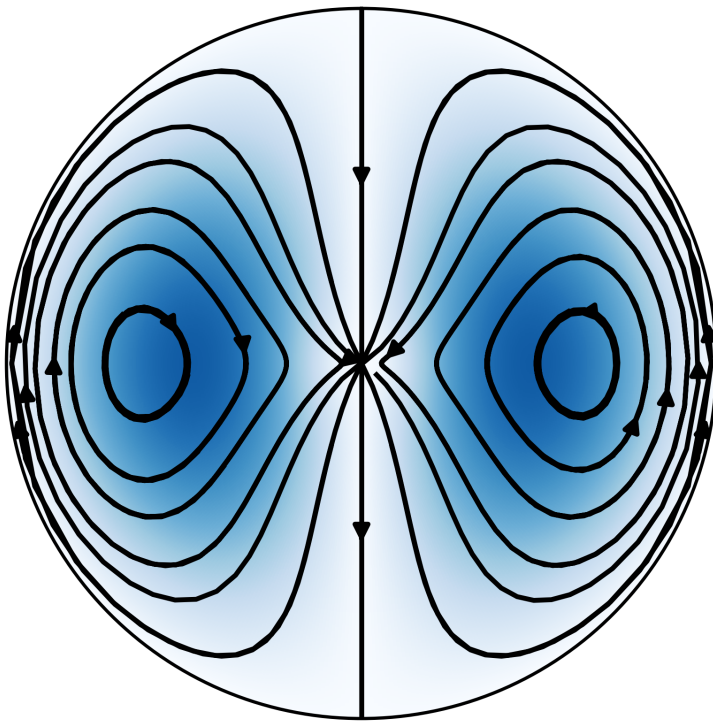


Hunting the internal magnetic field of the β Cep HD 192575



Jelle VANDERSNICKT

Supervisor: Prof. Dr. Conny Aerts
Institute of Astronomy, KU Leuven

Mentor: Vincent Vanlaer
Institute of Astronomy, KU Leuven

Thesis presented in
fulfillment of the requirements
for the degree of Master of Science
in Astronomy & Astrophysics

Academic year 2024-2025

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Preface

The completion of this thesis signifies the end of a major chapter in my life. This work is the result of a six-year long journey at KU Leuven, where I've learned a lot about the Universe and myself. I am immensely proud of what I have accomplished during this time, and I am grateful for every opportunity along the way. There are many people to thank for their support along the way.

This thesis would not have been possible without the guidance of my supervisors, Prof. Dr. Conny Aerts and Vincent Vanlaer. Their expertise and support have been invaluable throughout this process. Special thanks to Vincent for making time every single week to discuss my progress and answering my questions. I left our meetings every time with a clearer view of the work and I can only hope that some of your mathematical prowess and scientific rigor have rubbed off on me. I am also grateful to Prof. Dr. Conny Aerts for her guidance and for the opportunity to work on this project. You allowed to present my work at the group meetings multiple times and meet other researchers in the field. I am very thankful that you shared the results of my work with the community halfway through the project, showing great confidence in my abilities. I am very much looking forward to continuing working together with both of you next year, as I will join the group as a PhD student.

To my friends, I am incredibly grateful for your support and encouragement throughout all these years. We have had a lot of fun over the years, and I am looking forward to many more adventures in the future.

Ik wil ook graag mijn ouders bedanken voor hun onvoorwaardelijke steun. Jullie gaven me de kans om mijn dromen keer op keer na te jagen, startende met mijn keuze om fysica te studeren en jullie zorgden ervoor dat ik elke kans met beide handen kon grijpen. Zonder jullie was ik nooit geraakt waar ik nu sta, met diplomas in de hand en een PhD in het vooruitzicht. Jullie steun bracht mij de laatste jaren in Canada voor mijn uitwisseling en in Hawaii voor mijn internship. Dit zijn ongelooflijke avonturen die ik zal koesteren voor de rest van mijn leven.

Jeroen, schattie, bedankt voor de fantastische drie jaar die we al samen hebben doorgebracht. Je steunt me door dik en dun, en zeker wanneer deze thesis me even zwaar viel kon ik altijd op jou rekenen. Ik kan niet wachten op wat de toekomst ons nog zal brengen.

Scientific summary

Massive stars are the major engines of the Universe, driving the chemical and dynamical evolution of their host galaxies. After their supernovae deaths, they leave behind a compact object such as a neutron star or a black hole, the most extreme objects in the galaxy and possible gravitational wave sources. However, there are still many questions about the evolution and internal structure of massive stars. Asteroseismology is the most powerful tool known to date to investigate the internal structure of stars. Nevertheless, only a handful of massive stars have undergone detailed asteroseismic modelling.

This thesis aims to perform the first detailed magnetic asteroseismic modelling of a supernova progenitor star, HD 192575. We build upon previous asteroseismic analysis of the stellar structure models of HD 192575. By including the effects of misaligned rotation and magnetic field, we strive to explain the frequency splittings of the oscillation modes that have not yet been interpreted for this rotating massive pulsator.

We consider 600 stellar structure models calculated for a previous asteroseismic analysis of HD 192575. We predict their frequency splittings and amplitudes due to the effects of rotation and a magnetic field for different values of solid body rotation, magnetic field strengths and misalignment. By repeating this process for each stellar model, we obtain a grid of predicted splittings and amplitudes in which we search for the best fitting model.

We find that we can explain the observed frequency splittings using a misaligned magnetic field and rotation axis. The obtained values of the magnetic field strength and rotation rate depend heavily on the probing depth of the oscillation modes, making the correct identification of the modes crucial. Depending on the stellar model used, we require a magnetic field strength of a few kG near the surface and tens to hundreds of kG near the core to explain the observed splittings.

This thesis presents the first detection and asteroseismic modelling of an internal magnetic field in a massive star. We highlight how a misaligned magnetic field with respect to the rotation axis allows us to include the amplitudes of the oscillation modes in the analysis, a feat that is currently not possible otherwise.

Summary for a general audience

Stars come in many sizes, from small stars like our Sun to massive stars tens or even a hundred times more massive. These massive stars play an important role in the Universe. They can produce elements from helium to iron and enrich the interstellar medium with these heavier elements during both their relatively short lives (~ 10 million years) and their explosive deaths as supernovae. After their deaths, they leave behind a neutron star or a black hole, the densest objects known in the Universe. These heavy objects are possible gravitational wave sources, which is a field of research that has gained a lot of interest in recent years. Understanding the evolution of massive stars is thus fundamental to many different fields of astrophysics.

Asteroseismology studies the variations in brightness of stars over time. These variations are caused by internal waves in the star and are periodic in nature, similar to how earthquakes cause waves in the Earth. These *starquakes* can be studied to learn about the star's internal structure and evolution. Even though massive stars are major players in the Universe, only a handful of them have been studied in detail using asteroseismology.

This thesis offers the first in-depth magnetic asteroseismic analysis of a massive star, HD 192575. This star has been observed for a long time with NASA's space mission TESS and shows patterns in its starquakes that are currently not fully understood. We hypothesise that this pattern is caused by the presence of an internal magnetic field, in addition to the effects of rotation. Using previous models of the internal structure of the star, we include the effects of rotation and an internal magnetic field to try and match the observed frequency patterns to those predicted by the theoretical model.

We find that the observed patterns can be explained using a misaligned magnetic field with respect to the rotation axis. That is, the axis around which the star rotates is at an angle with the central axis (between the south and north pole) of the magnetic field. The derived rotation rates are similar to previous estimates, while the magnetic field strengths are in line with those expected from theory. This is the first instance of an asteroseismic analysis of a massive star which includes the effects of a magnetic field in addition to rotation. This opens the way for using asteroseismology to study the internal magnetic fields of massive stars, which are thought to play an important role in the evolution and end product.

Contribution statement

The hypothesis of this thesis was original conceived by the supervising team, Prof. Dr. Conny Aerts and PhD student Vincent Vanlaer, who phrased the research question. The implementation of the theoretical framework and selection of the magnetic field topology was performed by the author. The stellar structure models used for the fitting of HD 192575 in this thesis were calculated by Dr. Siemen Burssens and reevaluated by Vincent Vanlaer. The extraction of the observed frequencies and amplitudes from the TESS light curve was performed by PhD student Mathijs Vanrespaille, who also provided the stellar models used in the analyses presented in Chapter 4. All other aspects, unless otherwise stated, were performed by the author. The author is thankful to everyone involved in the project for their help and support.

In addition to the contributions, the author would like to acknowledge the use of the following software packages not mentioned in the main body. The implementation of the theoretical framework was accomplished using NumPy ([Harris et al., 2020](#)), SciPy ([Virtanen et al., 2020](#)), and Astropy ([Astropy Collaboration et al., 2013, 2018, 2022](#)). Visualisations were created using Matplotlib ([Hunter, 2007](#)). Generative artificial intelligence (GenAI) tools were used minimally in the completion of this thesis, with the sole use of Microsoft Copilot for short-form code completion (i.e. autocompletion of terms and single lines of code). No GenAI tools were used for writing or analysis.

Commonly used symbols and acronyms

Acronyms

β Cep star	β Cephei star
CMB	Convective boundary mixing
<i>g</i> -mode	Gravity mode
MESA	Modules for Experiments in Stellar Astrophysics
MLT	Mixing length theory
<i>p</i> -mode	Pressure mode
SPB star	Slowly Pulsating B-type star
TESS	Transiting Exoplanet Survey Satellite
ZAMS	Zero age main sequence
MAMS	Middle age main sequence
TAMS	Terminal age main sequence

Units and constants

d	Days ($8.64 \cdot 10^4$ s)
G	Gravitational constant ($6.674 \cdot 10^{-8}$ cm ³ g ⁻¹ s ⁻²)
G	Gauss (10^{-4} T)
$L_{\odot}$	Solar luminosity ($3.839 \cdot 10^{33}$ erg/s)
$M_{\odot}$	Solar mass ($1.989 \cdot 10^{33}$ g)
$R_{\odot}$	Solar radius ($6.957 \cdot 10^{10}$ cm)
pc	Parsec ($3.086 \cdot 10^{18}$ cm)
K	Kelvin
μ Hz	Microhertz (10^{-6} Hz)

Symbols

$\mathbf{B}$	Magnetic field vector
f	Cyclic frequency of the oscillation mode
l	Angular degree of the oscillation mode
L_{star}	Stellar luminosity
m	Azimuthal order of the oscillation mode
M_{star}	Stellar mass
n	Radial order of the oscillation mode
p	Pressure
r	Radial coordinate
R_{star}	Stellar radius
T_{eff}	Effective temperature
Y_l^m	Spherical harmonic of angular degree l and azimuthal order m
β	Obliquity angle between the magnetic field and rotation axes
θ	Polar coordinate
$\boldsymbol{\xi}, \delta\omega$	Displacement vector of the oscillation mode
ξ_r	Radial component of the displacement vector
ξ_h	Horizontal component of the displacement vector
ρ	Density
ϕ	Longitudinal coordinate
Φ	Gravitational potential
ω	Angular frequency of the oscillation mode
Ω	Rotation frequency of the star (often in units of the critical Keplerian frequency Ω_{crit})

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Chapter 1

Introduction

Stars are energetic, dynamic, and lively objects. Astronomers have tried to understand the story of stars and other celestial bodies for thousands of years. With the advent of the camera and the computer, we have been able to record the light of stars with incredible precision and study them in great detail. Not only did we learn about the incredibly long timescales on which the stars evolve, but we also discovered that they are not as static as they seem. Many stars show periodic brightness variations on timescales of minutes, hours, days, or months: scales that are very familiar to us. These variations, which are caused by oscillations in the entire interior of the star, depend on the internal structure and composition of the star.

Asteroseismologists use these oscillations to study the internal structure of stars. Similar to a musical instrument, each star has its own unique oscillation frequencies, depending on its structure. Just as a violin sounds different to a cello, a young massive star will oscillate at different frequencies than an old low-mass star.

