aligned} \tag{D.19}$$

We can therefore identify the Bekenstein-Hawking entropy [8] as

$$S_{BH} = \frac{A}{4G_N}. \tag{D.20}$$

D.4 Euclidean Path Integral

The Wick rotation [50] is a procedure that allows to replace a Lorentzian metric by a Euclidean metric, by analytically continuing from real to complex coordinates $t \to -i\tau$. This may avoid complications in the former case due to the fact that the metric is not positive-definite. Under this transformation, the metric transforms as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\vec{x}^2 \;\to\; ds_E^2 = \delta_{\mu\nu}dx^\mu dx^\nu = d\tau^2 + d\vec{x}^2.$$

The combination $\sqrt{-g}d^Dx$ is a tensor density, and is therefore invariant under oriented (complex) coordinate transformations; $\sqrt{-g}d^Dx = \sqrt{-g_E}d^Dx_E$. I denoted x_E and g_E as the Euclidean Wick-rotated coordinates and metric respectively. The volume element itself transforms $d^Dx = -id^Dx_E$. Therefore, $\sqrt{-g} = i\sqrt{-g_E}$. To yield a positive-definite metric, we use the branch $\sqrt{-1} = -i$ to obtain $\sqrt{-g} = \sqrt{g_E}$.

Therefore, the action transforms as

$$S = \int d^D x \, \sqrt{-g}\mathscr{L} = -i \int \, d^D x_E \sqrt{g_E}\mathscr{L} = i \int \, d^D x_E \sqrt{g_E}\mathscr{L}_E.$$

$\mathscr{L}_E = -\mathscr{L}$ is the positive-definite Lagrangian, which is in turn related to the energy-momentum tensor. One can therefore identify $I = \int \, d^D x_E \sqrt{g_E}\mathscr{L}_E$, and $S = iI$. In the following, I will always denote the Euclidean action as I, to distinguish from the Lorentzian signature action S.

As elaborated by the Hawking-Gibbons prescription in [65], the real time path integral is a rewriting of the transition function between two states $\langle \phi_2, t_2 \mid \phi_1, t_1 \rangle$ in terms of a functional integral over all configurations ϕ with fixed boundary conditions ϕ_1 at t_1 and ϕ_2 at t_2, weighted by the action $S[\phi]$;

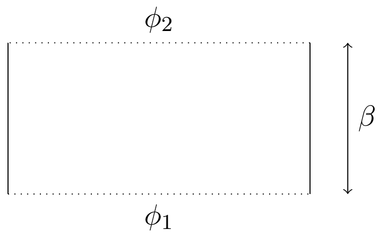
$$\langle \phi_2, t_2 \mid \phi_1, t_1 \rangle = \int [d\phi] \, e^{iS[\phi]}. \tag{D.21}$$

The overlap can be written in terms of the real time propagator

$$\langle \phi_2, t_2 \mid \phi_1, t_1 \rangle = \langle \phi_2 | \, e^{-iH(t_2 - t_1)} \, | \phi_1 \rangle \, .$$

By applying a Wick rotation $\tau \equiv i(t_2 - t_1)$, the Euclidean action transforms as $S[\phi] \to iI[\phi]$. The Euclidean path integral measures the transition from $|\phi_1\rangle$ to $|\phi_2\rangle$. If these states are defined on a plane, we can give it a graphical representation [62]:

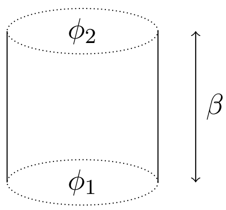
$$\langle \phi_2 | \, e^{-\beta H} \, | \phi_1 \rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=\phi_2} [d\phi] e^{-I[\phi]} \quad =$$



(D.22)

If these states are defined on a sphere, the path integral has the topology of a cylinder instead:

$$\langle \phi_2 | \, e^{-\beta H} \, | \phi_1 \rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=\phi_2} [d\phi] e^{-I[\phi]} \quad =$$



(D.23)

The thermal density matrix $\rho = e^{-\beta H}$ on a plane is likewise identified as a rectangle, without predefined boundary states $\phi_{1,2}$;

$$e^{-\beta H} \quad =$$

(D.24)

This just represents that the matrix element $\langle \phi_2 \mid \rho \mid \phi_1 \rangle$ is computed by the path integral with boundary conditions $\phi_{1,2}$.