1.1 Massive stars and their oscillations

Stars come in a wide range of masses, from small low-mass stars like our Sun to large stars that are tens of times more massive. For example, Betelgeuse in the constellation of Orion has about 15 times more gas than the Sun. These massive stars are the fundamental engines of the Universe, emitting radiation and matter into their surroundings and driving the chemical and dynamical evolution of the Universe during their lifetimes. Massive stars ($M \gtrsim 8 M_{\odot}$) are the most powerful engines of the Universe, with surface temperatures up to tens of thousands of Kelvin. Due to their large mass, they can burn heavy elements in their core. This makes them the main source of chemical enrichment in the Universe (Kobayashi et al., 2020). They eject their material through strong stellar winds (Björklund et al., 2023) and through powerful supernova explosions at the end of their lives (Wang and Burrows, 2024).

While only a small fraction of stars are massive, they are the most luminous stars in the Universe (Salpeter, 1955). As massive stars are short-lived compared to their low-mass counterparts, their presence in a galaxy is a good indicator of star formation activity (Schneider, 2015). Additionally, their hot surface temperatures make them a source of

strong ionizing radiation, which can heavily affect the surrounding interstellar medium and lead to the formation of H II regions (Strömgren, 1939).

Massive stars die impressive deaths, exploding as supernovae and leaving behind compact objects such as neutron stars or black holes. Most massive stars are born in a multiple system (Sana et al., 2014, Villaseñor et al., 2021, Banyard et al., 2022) and their internal processes have a strong influence on the evolution of the binary, especially in a contact system (Vandersnickt and Fabry, 2025). As compact objects in a multiple system can create gravitational waves, massive binary systems are considered to be one of the main progenitors of gravitational wave sources (Marchant et al., 2016). In short, massive stars play a vital role in the Universe, and better understanding their structure and evolution is fundamental to many aspects of astrophysics.

The internal structure of stars is mainly determined by the gravitational attraction of their own mass, counteracted by the pressure gradient throughout their structure (Kippenhahn et al., 2013). When these forces are in balance, the star is in *hydrostatic equilibrium*. The pressure inside the star is maintained by the heat generated in the core, which is the result of nuclear fusion reactions. In a massive star, the energy transport from the core to the surface occurs through two different mechanisms. In the stellar envelope, energy is transported by radiation, while in the core itself, the radiative transport is not efficient enough and large scale motions in the plasma occur. Hot plasma rises up to regions of lower temperature, mixing with the surroundings when they dissolve. This process is called convection and it is a very efficient method of mixing and energy transport, leading to homogeneous layers where convection occurs. Massive stars are said to have a convective core and a radiative envelope, such that their core can be considered homogeneously mixed.

For about 90% of the stars' lifetime, it will burn hydrogen into helium in the core, and the star is said to be on the *main sequence*. After the main sequence, heavier elements are burned, which is the driving force behind the further evolution of the star and the chemical enrichment of the Universe. This thesis will focus on a massive star during its main sequence lifetime.

Stellar structure and evolution models consider the equilibrium states of stars, where the forces are in balance. Small changes in the structure perturb this equilibrium and can cause waves to propagate through the star. Only standing waves can exist in the star for extended periods of time. These standing waves are the oscillation modes of the star and are expressed as small, local changes in density, pressure, and the gravitational potential of the star. The oscillation modes that reach the stellar surface can be observed as periodic brightness variations.

Within this thesis, we will consider stars in equilibrium to be perfectly spherical and focus on *non-radial* oscillations. These are small non-radial perturbations in the sense that material is not only displaced in the radial direction, but also in the horizontal direction. This is in contrast to *radial* oscillation modes, where the displacement of the material occurs solely in the radial direction. We can describe the angular dependence of the oscillation modes with spherical harmonics Y_l^m (see Chapter 2). The spherical harmonics are uniquely described by their angular degree l and azimuthal order m . The mode with angular degree $l = 1$ is called the *dipole* mode, $l = 2$ is the *quadrupole* mode.

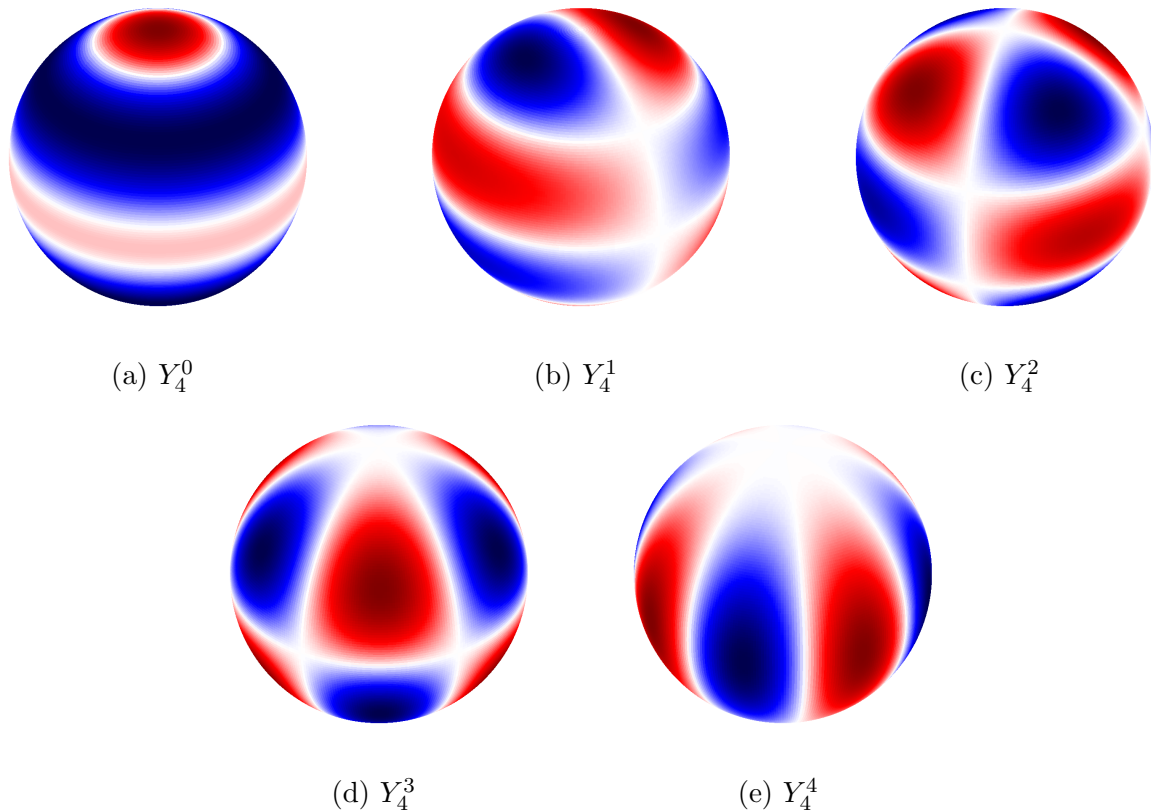


Figure 1.1: Visualisation of the different radial components of the oscillation modes for angular degree $l = 4$ and azimuthal orders $m = 0, 1, 2, 3, 4$ as given by the spherical harmonics Y_l^m .

The radial dependence of the mode is identified by the radial order n , which is the number of nodes of the mode's displacement vector in the radial direction. Under spherical symmetry, the oscillation modes of the star can be identified by the radial order n , angular degree l , and azimuthal order m . Fig. 1.1 shows the radial components of the oscillation modes for $l = 4$ and different azimuthal orders. The angular degree l gives the amount of nodal lines on the surface of the star, while the azimuthal order m gives the number of nodal lines crossing the poles. These nodal lines extend throughout the radial direction, forming nodal planes for the nodal lines through the poles and cones for the nodal lines parallel to the equator. The azimuthal order m ranges from $-l$ to l , leading to a total of $2l + 1$ possible azimuthal orders for a given angular degree l . The sign of the azimuthal order m denotes the direction of propagation, keeping the form of the spherical harmonic constant between modes of same absolute azimuthal order $|m|$. Modes with a non-zero azimuthal order propagate over the surface, while modes with $m = 0$ appear to be 'breathing'.

1.1.1 Dependence of oscillations on the internal structure

Stars are notably opaque and their strong radiation fields make investigating the internal structure of stars impossible with direct observations. Asteroseismology uses stellar oscillations to probe the internal structure of stars. As waves propagate through the star, their

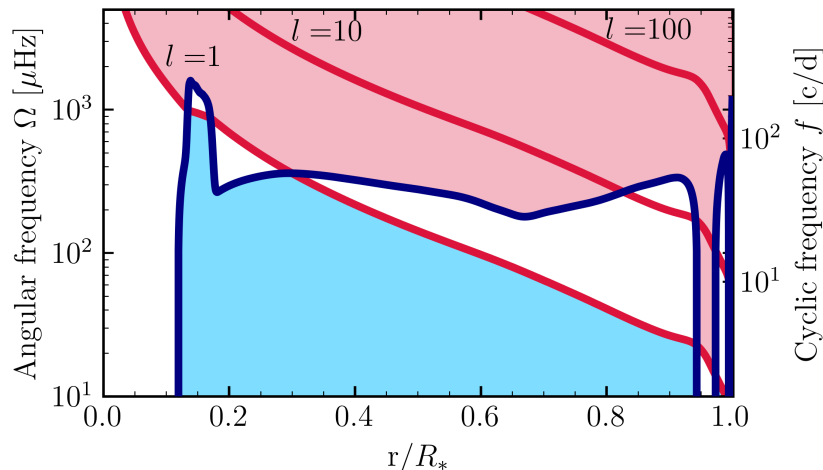


Figure 1.2: Propagation diagram for a $13 M_{\odot}$ star on the main sequence. The Brunt-Väisälä frequency N is shown as a blue line, while the characteristic acoustic frequency S_l is shown as a red line for three angular degrees. The blue region indicates the g -mode cavity, while the red region gives the p -mode cavity of the dipole modes.

frequencies and amplitudes are determined by the internal structure and composition of the star. Oscillation modes are classified by their restoring force, which is most commonly either gravity (in the form of buoyancy) or pressure. The gravity modes (g -modes, $n < 0$) and pressure modes (p -modes, $n > 0$) are the two main classes of oscillation modes, which cover different frequency regimes and are separated by the fundamental modes (f -modes, $n = 0$). The difference in sign between the radial order n of g - and p -modes is not a physical aspect, but rather a notation convention. Other restoring forces are possible in for example rotating stars, where the Coriolis force can enable gravito-inertial and Rossby modes (Townsend, 2003). The dependence on the restoring force leads to different propagation regions in the star where the waves can resonate.

To investigate where the different modes can exist in the stellar interior, a simplified model of the stellar oscillations is used, where the perturbations in the gravitational potential are ignored (Cowling, 1941). This approximation is strictly only valid for high angular degree l and high radial order $|n|$ modes, but offers valuable insights into the propagation regions of the different modes. Under this approximation, the radial displacement of the mode ξ_r is given by

$$\frac{d^2 \xi_r}{dr^2} = \frac{\omega^2}{c_s^2} \left(1 - \frac{N^2}{\omega^2} \right) \left(\frac{S_l^2}{\omega^2} - 1 \right) \xi_r, \quad (1.1)$$

where ω is the frequency of the mode, c_s is the local sound speed, N is the Brunt-Väisälä frequency, and S_l is the characteristic acoustic frequency. Each of N , S_l , and c_s vary throughout the radial structure. The characteristic acoustic frequency is additionally dependent on the angular degree l of the mode. For the oscillation to exist, the right hand side of the equation must be negative, which leads to conditions on the frequency ω based on the Brunt-Väisälä frequency and the characteristic acoustic frequency.

Figure 1.2 shows where the different modes can exist in a massive star ($13 M_{\odot}$). The figure shows the Brunt-Väisälä frequency N (blue line) and the characteristic acoustic frequency S_l (red line) for three different angular degrees throughout the radial structure

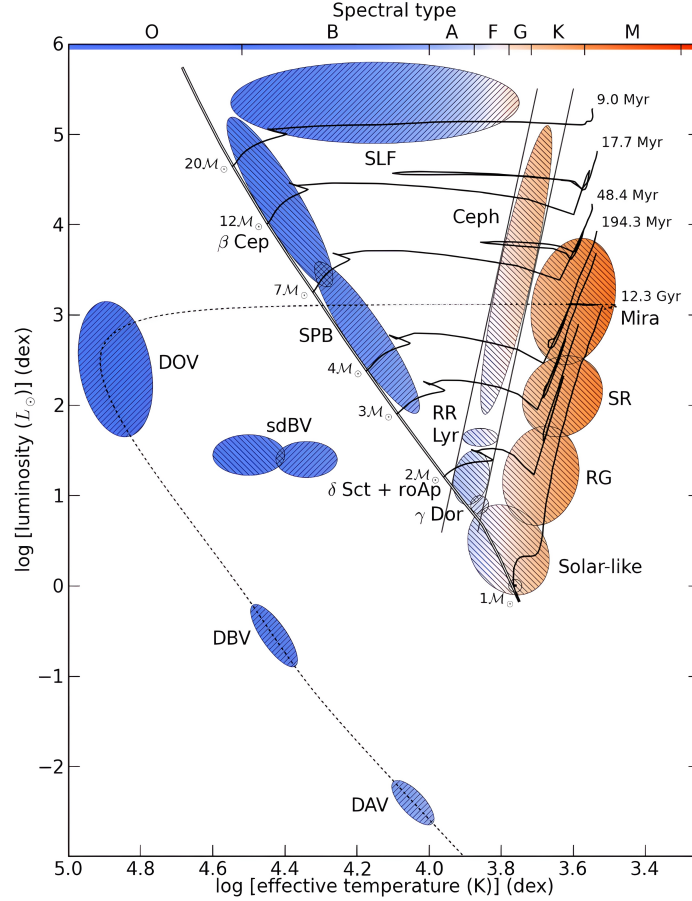


Figure 1.3: Hertzsprung-Russell diagram showing the different types of pulsating stars (from [Aerts, 2021](#)).

of the star. By applying the conditions on these parameters as given by Eq. (1.1), we can identify different propagation regions (called ‘cavities’) for the different oscillation modes that can exist in the star. These cavities are denoted by the shaded regions in the figure, coloured according to the type of mode that can exist in the cavity. The frequencies are given in both the angular frequency Ω , which describes the frequency in terms of angular progression, and the cyclic frequency f , which describes the frequency in terms of completed oscillation cycles. Both frequencies are connected by the relation $\Omega = 2\pi f$ (we refer to [Aerts, 2021](#), for the full explanation of the nomenclature).

The g -mode cavity (blue) is given by the Brunt-Väisälä frequency N , connecting the stability to the buoyancy. The Brunt-Väisälä frequency is only positive for non-convective regions of the star. The presence of the convective core and near-surface convective region are clearly visible in the g -mode cavity. The broad near-core peak in N is a consequence of the evolution of the star. As the core recedes, a composition gradient is created with an increasing molecular weight due to the presence of helium as a product of hydrogen burning. The lighter material (helium-poor) envelope above leads to a stronger stability against convection and an increase in the Brunt-Väisälä frequency.

The p -mode cavity (red) is given by the characteristic acoustic frequency S_l . This frequency is connected to the sound speed c_s , which connects pressure perturbations to the

local properties in the star. It depends additionally on the angular degree l of the mode, with a larger cavity for lower l . The areas outside the cavities are evanescent zones, where the oscillation modes are exponentially damped. The oscillations can, however, still cross these regions, which is important for the detection of g -modes, which are often only propagating deep in the interior of a star. Additionally, the g - and p -modes cavities can couple through the evanescent zones, which leads to *mixed modes*. These mixed modes can play an important role in evolved stars. The g -mode cavity is far from the surface in a star with a convective envelope, such that its g -modes are not observable. Mixed modes can in that case couple a p -mode to a g -mode and make the mixed mode visible at the surface, as happens in red giants (Aerts et al., 2010).

In addition to the propagation regions, Fig. 1.2 is a first indication of the sensitivity of the different oscillation modes to the internal structure of a massive star. The g -modes are in general sensitive to the internal structure close to the core of the star, while the p -modes are mostly sensitive to the outer layers of the star.

While not all stars oscillate, those that do can be classified by their oscillation properties. An overview of the different types of pulsating stars throughout the Hertzsprung-Russell diagram is shown in Fig. 1.3. The two major types for massive, main sequence oscillators are Slowly Pulsating B (SPB) stars and β Cephei (β Cep) stars. SPB stars (Waelkens, 1991) have a mass between $2 M_{\odot}$ and $7 M_{\odot}$ and pulsate with periods between 0.8 days (d) and 3.0 days. β Cep stars are more massive, with masses between $8 M_{\odot}$ and $25 M_{\odot}$, and pulsate in low order g -modes and p -modes with periods between 2 hours (h) and 8 hours (Stankov and Handler, 2005). A recent study by Fritzewski et al. (2025) reported on the first homogeneous analysis of 119 β Cep stars from TESS space photometry. However, detailed seismic modelling of massive stars has only been accomplished for a few stars (for a detailed overview see Bowman, 2020).

The propagation diagram above shows approximate predictions of which modes can exist in a star. However, the diagram does not predict whether the modes that can exist will actually be excited and observable. The actual mode selection of which modes will appear in the light curves is still a little understood problem, as this requires non-linear mode coupling for massive stars (Beeck et al., 2024). Similarly, many questions remain about the driving of the modes, which determines the amplitudes of the oscillation modes (Aerts et al., 2010).

1.1.2 Internal rotation and its effects on the frequencies

The oscillations of a star depend on the restoring forces and are thus sensitive to the internal structure of the star. For a spherically symmetric star, there is no preferential axis to define the poles. Since the azimuthal order m depends on this axis, the oscillation equations and thus the frequencies must be independent of m . In other words, in a spherically symmetric star, each mode with the same angular degree l and radial order n will have the same frequency, regardless of the azimuthal order m . Any effect on the stellar oscillations that breaks the spherical symmetry and introduces a preferential axis will break the degeneracy and lead to a splitting of the oscillation frequencies with respect to the azimuthal order m .

Rotation breaks the spherical symmetry of a star and leads to a splitting of the oscillation

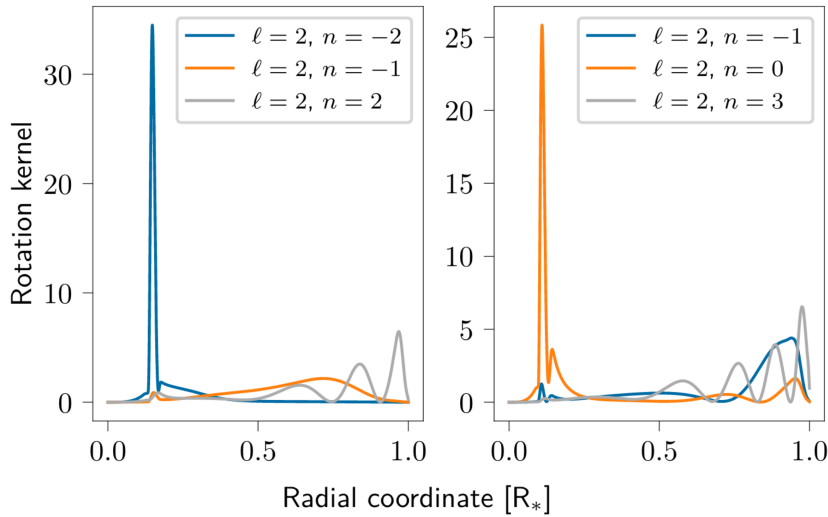


Figure 1.4: Rotation kernels calculated for two different stellar models of a $12 M_{\odot}$ star. Figure is from Vanlaer et al. (subm.).

frequencies into $2l + 1$ components as it introduces the rotation axis as the preferential axis. Additionally, it introduces a preferential direction in the propagation of the waves, as some waves will propagate with the direction of rotation and others against it. The effect of rotation on the oscillations can be understood from a simple geometrical model where only the transformation between the rotational frame of the star and the inertial frame of the observer is accounted for. Consider a star rotating as a solid body with an angular frequency Ω , then the oscillation frequency in the corotating frame $\omega'(n, l, m)$ is related to the oscillation frequency in the inertial frame $\omega(n, l, m)$ by

$$\omega(n, l, m) = \omega'(n, l, m) + m\Omega. \quad (1.2)$$

This transformation follows naturally when accounting for the propagation direction of the oscillation modes. If the mode propagates in the direction of the rotation (positive m), the observed mode period will be shortened, while the opposite is true for a mode propagating against the rotation (negative m). The dependence on the azimuthal order m can be understood from Fig. 1.1, where a higher azimuthal order mode has more nodal lines in the direction of the rotation axis and a full rotation period is not required to complete an oscillation cycle in the inertial frame. A more detailed derivation of this property is given in Chapter 2.

Aside from a geometrical effect, the Coriolis force also comes into play, altering the corotating frequency ω' compared to the frequency of the same mode without rotation. Moreover, uniform rotation is an assumption that does not apply to most stars. A first complication is thus to consider differential rotation in the radial direction (i.e. $\Omega = \Omega(r)$) and include the effects of the Coriolis force. This general rotation law in one dimension leads to the frequency splitting following (Aerts et al., 2010)

$$\omega = \omega_0 + m(1 - C_{nl}) \int_0^{R_{\text{star}}} K_{nl}(r) \Omega(r) dr, \quad (1.3)$$

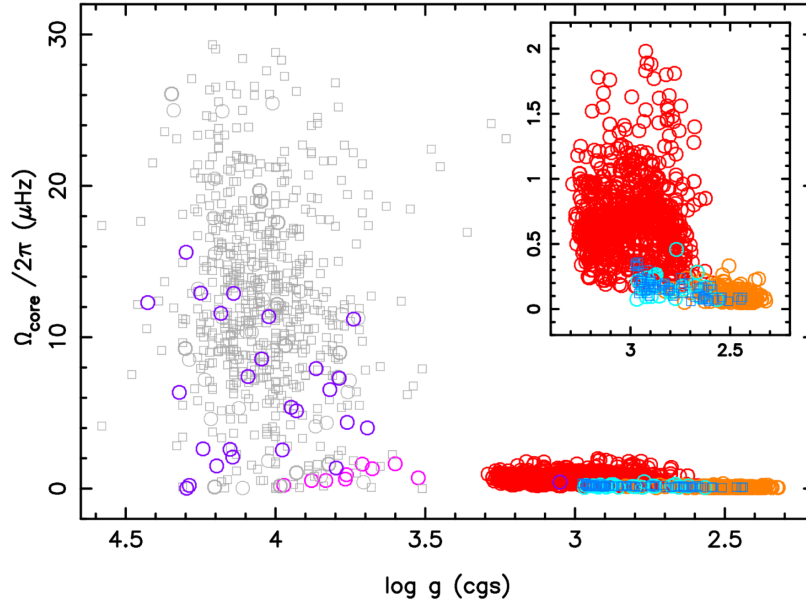


Figure 1.5: Core (near-core) rotation rates derived from mixed (gravity) modes for stars in core-hydrogen burning (purple and gray), hydrogen-shell burning (pink and red), and core-helium burning (after the helium flash in orange and avoiding the helium flash in cyan). Figure is from [Aerts \(2021\)](#).

up to first order in the rotation frequency, where ω_0 is the frequency without considering rotation and ω is the frequency in the inertial frame corrected for rotation. C_{nl} is the Ledoux constant ([Ledoux, 1951](#)) and K_{nl} is the normalized rotational kernel. The Ledoux constant is the integrated first-order effect of the Coriolis force on the mode and must be included for low order g - and p -modes, even in the case of solid body rotation. Both terms carry dependence on the radial and horizontal displacement of the oscillation mode and are thus sensitive to the internal structure of the star. The stellar model used to determine the kernel and Ledoux constant must be carefully considered before comparing the observed frequencies to the theoretical predictions.