D.4.1 Thermal partition function

The thermal partition function $\text{Tr}(e^{-\beta H}) = \sum_\phi \langle \phi \mid e^{-\beta H} \mid \phi \rangle$ is obtained from the thermal density matrix $\rho = e^{-\beta H}$ Eq D.24 by gluing the opposite ends together

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_\phi \langle \phi \mid e^{-\beta H} \mid \phi \rangle \,. \tag{D.25}$$

The path integral is now taken over all fields which are periodic in imaginary time β;

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \oint [d\phi] e^{-I[\phi]} \qquad t_E \sim t_E + \beta. \tag{D.26}$$

Since the left-hand side is just the partition function of a canonical ensemble consisting of fields at temperature $T = 1/\beta$, one can use the path integral over periodic fields to obtain thermodynamic quantities. Recall that in ordinary thermodynamics the free energy F is derived from the partition function: $Z = e^{-\beta F}$, with $F = E - TS$. Therefore, the most relevant derivations are the entropy S and energy E;

$$S = (1 - \beta \partial_\beta) \ln Z(\beta), \tag{D.27}$$
$$E = -\partial_\beta \ln Z(\beta). \tag{D.28}$$

D.4.2 Gravitational path integral

In quantum gravity, the manifold on which the fields are defined is allowed to fluctuate itself. The only restriction on the manifold in the thermal partition function $Z(\beta)$ are the periodic boundary conditions of the timelike coordinates near infinity: $g_{00} \to 1$, $t_E \sim t_E + \beta$. Separating the dynamical field configurations into matter fields ϕ and the metric g, the Euclidean path integral is given by [65];

$$Z(\beta, \phi_b) = \int \mathcal{D}g \mathcal{D}\phi e^{-I[g,\phi]}. \tag{D.29}$$

$I[g, \phi]$ specifies the Euclidean action due to gravity and matter fields. This is usually a sum of the pure-gravity Einstein-Hilbert action and matter fields minimally coupled to gravity. More often than not, we do not know how to properly define this integral, or how to perform perturbative calculations. Furthermore, there is the problem that the gravitational action is often unbound from below, making the path integral to diverge.
In the semiclassical analysis, one presumes that the dominant contribution to the path integral comes from the metric g and matter fields ϕ near the on-shell background solutions $g = g_0 + \delta g$, $\phi = \phi_0 + \delta\phi$. By expanding the action into a Taylor series around the background solutions $I[g, \phi] = I[g_0, \phi_0] + I_2[\delta g] + I_2[\delta\phi] +$ higher order corrections, the path integral can be written as

$$\ln Z = -I[g_0, \phi_0] + \ln \int [dg] e^{-I_2[\delta g]} + \ln \int [d\phi] e^{-I_2[\delta\phi]}. \tag{D.30}$$

I_2 are the quadratic contributions in the fluctuations, which lead to one-loop determinants. In the saddle-point approximation, one further neglects the background fluctuations and writes

$$Z(\beta) = e^{-I[g_0,\phi_0]}. \tag{D.31}$$

As will be explained in the following section, the semiclassical approximation is often useful to calculate correlators in the dual strongly coupled field theory.

D.5 The holographic dictionary: field/operator correspondence

The holographic duality is a duality between a theory of gravity in D dimensions, and a theory without gravity in $d = D - 1$ dimensions. More precisely, the novel example is the AdS_5/CFT_4 conjecture of Maldacena which can be formulated as:

$\mathcal{N} = 4$ *SUSY Yang-Mills is dual to type IIB superstring theory on asymptotically* $AdS_5 \otimes S_5$

This correspondence was first argued by Maldacena in [4] by looking at the low energy excitations of D-branes in a closed type IIB string background in different regimes of 't Hooft coupling constant. The low energy spectrum describes two decoupled sectors in both theories, one of which are always the background type IIB excitations. It makes sense to identify the other sectors in both theories, which are exactly the excitations of $\mathcal{N} = 4$ SUSY Yang-Mills theory and type IIB string theory on $AdS_5 \otimes S_5$. By the UV/IR duality, the 3+1d field theory is placed at the asymptotic boundary of the AdS_5-universe.
The path integral on $AdS_5 \otimes S_5$ is determined by integrating over all possible fluctuations in the string theory in the bulk, weighting each contribution with the respective gravitational action $S_{grav}[g, \phi_0]$;

$$Z[g, \phi_0] = \int \mathcal{D}g\mathcal{D}\phi \; e^{iS_{grav}[g,\phi_0]}. \tag{D.32}$$