Different oscillation modes are sensitive to the stellar structure (including rotation) at different radii. This dependence on the internal rotation is expressed by the rotational kernel $K_{nl}(r)$ for a given mode and combining information of different types of modes can be used to infer the internal rotation profile of the star. Fig. 1.4 shows the large dependence of the rotational kernel on the model of a star. The kernels show the sensitivity of the modes to the rotation at different radii in the star. It is clear from the figure that large differences in sensitivity exist between different modes, and between different stellar models for a similar star.

[Aerts \(2021\)](#) reports on the near-core rotation rates Ω_{core} derived from a large sample of stars using asteroseismology and the near-core probing power of g -modes. The masses of these stars range from $0.8 M_{\odot}$ to $3.3 M_{\odot}$, which were found to rotate nearly uniformly throughout their main sequence phase (see Fig. 1.5). More recently, [Aerts et al. \(2025\)](#) found similar results for masses up to $7 M_{\odot}$. These results call for a reevaluation of the theory of angular momentum transport, as they currently cannot be explained. The

predicted angular momentum transport between the core and the envelope is not sufficient to explain these close values of the rotation rates of the core and the envelope.

1.2 Internal magnetic fields

One proposed solution for the angular momentum transport problem is the presence of magnetic fields in the interior of stars (Aerts et al., 2019). Little is known about these internal magnetic fields as they are notoriously difficult to detect and characterize, being mostly confined to the stellar interior. Magnetic fields at the surface of stars have been measured for most types of stars (Landstreet, 1992, Donati and Landstreet, 2009). About 10% of stars with spectral types O, B, or A ($T_{\text{eff}} \gtrsim 7500$ K and $M \gtrsim 1.4 M_{\odot}$) have large-scale magnetic fields at their surface, with strengths ranging from hundreds of gauss (G) to tens of kilogauss (Shultz et al., 2018). These magnetic fields are thought to be of ‘fossil’ origin (Mestel, 1999, Neiner et al., 2015), where the field is a remnant of a magnetic field present during the star’s formation.

Strong, stable surface fields on the order of kilogauss in strength have been observed for 5 – 10% of intermediate- and high-mass stars (Wade, 2012, Braithwaite and Spruit, 2017). Another type of magnetic intermediate-mass stars are those with weak surface fields of about 1 G (Blazère et al., 2016). These stars have radiative envelopes, which can host fossil magnetic fields. These magnetic intermediate- and high-mass stars are proposed to be the progenitors of magnetic white dwarfs and neutron stars (Ferrario et al., 2020), the detection of which is a strong argument for internal magnetic fields in stars.

The presence of magnetic fields in the interior of stars can have a profound impact on the stellar structure and evolution throughout the full lifetime and even death of the star. Efforts are being made to include it in state-of-the-art 3D simulations of supernovae (Varma et al., 2023). The magnetic fields present during a supernova are dependent on the internal magnetic fields during the star’s lifetime. Understanding and measuring these fields is thus crucial both during and after the star’s lifetime.

Asteroseismology provides a unique and promising opportunity to study these fields by probing the internal structures through stellar oscillations. As an internal magnetic field breaks the spherical symmetry of the stellar interior, the oscillation frequencies of the star are split in their azimuthal order. In contrast to rotation, the magnetic field is only sensitive to the absolute value of the azimuthal order, which leads to different splittings compared to rotation. While the theory has long predicted the effect of magnetic fields on the oscillation frequencies (Gough and Thompson, 1990, Hasan et al., 2005), the first detections of internal magnetic fields through asteroseismology have only recently been made in red giants (Li et al., 2022, 2023, Deheuvels et al., 2023). The results relied on the theoretical frameworks developed by Gomes and Lopes (2020), Mathis et al. (2021), Loi (2021), Bugnet (2022), and Mathis and Bugnet (2023). Following these successes, the detection of internal magnetic fields in low- and intermediate-mass stars followed (Lecoanet et al., 2022, Hatt et al., 2024). Currently, there has been no direct detection of internal magnetic fields in the oscillation frequencies of massive stars.

Internal magnetic fields can have different origins and extents in the internal structure of a star. Firstly, the radiative envelope can host a *fossil magnetic field* that is a remnant

of the formation process, where an initial magnetic field was present in the molecular cloud from which the star formed. These fossil fields have a decay time scale given by $\tau_{\text{decay}} \approx R^2/\eta$, where R is the stellar radius and η is the magnetic diffusivity. These time scales are generally between 10^9 and 10^{10} years, which is long enough to be present throughout the evolution of most stars with a radiative envelope (Aerts, 2021). Mestel and Weiss (1987) showed that even a weak magnetic field ($B \gtrsim 3 \cdot 10^{-2}$ G) can efficiently transport angular momentum and enforce uniform rotation in the radiative envelope on a timescale of 10^4 years. Aerts et al. (2018) discuss how this timescale still allows for a small degree of differential rotation due to the continually slowing down of the stellar surface due to magnetic breaking via a stellar wind.

Secondly, the field can be generated by the convective movement of ionized plasma in the core of the star. This induces a dynamo effect and can lead to a *dynamo-generated magnetic field*. These fields in the convective core are thought to not penetrate deeply into the radiative envelope (MacGregor and Cassinelli, 2003), leaving them detectable only by modes that deeply penetrate the stellar interior. Recent 3-dimensional simulations of dynamo-generated magnetic fields in a $7 M_{\odot}$ star by Ratnasingam et al. (2024) have shown that these fields can have a large impact on the rotation of massive stars as the authors found evidence for the rotational shear layer to have a width of the Brunt-Väisälä frequency peak (c.f. Fig. 1.2). This rotational shear layer is the region in the stellar interior that connects the core rotational velocity to the envelope rotational velocity, and is thus a crucial region for the angular momentum transport in the star.

1.3 HD 192575

HD 192575 is an early type B0.5V star with a seismically derived mass of $12 \pm 1.5 M_{\odot}$ (Burssens et al., 2023). Spectroscopic observations lead to a similar mass estimation of $12.5 M_{\odot}$, as derived from the effective temperature $T_{\text{eff}} \approx 27\,700$ K and luminosity $\log(L/L_{\odot}) = 4.26$ (Hohle et al., 2010). Bailer-Jones et al. (2021) determined the distance to the star to be 633 ± 7 pc, for which they used the parallax measured in Gaia-DR3 (Vallenari et al., 2023). The pulsations of HD 192575 were only recently discovered by Burssens et al. (2023), who classified the star as a new β Cep star. Using the High Efficiency and Resolution Mercator Spectrograph (HERMES) mounted on the 1.2 m Mercator telescope on La Palma (Raskin et al., 2011), they derived an effective temperature of $T_{\text{eff}} = 23\,900 \pm 900$ K and a surface gravity of $\log g = 3.65 \pm 0.15$. The rotational line broadening gave a projected rotational velocity of $v \sin i = 27_{-8}^{+6}$ km/s, which shows that the star is rotating relatively slowly although it was classified as moderate rotator with the definition of Aerts and Tkachenko (2024). In addition to the spectroscopic analysis, Burssens et al. (2023) analysed the light curve obtained with the Transiting Exoplanet Survey Satellite (TESS, Ricker et al., 2015) to investigate the internal properties of the star.

Figure 1.6 shows the Lomb-Scargle periodogram of HD 192575. This periodogram was obtained from the TESS light curve with a two minute cadence. The total light curve is a combination of data from cycles 2, 4, and 5. The light curve has a time base of 1951.46 days with a duty cycle of 43.1%. The star shows a rich frequency spectrum with a large number of oscillation frequencies with clear signs of rotational splitting. In particular, the

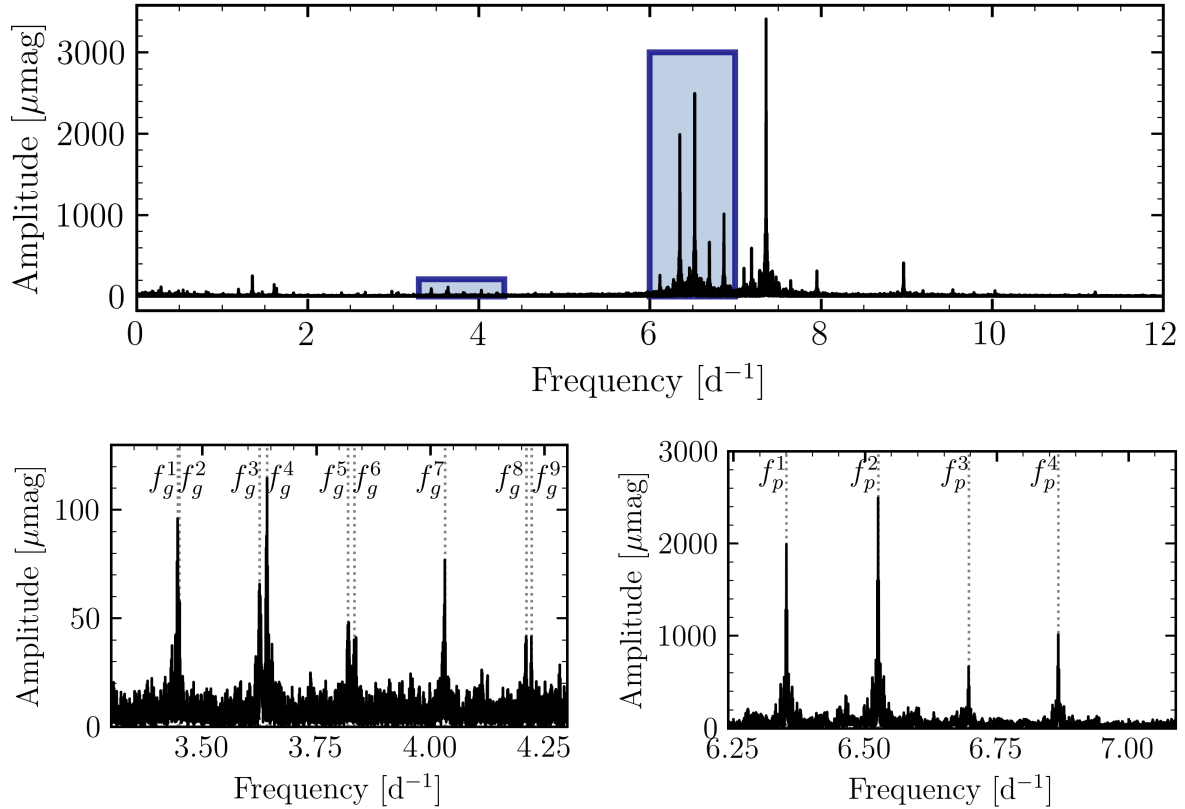


Figure 1.6: Lomb-Scargle periodogram of HD 192575 showing the oscillation frequencies of the star. The top panel shows the full periodogram, while the bottom panels show a zoom-in of the g -mode (left) and p -mode (right) regions denoted in blue. The dotted lines in the bottom panels show the frequencies extracted from the data by M. Vanrespaille.

pulsations in the g -mode regime around 3.8 d^{-1} have gained attention due to its interesting splitting pattern. Instead of a cleanly spaced splitting (as can be seen for the p -modes around 6.5 d^{-1}), the g -modes show a complex splitting pattern of seemingly doubly split frequencies.

Burssens et al. (2023) interpreted these oscillations (as extracted from only cycle 2 TESS data) as a so-called *avoided crossing* (Aizenman et al., 1977), where two modes of the same angular degree l are close in frequency. This is a consequence of the evolution of the star, where the frequencies of the modes approach each other, but cannot cross. This interpretation put a tight constraint on the predicted parameters of the star, as avoided crossings at specific frequencies only occur for limited regions of the parameter space. Notably the evolutionary stage is strongly limited by this assumption. They found that HD 192575 is a supernova progenitor of $12_{-1.5}^{+1.5} M_{\odot}$ with a convective core mass of $M_{\text{cc}} = 2.9_{-0.8}^{+0.5} M_{\odot}$. The differential rotation profile was approximated as a two-step profile connected by a shear layer, which led to the conclusion that the core rotates between 1.4 and 6.3 times faster than the envelope, depending on the location and width of the shear layer. Mombarg et al. (2024) managed to pinpoint this value to 1.4 from two-dimensional evolution models for the properties of the star.

Vanlaer et al. (subm.) revisited the analysis of HD 192575 with TESS lightcurves from

cycles 2 and 4 and were able to determine six sets of possible indentifications of the oscillation modes in the regime of interest without assuming an avoided crossing. These sets of different identifications were obtained by considering $f_g^1, f_g^3, f_g^5, f_g^7$, and f_g^9 (see Fig. 1.6) to form a quadrupole ($l = 2$) g -mode and f_p^1, f_p^2, f_p^3 , and f_p^4 to form a quadrupole p -mode, leaving the interpretation of the additional frequencies open. Together with two additional p -modes at higher frequency, they obtained six sets of different mode identifications and stellar parameters estimations. Forgoing the assumption of the avoided crossing allowed for a more general exploration of the parameter space, leading to a more robust estimation of the stellar parameters and an internal rotation profile from frequency inversion. They found the models to favour weak differential rotation and showed that the splittings of the mode require additional effects such as magnetic fields to explain the observations.

1.4 The role of this thesis

This thesis will revisit the interpretation of the g - and p -mode oscillations of HD 192575 as shown in Fig. 1.6. We will extend upon the work of Vanlaer et al. (subm.) and investigate the combined effects of rotation and magnetic fields to model the observed splittings of the oscillation modes. This interpretation forgoes the assumption of the avoided splitting of Burssens et al. (2023) and allows for a more general interpretation of the oscillation modes in terms of the first magneto-asteroseismology of a β Cep star. We will consider a seismic model where the magnetic field can be misaligned with respect to the rotation axis, as put forward by Loi (2021) within the context of red giant asteroseismology. This model will allow us to predict the frequency splittings and relative amplitudes of the modes, which can be compared to the frequencies and amplitudes observed in the light curve of HD 192575. We will apply the identification of the modes as proposed by Vanlaer et al. (subm.), relying on their determination of the stellar interior and extending the interpretation to include the effects of the internal magnetic field. The model used in this thesis allows us to incorporate the amplitudes in the asteroseismic analysis of the star, extending beyond current methods that consider only mode frequencies. In addition, this thesis delivers the first successful asteroseismic modelling of magnetic fields in a massive star (Vandersnickt et al., to be submitted to A&A).

The thesis is structured as follows: Chapter 2 extensively discusses the theoretical background of the mode splitting in rotating stars and the effects of magnetic fields on the oscillation frequencies. Chapter 3 expands upon the magnetic field topology used in this thesis. Chapter 4 discusses the splittings of the oscillation modes in the presence of both rotation and an internal magnetic field for different stellar models. In Chapter 5, we will use the model to interpret the modes of interest in HD 192575 by fitting the observed splittings to the theoretical predictions. We summarize our conclusion and discuss future work in Chapter 6.

Chapter 2

Stellar Oscillation Equations

Stellar oscillations are a powerful tool to examine the interior of stars, as the observed frequencies depend on the internal structure. The oscillations are small compared to the large-scale structure of the stars, and they can be seen as periodic perturbations of the equilibrium structure of the star. In this chapter, we¹ will derive the equations of stellar oscillations using linear perturbation theory. We will start by describing the basic equations of stellar structure in Section 2.1, which we perturb to obtain the standard oscillation equations in Section 2.2. Using this framework, we will consider the effects of large-scale rotation and the inclusion of a magnetic field in Section 2.3. We discuss the validity of the assumptions made in this chapter in Section 2.4.

2.1 Stellar structure equations

We can describe the equilibrium state of a star on a basic but very effective level as a hydrodynamical system governed by four equations: the continuity equation, the Poisson equation of the gravitational potential, the equation of motion, and the energy equation. These equations assume the properties of the gas to be continuous and differentiable. When considering the gas as seen by a stationary observer, the physical properties of the gas are given as a function of position and time, i.e. $\rho = \rho(\mathbf{r}, t)$, which is referred to as the *Eulerian* description. We can also consider labeling the fluid elements inside the star, for example by the initial position $\mathbf{r}_0$, and describe the physical properties as a function of this label and time. This is called the *Lagrangian* description, in which the position of the labelled fluid element is then given as $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$.

These descriptions must be equivalent, which is enforced by demanding

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{dt} - \mathbf{v} \cdot \nabla \phi, \quad (2.1)$$

for any quantity ϕ , where $\mathbf{v}$ describes the fluid velocity. The right-hand side of this equation describes the Lagrangian formulation with the material time derivative. The left-hand side gives the Eulerian equivalent formulation, where $\frac{\partial \phi}{\partial t}$ is the local time derivative.

¹I have selected to write my thesis using the plural pronoun “we” in the belief that you and I, dear Reader, are on this adventure together.

The four stellar structure equations are given by (Kippenhahn et al., 2013)

$$\begin{cases} \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \\ \rho \frac{d\mathbf{v}}{dt} = -\nabla p - \rho \nabla \Phi, \\ \nabla^2 \Phi = 4\pi G \rho, \\ \rho \frac{dq}{dt} = \rho \epsilon - \nabla \cdot \mathbf{F}, \end{cases} \quad (2.2)$$

where we formulated the equations in the Lagrangian description. The first equation is the continuity equation, where ρ is the density and $\mathbf{v}$ is the velocity of the fluid. This equation describes the conservation of mass in a star, as the rate of change of the density must be equal to the flux. The second equation describes the motion of the fluid and is a formulation of Newton's second law of motion. The change in velocity is proportional to the forces acting on the fluid, with the density acting as the proportionality constant. The forces present are the pressure p differences and the self-gravity of the fluid, with the gravitational potential given by Φ . We will later add additional forces to the system, which will change this expression. The third equation describes the form of the gravitational potential Φ following the Poisson equation. The final equation is the energy equation, where the change in heat q is given by the local energy production ϵ and the energy transport $\mathbf{F}$. The energy production in a star is mainly due to nuclear fusion, where for a main sequence star hydrogen is turned into helium in the core region. The energy transport in the stellar interior is either by radiation or convection. Massive stars such as HD 192575 have a convective core and a radiative envelope.

These equations deliver a basic description of the stellar structure, with a few important notes. We did not describe the chemical evolution of the star as a consequence of nuclear fusion. As the timescales of the oscillations are much shorter than the evolutionary timescale, which can be described by the nuclear timescale, we can safely neglect the changes in composition for the derivation of the oscillation equations. Additionally, this formulation ignores magnetic fields, as this requires the inclusion of the Maxwell equations. Rotation can be included through the velocity in the equations, but this does not include the rotational mixing of the material. Often, the effects of rotation are included perturbatively only in the oscillation equations, while the equilibrium model is considered to be stationary. We will add both rotation and magnetic effects below using such a perturbative approach, in which we assume their effects the background stellar structure and the oscillation modes are small.

2.2 Oscillation equations

Before we perturb the equation above to obtain a description of the oscillations, we need to describe the unperturbed equilibrium structure of the star. This will give us useful tools to connect the unperturbed quantities. To determine the equilibrium structure, and the perturbed equations later on, we will assume the star to be spherically symmetric. This assumption is valid for small effects of rotation and magnetism, which break the spherical symmetry.

The equilibrium structure is static and has no velocities, which means that all time deriva-

tives vanish in the above formulation. This gives us the following equilibrium equations

$$\begin{cases} \mathbf{v}_0 = 0, \\ \nabla p_0 = -\rho_0 \nabla \Phi_0, \\ \nabla^2 \Phi_0 = 4\pi G \rho_0, \\ \rho_0 \epsilon_0 = \nabla \cdot \mathbf{F}_0, \end{cases} \quad (2.3)$$

where the subscript 0 indicates the equilibrium quantities. The second equation describes the balance between the gravitational force and the pressure gradient, denoting that the equilibrium structure is in hydrostatic equilibrium.

2.2.1 Linear perturbation of the equilibrium structure

We can now perturb the stellar structure equations around the equilibrium state. Similar to above, we can consider the perturbations in the Eulerian and Lagrangian descriptions. Consider a quantity $\phi(\mathbf{r}, t)$, the Eulerian perturbation is given by

$$\phi'(\mathbf{r}, t) \equiv \phi(\mathbf{r}, t) - \phi_0(\mathbf{r}, t), \quad (2.4)$$

where $\phi_0(\mathbf{r}, t)$ is the equilibrium quantity. This is the local perturbation at any given point $\mathbf{r}$, and it commutes with the local derivatives. The Lagrangian perturbation is defined for a fluid element at equilibrium position $\mathbf{r}_0$ that is moved to the perturbed position $\mathbf{r}_0 + \delta\mathbf{r}$

$$\delta\phi(\mathbf{r}) \equiv \phi(\mathbf{r}_0 + \delta\mathbf{r}, t) - \phi_0(\mathbf{r}_0, t). \quad (2.5)$$

The Lagrangian perturbation commutes with the material derivatives.

For this thesis, we will only consider linear perturbation theory. When the equations are perturbed, only the terms containing no products or powers higher than one of perturbations are retained. To connect the Lagrangian and Eulerian formulations, we return to the definition of the Lagrangian perturbation moving an element from $\mathbf{r}_0$ to $\mathbf{r}_0 + \delta\mathbf{r}$

$$\delta\phi = \phi(\mathbf{r}_0 + \delta\mathbf{r}) - \phi_0(\mathbf{r}_0) \quad (2.6)$$

$$= \phi(\mathbf{r}_0) + \delta\mathbf{r} \cdot \nabla \phi_0 - \phi_0(\mathbf{r}_0) \quad (2.7)$$

$$= \phi' + \delta\mathbf{r} \cdot \nabla \phi_0. \quad (2.8)$$

This is the perturbation equation connecting the two frameworks. Using this equation, we find that $\frac{d\delta\mathbf{r}}{dt} = \frac{\partial\mathbf{r}'}{\partial t} = \mathbf{v}'$.

We now have the necessary tools to derive the oscillation equations for a static, non-magnetic star. In this approximation, stars are perfect spheres, and we take the natural choice of working in spherical coordinates (r, θ, ϕ) , where r is the radial coordinate, θ is the colatitude (polar angle) and ϕ is the longitude (azimuthal angle), as visualized in Fig. 2.1. At a certain moment t in time, all equilibrium quantities are sole functions of the radial coordinate r as a consequence of the static and spherically symmetric assumptions. It follows that the solution must be separable in θ and ϕ (i.e. the “horizontal variables”):

$$\delta\mathbf{r} = \xi_r \hat{\mathbf{e}}_r + \boldsymbol{\xi}_h, \quad (2.9)$$

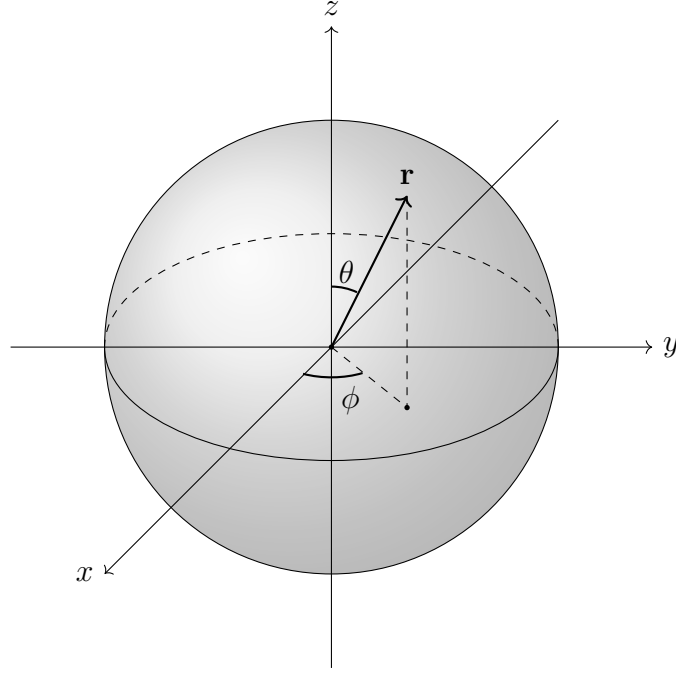


Figure 2.1: Spherical coordinate system.

where $\hat{\mathbf{e}}_r$ is the radial unit vector such that ξ_r is the radial displacement. The horizontal displacement is combined in the vector ξ_h , which is perpendicular to the radial displacement. Even though we have split the vector in radial and horizontal components, both still contain a dependency on all three coordinates.