This should be related to the generating function of correlation functions in the dual field theory.
In the holographic dictionary, Witten [10], Gubser, Klebanov and Polyakov [11] proposed that the relation between the two path integrals uses the boundary values of the bulk fields (generically denoted $\phi_0(x^\mu) = \phi(x^\mu, z)|_{z=0}$) at the holographic boundary $z = 0$ as coupling constants of corresponding operators $\mathcal{O}(x^\mu)$ in the field theory:

$$\boxed{\langle e^{\int d^{D-1}dx^\mu \; \phi_0(x^\mu)\mathcal{O}(x^\mu)}\rangle = Z_{string}\left[\phi(x^\mu, z)|_{z=0} = \phi_0(x^\mu)\right] \approx Z_{grav}[\phi_0(x^\mu)] = e^{iS_{grav}[\phi_0]}}. \tag{D.33}$$

In most cases, there is no known technique to actually evaluate the path integral D.32. In the semiclassical approximation, the right-hand side of the identity corresponds to passing from the gauge-string to the gauge-gravity duality in the appropriate limit of large 't Hooft coupling $N \to \infty$, $g^2N \to \infty$.
This limit localizes the fields in the bulk to the on-shell solutions of the classical supergravity action with appropriate boundary conditions. The equivalence essentially implies that the classical gravity action effectively serves as a generating functional for correlation functions of gauge-invariant operators $\mathcal{O}$ in the dual gauge

theory. In this limit, the gauge theory is necessarily strongly interacting.
In the bottom up approach to holography, one usually takes the holographic dictionary as a given and assumes that we have a CFT dual to higher-dimensional gravity.

The duality is most convenient in a Euclidean setting, where one Wick-rotates the time coordinate to obtain a Euclidean signature manifold. The Euclidean version of AdS has the topology of a ball. Calculating correlators in a Minkowski setting is more subtle, and is resolved in [120][121].

In a Euclidean setting, the on-shell solution $Z_{grav}[\phi_0] \approx e^{iS_{grav}[\phi_0]}$ transforms to

$$Z_{grav}[\phi_0] \approx e^{-I[\phi_0]}$$

, and acts as the generating functional for the gauge theory correlators. The correlators are computed in the usual way [62], where the boundary values of the on-shell gravity fields act as sources of the corresponding operators in the boundary theory;

$$\langle \mathcal{O}_1(x_1^\mu)\mathcal{O}_2(x_1^\mu)\dots\mathcal{O}_n(x_1^\mu)\rangle \sim \frac{\delta^n}{\delta\phi_0^1\,\delta\phi_0^2\dots\delta\phi_0^n} e^{-I[\phi_0]}|_{\phi_0^i=0} \tag{D.34}$$

Each differentiation brings down an operator insertion $\mathcal{O}$ that sends a ϕ-particle into the bulk. This can be visualized via Witten diagrams (see figure D.4):

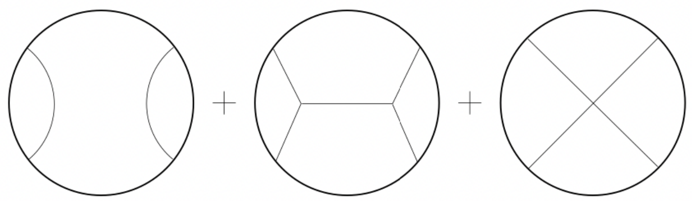


Figure D.4: Tree-level contributions to four-point correlation functions. The interior and boundary of each disk denote the interior and boundary of the AdS geometry. The boundary points are the points in flat Euclidean space where field theory operators are inserted. The four boundary points denote four-point correlation contribution. [122]

The semiclassical limit turns off quantum corrections, such that only the tree-level diagrams of the gravity theory contribute. In the context of JT gravity, these diagrams can be calculated exactly on a quantum mechanical level.
The correspondence between fields on AdS and operators in the field theory is very general. However, as a concrete and relevant example in the main text (c.f. section 1.8), consider a scalar field in an AdS background $\phi(x^\mu, z)$. Since the equations of motion in AdS are second order, we usually need to specify two boundary conditions in order to find a unique solution. One is the regularity of the bulk fields at the center of the ball. The correct boundary condition at the asymptotic boundary ($z = 0$) is a separation of variables between the radial z and asymptotic x^μ coordinates; $\phi(x^\mu, z) = f(z)\tilde{\phi}(x^\mu)$. The specific form of $f(z)$ is obtained by solving the

equation of motion in the bulk AdS-space. The Euclidean Poincaré patch corresponds to:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_0^2 + d\vec{x}^2). \tag{D.35}$$

Taking $L = 1$, the Klein-Gordon action corresponding to a massive scalar in Euclidean AdS_D is

$$S \sim \int d^D x \sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2) = \int dz d^d x \, \frac{1}{z^D}(z^2(\partial_z\phi)^2 + z^2(\partial_\mu\phi)^2 + m^2\phi^2). \tag{D.36}$$