We now perturb the stellar structure equations using the framework described above.

1. *The continuity equation.* We consider the Eulerian perturbation and obtain

$$\begin{aligned}
 0 &= \left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} \right)', \\
 0 &= \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} \right)', \\
 0 &= \frac{\partial \rho'}{\partial t} + \nabla \cdot \rho_0 \mathbf{v}' + \nabla \cdot \rho' \mathbf{v}_0, \\
 0 &= \rho' + \nabla \cdot \rho_0 \delta \mathbf{r},
 \end{aligned}$$

where we used that $\mathbf{v}_0 = 0$ and integrated over time in the last step.

2. *The equation of motion.* We consider the Eulerian perturbation and obtain

$$\begin{aligned}
 \left(\rho \frac{d\mathbf{v}}{dt} \right)' &= -\nabla p' - \rho_0 \nabla \Phi' - \rho' \nabla \Phi_0, \\
 \rho_0 \frac{\partial^2 \delta \mathbf{r}}{\partial t^2} &= -\nabla p' - \rho_0 \nabla \Phi' - \rho' \nabla \Phi_0, \\
 \rho_0 \frac{\partial^2 \delta \mathbf{r}}{\partial t^2} &= -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0,
 \end{aligned}$$

where $\mathbf{g} \equiv -\nabla\Phi$ is the gravitational acceleration. We work again under the assumption $\mathbf{v}_0 = 0$.

3. *The Poisson equation.* We consider the Eulerian perturbation and obtain immediately

$$\nabla^2\Phi' = 4\pi G\rho'.$$

4. *The adiabatic energy equation.* The stellar oscillations occur on short timescales compared to the heat transport. We will therefore assume the oscillations to be adiabatic. This simplifies the energy equation above to the adiabatic energy equation (as worked out in, for example, [Aerts et al., 2010](#)). We consider the Lagrangian perturbation of the adiabatic energy equation and obtain

$$\begin{aligned}\delta\left(\frac{dp}{dt}\right) &= \delta\left(\frac{\Gamma_1 p}{\rho} \frac{d\rho}{dt}\right), \\ \frac{d\delta p}{dt} &= \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{d\delta\rho}{dt}, \\ \delta p &= \frac{\Gamma_{1,0} p_0}{\rho_0} \delta\rho,\end{aligned}$$

as all equilibrium quantities are static. Translating to Eulerian formulation, we get

$$p' + \delta\mathbf{r} \cdot \nabla p_0 = \frac{\Gamma_{1,0} p_0}{\rho_0} (\rho' + \delta\mathbf{r} \cdot \nabla \rho_0).$$

In this formulation, time is an independent variable, which only appears in the perturbed equation of motion as a second derivative. Hence, we can write the time dependence of the linear oscillation mode as $e^{-i\omega t}$, where ω is the oscillation frequency of the mode and all solutions are of the form $\delta\mathbf{r}(r, \theta, \phi, t) = (\xi_r \hat{\mathbf{e}}_r + \boldsymbol{\xi}_h) e^{-i\omega t}$. Using this formulation, we can rewrite the perturbation equations in a time-independent form, which will allow for an elegant description as a linear eigenvalue problem in the next section. To obtain this formulation, we return to the perturbed equation of motion and substitute our time-separated formulation. We get

$$\begin{aligned}\rho_0 \frac{\partial^2 \delta\mathbf{r}}{\partial t^2} &= -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0, \\ \rho_0 \frac{\partial^2}{\partial t^2} \delta\mathbf{r} e^{-i\omega t} &= -\nabla p' e^{-i\omega t} + \rho_0 \mathbf{g}' e^{-i\omega t} + \rho' \mathbf{g}_0 e^{-i\omega t}, \\ \rho_0 \omega^2 \delta\mathbf{r} &= \nabla p' - \rho_0 \mathbf{g}' - \rho' \mathbf{g}_0.\end{aligned}\tag{2.10}$$

Replacing the time-dependent formulation of the momentum equation with the time-independent formulation of a mode's displacement, the equations lose their time dependence. Using the separation of variables, the spatial gradient can be written as

$$\nabla = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \nabla_h,\tag{2.11}$$

which separates the radial and horizontal gradients. Using the time-independent perturbation equations, we can describe the displacement vector as a solution of the following

equations

$$\begin{cases} \omega^2 \left(\rho' + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_0 \xi_r) \right) = \nabla_h^2 (p' + \rho_0 \Phi') \\ -\omega^2 \rho_0 \xi_r = -\frac{\partial p'}{\partial r} - \rho' g_0 - \rho_0 \frac{\partial \Phi'}{\partial r} \\ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Phi'}{\partial r}) + \nabla_h^2 \Phi' = 4\pi G \rho' \end{cases} \quad (2.12)$$

As described in [Aerts et al. \(2010\)](#), this leads to the angular dependence being described by the spherical harmonics

$$Y_l^m(\theta, \phi) = (-1)^m c_{lm} P_l^m(\cos \theta) e^{im\phi}, \quad (2.13)$$

where P_l^m are the associated Legendre polynomials and c_{lm} are normalization constants such that

$$|Y_l^m|^2 = \int_{\text{sphere}} Y_l^m (Y_l^m)^* \sin(\theta) d\theta d\phi = 1, \quad (2.14)$$

where the asterisk (*) denotes the complex conjugate. The spherical harmonics are uniquely defined by the angular degree l and azimuthal order m for a chosen reference axis, where l is a non-negative integer, and m is an integer such that $-l \leq m \leq l$. The spherical harmonics form an orthonormal basis for a sphere and they are eigenfunctions of the horizontal part of the Laplacian operator, which allows us to write

$$\nabla_h^2 Y_l^m = -\frac{l(l+1)}{r^2} Y_l^m. \quad (2.15)$$

It follows that the dependent variables of Eq. (2.12) can be written as, for example,

$$\rho'(r, \theta, \phi, t) \propto \rho'(r) Y_l^m(\theta, \phi) e^{-i\omega t}, \quad (2.16)$$

where the proportionality is given by a normalization constant. We have similar expressions for ξ_r , p' , and Φ' . The radial dependence of these variables can be solved for by using the system of equations in Eq. (2.12), together with the perturbed energy equation and Eq. (2.15). Notice that while the spherical harmonics are characterized by both the angular degree l and azimuthal order m , only the angular degree is present in the system of equations. This leads to a degeneracy in azimuthal order for the obtained frequencies ω , which is a direct consequence of the spherical symmetry of the star. This spherical symmetry – and consequently the azimuthal degeneracy – can be broken by including non-spherically symmetric effects, such as rotation or magnetic fields.

The horizontal component of the displacement vector ξ_h is obtained through the horizontal part of the equation of motion, which shows that ([Aerts et al., 2010](#))

$$\xi_h \propto \xi_h(r) \nabla_h Y_l^m, \quad (2.17)$$

and the radial component $\xi_r(r)$ is a linear combination of $p'(r)$ and $\Phi'(r)$. Combining both results for the displacement vector, we can write the full displacement vector as

$$\begin{aligned} \delta \mathbf{r} &\propto \xi_r(r) \hat{\mathbf{e}}_r + \xi_h(r) \nabla_h Y_l^m \\ &\propto \mathcal{R} \left[\left(\xi_r Y_l^m \hat{\mathbf{e}}_r + \xi_h(r) \left(\frac{\partial Y_l^m}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{1}{\sin \theta} \frac{\partial Y_l^m}{\partial \phi} \hat{\mathbf{e}}_\phi \right) \right) e^{-i\omega t} \right], \end{aligned} \quad (2.18)$$

where we expanded the horizontal gradient in spherical coordinates and only consider the real part of the solution. For the spherically symmetric case, each spherical harmonic Y_l^m leads to a solution of the oscillation equations for an obtained radial solution $\xi_r(r)$ and a horizontal solution $\xi_h(r)$. In general, the total perturbation of the star will be a linear combination of these solutions, as the spherical harmonics form a basis for a sphere.

As the oscillations are described by a system of first-order differential equations, we require boundary conditions. We consider both boundary conditions at the centre of the star and at the surface. At the centre ($r = 0$), we require the solution to be regular, which can be expressed as (Unno et al., 1989, Aerts et al., 2010)

$$\xi_r \approx l\xi_h \quad \text{for } r = 0, \quad (2.19)$$

$$\frac{d\Psi'}{dr} - \frac{l}{r}\Psi' = 0 \quad \text{for } r = 0. \quad (2.20)$$

At the surface ($r = R$), we require the perturbation of the gravitational potential and its derivative to be continuous. This translates to the perturbation Φ' satisfying

$$\frac{d\Phi'}{dr} + \frac{l+1}{r}\Phi' = 0 \quad \text{for } r = R. \quad (2.21)$$

Additionally, we will consider the star to have a free upper boundary such that no forces are acting on the surface (i.e. we ignore the effects of a stellar atmosphere). This translates to the Lagrangian pressure perturbation being zero at the surface

$$\delta p = p' + \xi_r \frac{dp}{dr} = 0 \quad \text{for } r = R. \quad (2.22)$$

These boundary conditions are known as the *vacuum boundary conditions*.

The oscillation equations are homogeneous differential equations, which means the solutions are defined up to a scaling factor. In physics terms, this means that while we can calculate the form of the displacement and the frequency, the amplitude of the oscillation is not determined by the equations. This is true for all independent solutions. When a solution is a linear combination of different spherical harmonics with different frequencies (e.g. for misaligned rotational and magnetic axes), the relative amplitudes of the different components are fixed. We will see examples of this later on.

To get a sense of the physical behaviour of the displacement, Fig. 2.2 shows the radial part of the displacement vector with $l = 2$ and $m = 0, 1, 2$. The radial displacement is given at four different steps in a single period $P = 2\pi/\omega$. While the figure might be suggestive of rotation, it shows the periodic nature of the oscillation through the star, where the displacement propagates over the surface. The displacement for identical $|m|$ are similar in form, but propagate in opposite directions. We can qualitatively describe the angular dependence using the angular degree l and the azimuthal order m . The angular degree describes the number of nodal lines on the stellar surface, of which m are longitudinal. For $m = 0$ (zonal mode), all nodal lines are latitudinal. At $t = \frac{P}{4}$ and $t = \frac{3P}{4}$ the radial displacement disappears and it passes through spherical symmetry with no radial displacement. For $m = \pm 1$ (tesseral mode), the sphere has one longitudinal nodal line and one latitudinal line, dividing the sphere in four separate sectors. For $m = \pm 2$ (sectoral mode), the sphere has two longitudinal nodal lines, such that the sphere is divided in longitudinal slices.

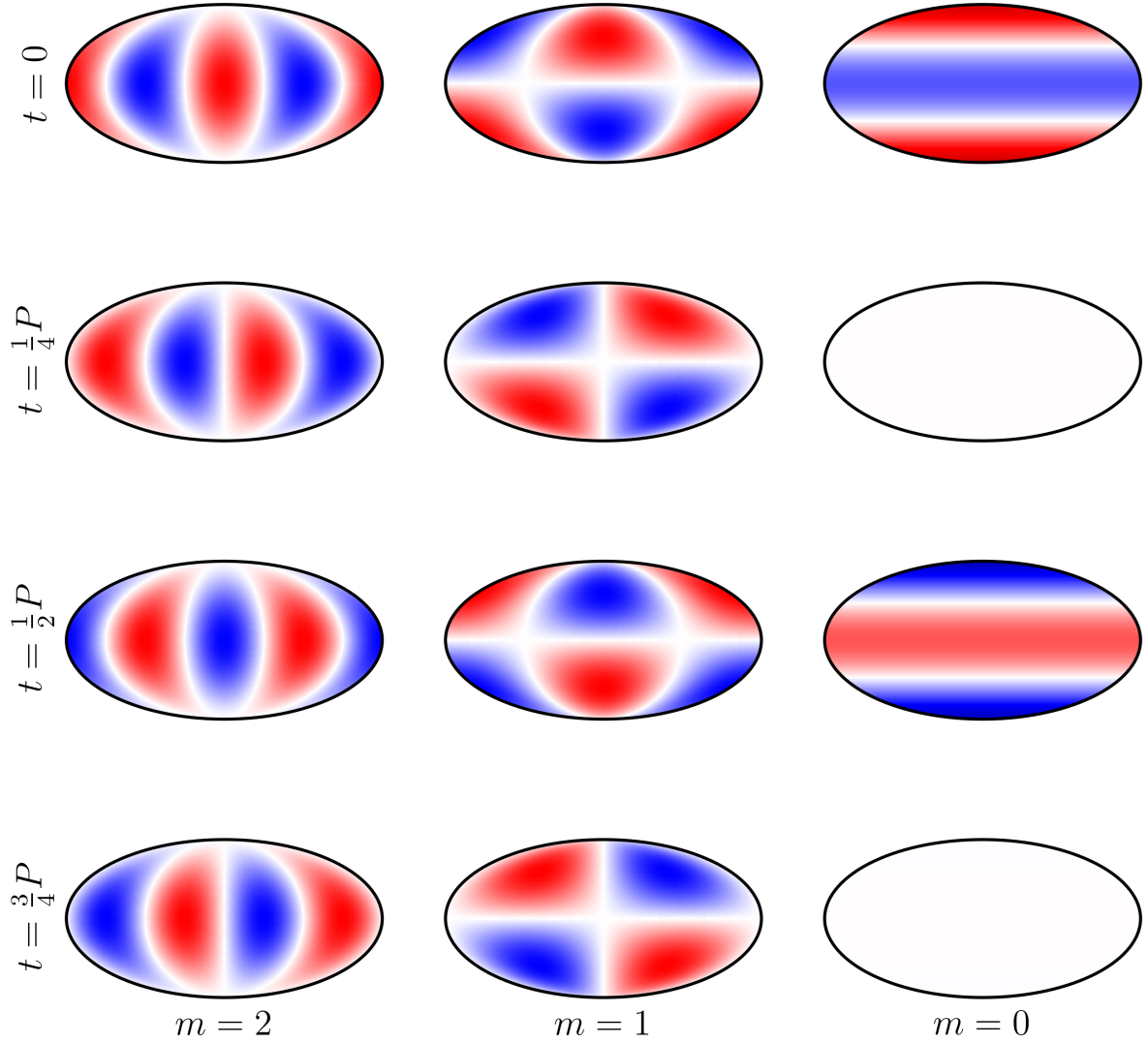


Figure 2.2: Radial displacement for $l = 2$ at four steps in a period. We show the behaviour for $m = 0, 1, 2$ over the full sphere with a Mollweide projection.

2.2.2 Perturbation equations as a linear eigenvalue problem

The perturbed quantities can all be written as functions of the displacement vector by use of the time separated perturbation equations. This allows us to formulate the oscillation frequency ω as an eigenfrequency of the operator² defined by the time-independent equation of motion following

$$\omega^2 \delta \mathbf{r} = \mathcal{F}(\delta \mathbf{r}), \quad (2.23)$$

where $\mathcal{F}(\delta \mathbf{r}) = \frac{1}{\rho_0} \nabla p' - \mathbf{g}' - \frac{\rho'}{\rho_0} \mathbf{g}_0$ (cf. Eq. 2.10) is a linear operator as all perturbations are linear. To do this, we require a Hilbert space, which is a complete metric space for a defined inner product. Consider $\mathcal{H}$ to be a vector space of vector functions which satisfy the boundary conditions, and have appropriate regularity conditions. The inner product

²Even though the actual eigenvalue is ω^2 , we will refer to ω as the eigenfrequency.

on the Hilbert space $\mathcal{H}$ is defined as

$$\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} : (\boldsymbol{\xi}, \boldsymbol{\eta}) \mapsto \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \int_V \rho_0 \boldsymbol{\xi}^* \cdot \boldsymbol{\eta} \, dV, \quad (2.24)$$

where again the asterisk (*) denotes the complex conjugate. Lynden-Bell and Ostriker (1967) showed that the operator $\mathcal{F}$ is symmetric for the inner product in the sense that

$$\langle \mathcal{F}(\boldsymbol{\xi}), \boldsymbol{\eta} \rangle = \langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\eta}) \rangle. \quad (2.25)$$

We will use this formalism in the following sections.

2.3 Perturbing the oscillation equations

Within this thesis, we want to determine the effects of rotation and magnetic fields on the oscillations of stars, where rotational and magnetic effects are similar in strength, but can be considered perturbatively compared to the effects of pressure and buoyancy. To do so, we will work within the eigenvalue formulation of the stellar pulsations as described in Sect. 2.2.2, where we found that

$$\omega_0^2 \boldsymbol{\xi}_0 = \mathcal{F}_0(\boldsymbol{\xi}_0), \quad (2.26)$$

where the 0 index denotes this to be a solution of the adiabatic oscillation equations for pressure and buoyancy perturbations for a fixed n , l , and m for a chosen reference axis. We replace $\boldsymbol{\xi} \equiv \delta \mathbf{r}$ for clarity in the following sections. We consider now a perturbation $\delta \mathcal{F}$ of the operator $\mathcal{F}_0$, and derive the change in eigenvalue for linear perturbation theory, such that $\mathcal{F} = \mathcal{F}_0 + \delta \mathcal{F}$. We will obtain a new eigenvalue $\omega^2 = \omega_0^2 + \delta \omega^2$ with corresponding eigenfunction $\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \delta \boldsymbol{\xi}$, following the framework described in Loi (2021). Hence, we find to the first order in linear perturbation theory that

$$\omega_0^2 \boldsymbol{\xi}_0 + \omega_0^2 \delta \boldsymbol{\xi} + \delta \omega^2 \boldsymbol{\xi}_0 = \mathcal{F}_0(\boldsymbol{\xi}_0) + \delta \mathcal{F}(\boldsymbol{\xi}_0) + \mathcal{F}_0(\delta \boldsymbol{\xi}). \quad (2.27)$$

We approximate $\delta \boldsymbol{\xi}$ as a linear combination in the set of all eigenfunctions $\{\boldsymbol{\xi}_0^{(i)}\}$ of $\mathcal{F}_0$ with the same n and l as $\boldsymbol{\xi}_0$ such that $\delta \boldsymbol{\xi} \approx \sum_i c_i \boldsymbol{\xi}_0^{(i)}$, where for each element we have $\mathcal{F}_0(\boldsymbol{\xi}_0^{(i)}) = \omega^{(i)2} \boldsymbol{\xi}_0^{(i)}$. We can rewrite Eq. (2.27) as

$$\delta \omega^2 \boldsymbol{\xi}_0 = \delta \mathcal{F}(\boldsymbol{\xi}_0) - \left[\sum_i c_i \left(\omega_0^2 - \omega^{(i)2} \right) \boldsymbol{\xi}_0^{(i)} \right]. \quad (2.28)$$

This is a useful result, as the inner product of the second term of the right-hand side with any element of the set $\{\boldsymbol{\xi}_0^{(i)}\}$ is zero. Now imagine that $\boldsymbol{\xi}_0$ is not an element of the set $\{\boldsymbol{\xi}_0^{(i)}\}$, but rather a linear combination $\boldsymbol{\xi}_0 = \sum_i a_i \boldsymbol{\xi}_0^{(i)}$ (e.g. after switching between reference frames or by mixing modes due to including both rotation and magnetic effects). We would then obtain for each term in the decomposition

$$\delta \omega^2 \langle \boldsymbol{\xi}_0^{(i)}, \boldsymbol{\xi}_0^{(i)} \rangle a_i = \langle \delta \mathcal{F}(\boldsymbol{\xi}_0), \boldsymbol{\xi}_0^{(i)} \rangle = \langle \delta \mathcal{F}(\boldsymbol{\xi}_0^{(i)}), \boldsymbol{\xi}_0^{(i)} \rangle a_i. \quad (2.29)$$

We define $\mathbf{a}$ as the vector of all coefficients a_i for the vector $\boldsymbol{\xi}_0$ expressed in the orthogonal set $\{\boldsymbol{\xi}_0^{(i)}\}$, such that the following relation must hold

$$\delta \omega^2 \mathbf{a} = \mathbf{M} \mathbf{a}, \quad (2.30)$$

where the matrix $\mathbf{M}$ is defined as

$$\mathbf{M} = \frac{\text{diag} \left(\langle \delta \mathcal{F}(\xi_0^{(i)}), \xi_0^{(i)} \rangle \mid \text{for } \xi_0^{(i)} \in \{\xi_0\} \right)}{\langle \xi_0, \xi_0 \rangle}. \quad (2.31)$$

This shows that the perturbation of the oscillation frequency is the eigenvalue of the matrix $\mathbf{M}$, which holds the inner product of the perturbation for any basis set of solutions. We have derived the above result under very general conditions (e.g. $\delta \mathcal{F}$ is not axisymmetric). If $\delta \xi$ can be expressed with a single basis element, Eq. (2.30) simplifies to the scalar equation

$$\delta \omega^2 = \frac{\langle \delta \mathcal{F}(\xi_0), \xi_0 \rangle}{\langle \xi_0, \xi_0 \rangle}, \quad (2.32)$$

which is the result presented in [Schiff and Hamermesh \(1949\)](#) and [Aerts et al. \(2010\)](#). This is, for example, the case for the effect of only including rotation. For this thesis, we consider the perturbative effects of rotation and internal magnetic fields by inclusion of the operators $\delta \mathcal{F}_{\text{rot}}$ and $\delta \mathcal{F}_{\text{mag}}$, respectively. This leads to the following formulation of the equation of motion

$$\omega^2 \xi = \mathcal{F}_0 \xi + \delta \mathcal{F}_{\text{rot}} \xi + \delta \mathcal{F}_{\text{mag}} \xi. \quad (2.33)$$

The effect on the eigenfrequency of the mode can be determined using Eq. (2.30), by considering $\delta \mathcal{F} = \delta \mathcal{F}_{\text{rot}} + \delta \mathcal{F}_{\text{mag}}$. We will first determine expressions for $\delta \mathcal{F}_{\text{rot}}$ and $\delta \mathcal{F}_{\text{mag}}$ separately, after which we will discuss how to combine them to a single operator.