$d = D-1$ is the dimesnion of the dual field theory. The radius can be reintroduced in the final result by simple dimensional analysis. The equations of motion, obtained by varying the action, yield

$$\partial_z\left(\frac{1}{z^{D-2}}\partial_z\phi\right) + \sum_\mu \partial_\mu\left(\frac{1}{z^{D-2}}\partial_\mu\phi\right) = \frac{1}{z^D}m^2\phi. \tag{D.37}$$

A solution generically takes the form $\phi(x^\mu, z) = f(z)\tilde{\phi}(x^\mu)$, where we choose $\partial_\mu\left(\frac{1}{z^{D-2}}\partial_\mu\tilde{\phi}(x)\right) = 0$. The boundary condition then takes the form

$$z^D\partial_z(z^{2-D}\partial_z f(z)) = m^2 f(z).$$

The generic solution is readily seen to be a power law ansatz $f(z) \sim z^{\Delta'}$. Under a dilation $z \to \lambda z$, $x^\mu \to \lambda x^\mu$, Δ' is seen to be the scaling dimension of the field. This will indeed be related with the conformal dimension of the (quasi-)primary field in the dual CFT $\mathcal{O}(x^\mu)$.
Inserting the ansatz in the wave equation, the scaling dimension is fixed in terms of the mass m of the scalar field as:

$$m^2 = \Delta'(\Delta' - D + 1). \tag{D.38}$$

This obeys two solutions

$$\Delta'_\pm = \frac{D-1}{2} \pm \sqrt{\frac{(D-1)^2}{4} + m^2}. \tag{D.39}$$

We call the largest solution $\Delta := \Delta'_+$. Then, the other solution is $\Delta'_- = D - 1 - \Delta$. Denoting $d = D-1$ the dimension of the conformal field theory, the general solution near the boundary $z = 0$ can be written as:

$$\phi(x^\mu, z) \;\sim\; z^{d-\Delta}\tilde{\phi}_0(x^\mu) + z^\Delta\tilde{\phi}_1(x^\mu) \qquad (\text{at } z \approx 0). \tag{D.40}$$

Since the scaling dimension Δ is related to the conformal dimension of the primary field, it has to satisfy the unitary bound $\Delta \geqslant (d-2)/2$. However, the easiest approach is to restrict to the range to $\Delta > d/2$. In that case, the solution corresponding to $\tilde{\phi}_1$ is regular at the center but vanishes at the boundary. To retain interesting dynamics near the boundary, we consider only the solution corresponding to $\tilde{\phi}_0$, which diverges near $z \to 0$. [123][124] generalizes this to the range $\frac{d}{2} \geqslant \Delta \geqslant \frac{d-2}{2}$.
In the case of the former, the correct boundary condition of the field reaching the boundary is

$$\phi(x^\mu, z) \sim z^{d-\Delta}\tilde{\phi}_0(x^\mu). \tag{D.41}$$

In the range $\Delta > d/2$, the norm of this solution is not normalizable

$$\int dz dx^{\mu} \sqrt{g}\, |\phi(x^{\mu}, z)|^2 \sim \int dz dx^{\mu} \frac{1}{z^{d+1}} z^{2d-2\Delta} \to \infty.$$

To deal with the divergence, one introduces a cut-off parameter at the boundary $z = 0 \to \epsilon$, $\phi \sim \epsilon^{d-\Delta}\tilde{\phi}_0$ and eventually takes $\epsilon \to 0$ in the final expressions.
Once the value of $\tilde{\phi}_0(x^{\mu})$ is specified, we have a unique regular solution that extends to all of AdS space. In particular, $\tilde{\phi}_1(x^{\mu})$ will be determined as a functional of $\tilde{\phi}_0(x^{\mu})$ by imposing the equations of motion and regularity at the center.

We can now present a heuristic argument why Δ should indeed correspond to the conformal dimension in the field theory. From the Klein-Gordon action Eq D.36, we see that the total field $\phi(x^{\mu}, z)$ should be dimensionless. From Eq D.41, this implies that $\tilde{\phi}_0(x^{\mu})$ should scale with Length$^{\Delta-d}$. This boundary value will eventually couple in the term $\int d^{D-1}dx^{\mu}\ \tilde{\phi}_0(x^{\mu})\mathcal{O}(x^{\mu})$ in the generating function of the holographic dictionary Eq D.33. Since the action should be dimensionless, this in turn demands the conformal primary $\mathcal{O}$ associated with the field ϕ, to have the conformal scaling dimension of Δ.

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