2.3.1 Deriving the rotation operator

To derive an expression for the rotation operator, we return to the time-dependent momentum equation and consider a large-scale velocity $\mathbf{v}_0 \neq 0$, which describes a rotation of the star around the z -axis. We perturb the equation using the Lagrangian framework in the corotating frame and obtain

$$\begin{aligned} \delta \left(\rho \frac{d\mathbf{v}}{dt} \right) &= \rho_0 \frac{d\delta \mathbf{v}}{dt} = \delta (-\nabla p + \rho \mathbf{g}) \\ &= -\delta \nabla p + \delta \rho \cdot \mathbf{g} + \rho \cdot \delta \mathbf{g} \\ &= -\nabla p' - \delta \mathbf{r} \nabla^2 p_0 + \rho' \mathbf{g}_0 + (\delta \mathbf{r} \cdot \nabla \rho_0) \mathbf{g}_0 + \rho \mathbf{g}' + \rho_0 \delta \mathbf{r} (\nabla \cdot \mathbf{g}_0) \\ &= -\nabla p' + \delta \mathbf{r} \nabla \cdot (\nabla p_0 + \rho_0 \mathbf{g}_0) + \rho' \mathbf{g}_0 + \rho_0 \mathbf{g}' \\ &= -\nabla p' + \rho' \mathbf{g}_0 + \rho_0 \mathbf{g}' \end{aligned}$$

as we know from the equilibrium state Eq. (2.3) that $\nabla p_0 + \rho_0 \mathbf{g}_0 = 0$. By using the equality $\frac{d\delta \mathbf{v}}{dt} = \frac{d^2 \delta \mathbf{r}}{dt^2}$, we can write the final equation in both Lagrangian and Eulerian format

$$\rho_0 \frac{d^2 \delta \mathbf{r}}{dt^2} = -\nabla p' + \rho' \mathbf{g}_0 + \rho \mathbf{g}', \quad (2.34)$$

$$\rho_0 \frac{\partial^2 \delta \mathbf{r}}{\partial t^2} + 2\rho_0 (\mathbf{v}_0 \cdot \nabla) \left(\frac{\partial \delta \mathbf{r}}{\partial t} \right) = -\nabla p' + \rho' \mathbf{g}_0 + \rho \mathbf{g}', \quad (2.35)$$

where now the second term on the left-hand side does not vanish, leading to a change in the perturbation equation. We return to the time-independent formulation by considering $\delta \mathbf{r}(r, t) = \boldsymbol{\xi}(r)e^{-i\omega t}$. This gives

$$-\rho_0\omega^2\boldsymbol{\xi} - 2\rho_0\omega i(\mathbf{v}_0 \cdot \nabla)\boldsymbol{\xi} = -\nabla p' + \rho' \mathbf{g}_0 + \rho \mathbf{g}'. \quad (2.36)$$

As we take $\mathbf{v}_0$ such that the perturbation is small, we find $\delta \mathcal{F}_{\text{rot}} = -2\omega i(\mathbf{v}_0 \cdot \nabla)\boldsymbol{\xi}$.

We are considering a rotational velocity, this means that we can write $\mathbf{v}_0 = \boldsymbol{\Omega} \times \mathbf{r}$, where $\boldsymbol{\Omega}$ is the angular velocity of the star. Using this relation, we can rewrite the rotational perturbation operator in the corotating frame as

$$\delta \mathcal{F}_{\text{rot}} = -2i\omega \boldsymbol{\Omega} \times \boldsymbol{\xi}. \quad (2.37)$$

Note that we only consider linear terms in the expression above. The non-linear terms of higher order in $\boldsymbol{\Omega}$ are neglected in the derivation above.

Equation (2.37) expresses the effect of the Coriolis force on the oscillations of the star. We can investigate the effect of the operator on the eigenvalues using Eq. (2.32), as the operator is axisymmetric. We find the change in eigenvalue to be

$$\delta\omega^2 = -2m\omega_0\Omega \frac{\int_0^{R_{\text{star}}} \rho (2\xi_h\xi_r + \xi_h^2) r^2 dr}{\int_0^{R_{\text{star}}} \rho (\xi_r^2 + l(l+1)\xi_h^2) r^2 dr}, \quad (2.38)$$

where R_{star} is the stellar radius. The derivation of this expression can be found in Appendix A.

This equation describes the change in eigenvalue in the corotating frame. As observers, we are not in the corotating frame, and we have to transform the eigenvalue to the inertial frame. Additionally, we want to describe the change in eigenfrequency ω , rather than the change in ω^2 . This second step is easily done by realizing $\delta\omega^2 = 2\omega_0\delta\omega$. The transformation to the inertial frame can be described from simple considerations. The star is rotating around the z -axis with angular velocity Ω , any set of coordinates (r', θ', ϕ') in the corotating frame can be transformed to the inertial frame $(r, \theta, \phi - \Omega t)$. Applying this change for the relevant terms in Eq. (2.13) and Eq. (2.18), we find

$$e^{i(m\phi' - \omega't)} = e^{i(m\phi - m\Omega t - \omega t)}. \quad (2.39)$$

This shows that the change in frequency between the corotating (ω') and inertial frame (ω) is given by $\omega' \equiv \omega - m\Omega$. Referring back to Fig. 2.2, this is not a surprising result as we see that for $m \neq 0$, the displacement propagates over the surface of the star in a circular motion. Depending on the rotational direction, the period for a propagating displacement is either shortened (for

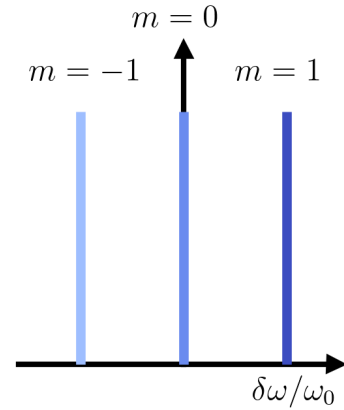


Figure 2.3: Example of the rotational splittings of a dipole mode ($l = 1$). The different colors give the different azimuthal orders. The vertical arrow shows the position of ω_0 .

prograde displacement with respect to the rotational direction; $m > 0$), or lengthened (for retrograde displacement with respect to the rotational direction; $m < 0$) for an observer in the inertial frame.

Using both results above, we can write the change in eigenfrequency in the inertial frame as

$$\delta\omega = m\Omega \left(1 - \frac{\int_0^{R_{\text{star}}} \rho (2\xi_h \xi_r + \xi_h^2) r^2 dr}{\int_0^{R_{\text{star}}} \rho (\xi_r^2 + l(l+1)\xi_h^2) r^2 dr} \right), \quad (2.40)$$

which is independent of the unperturbed frequency ω_0 . The second term is often referred to as the Ledoux constant β_{nl} (Ledoux, 1951), where the subscripts are used to indicate the dependence on the radial order n and the angular degree l of the mode. A graphical representation of the rotational mode splitting is shown in Fig. 2.3. Following the discussion in Aerts et al. (2010), the Ledoux term will be negligible for high radial order or high angular degree p -mode. The splitting of these frequencies are thus a direct probe of the rotational velocity of the star. However, for high radial order or high angular degree g -mode, the radial component of the displacement can be neglected, and we find $\delta\omega \approx m\Omega \left(1 - \frac{1}{l(l+1)} \right)$. The splitting of a dipole g -mode ($l = 1$) will thus be only half of the rotation rate and half the splitting of a corresponding p -mode. This shows that the inclusion of large-scale rotation breaks the azimuthal order degeneracy.

In this derivation, we assumed solid body rotation. If the rotation is dependent on the radial coordinate (i.e. $\Omega = \Omega(r)$), the corotating frame is not defined and we cannot get Ω out of the integral. This is discussed in Aerts et al. (2010) and is considered for HD 192575 by Burssens et al. (2023).

2.3.2 Deriving the magnetic field operator

The magnetic field operator can be derived using the magneto-hydrodynamic equations. This requires us to change the momentum equation by adding the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}$, which is the vector product of the current density $\mathbf{J} \equiv \nabla \times \mathbf{B}$ and the magnetic field $\mathbf{B}$, and to add the induction equation to the system of equations (Goedbloed and Poedts, 2009). For ease of writing, we consider the units of the magnetic field such that the magnetic permeability $\mu_0 = 1$. The momentum equation can be written as

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \rho \mathbf{g} + (\nabla \times \mathbf{B}) \times \mathbf{B} \quad (2.41)$$

and the induction equation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (2.42)$$

We derive the magnetic field operator under the assumption of no large-scale velocity. This means that the velocity vector is given by $\mathbf{v} = \mathbf{v}' = \frac{\partial \boldsymbol{\xi}}{\partial t}$. By perturbing the induction equation and integrating over time, we find

$$\mathbf{B}' = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0). \quad (2.43)$$

We now take the Eulerian perturbation of the momentum equation. We find

$$\begin{aligned}
\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \rho' \frac{\partial \mathbf{v}_0}{\partial t} &= -\nabla p' + \rho' \mathbf{g}_0 + \rho_0 \mathbf{g}' + [(\nabla \times \mathbf{B}) \times \mathbf{B}]' \\
&= \mathcal{F}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{B}' + (\nabla \times \mathbf{B}') \times \mathbf{B}_0 \\
&= \mathcal{F}_0 + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)) + (\nabla \times \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)) \times \mathbf{B}_0 \\
&= \mathcal{F}_0 + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)) - \mathbf{B}_0 \times [\nabla \times (\boldsymbol{\xi} (\nabla \cdot \mathbf{B}_0) \\
&\quad - \mathbf{B}_0 (\nabla \cdot \boldsymbol{\xi}) + (\mathbf{B}_0 \cdot \nabla) \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{B}_0)] \\
&= \mathcal{F}_0 + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)) - \mathbf{B}_0 \times [\nabla \times ((\mathbf{B}_0 \cdot \nabla) \boldsymbol{\xi} \\
&\quad - \mathbf{B}_0 (\nabla \cdot \boldsymbol{\xi}) - (\boldsymbol{\xi} \cdot \nabla) \mathbf{B}_0)].
\end{aligned}$$

Which shows that the perturbation of the magnetic field can be written as

$$\begin{aligned}
\delta \mathcal{F}_{\text{mag}} \boldsymbol{\xi} &= \frac{1}{\rho_0} [\mathbf{B}_0 \times [\nabla \times (-\mathbf{B}_0 (\nabla \cdot \boldsymbol{\xi}) + (\mathbf{B}_0 \cdot \nabla) \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{B}_0)]] \\
&\quad + \frac{1}{\rho_0} [(\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)) \times (\nabla \times \mathbf{B}_0)] \\
&\quad - \frac{\rho'}{\rho_0^2} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0.
\end{aligned} \tag{2.44}$$

This expression is less straightforward than the rotation operator. However, there are a few interesting properties to point out. Firstly, we remark that the full operator is not self-adjoint when ignoring the deformations in the geometry, which can possibly lead to complex eigenfrequencies in the current formulation. Numerical test show that the imaginary component is around 10^{-19} times smaller than the real component for a wide range of values and stellar models, which is negligible for the current purposes.

Furthermore, the first term of the operator is self-adjoint under the current assumptions. As this term is the dominant term for high-order g -modes, it can be used as a good approximation for the magnetic perturbations on these modes (Loi, 2021). Secondly, we can connect each term of the full operator to a quantity occurring in Eq. (2.41). The first term is the perturbation of the current density $\mathbf{J}$, the second term describes the perturbation of the magnetic field $\mathbf{B}$, and the last term gives the effect of a density perturbation under the magnetic field. A similar term for the density perturbation is absent for the rotation operator, as the Coriolis force depends on the position of the material element and not on its properties. Thirdly, as the unperturbed frequency is not present in the operator, we expect $\mathbf{M}_{\text{mag}} \propto \omega_0^{-1}$, such that for a pure magnetic perturbation, we have

$$\frac{\delta \omega}{\omega_0} \propto \frac{1}{\omega_0^2}. \tag{2.45}$$

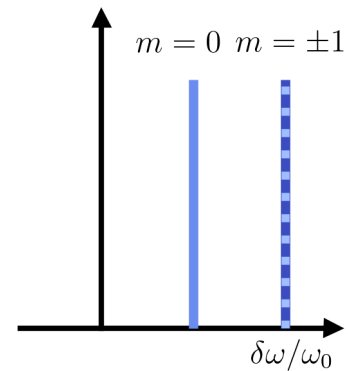


Figure 2.4: Example of the magnetic splittings of a dipole mode ($l = 1$). The different colors give the different azimuthal orders. The vertical arrow shows the position of ω_0 .

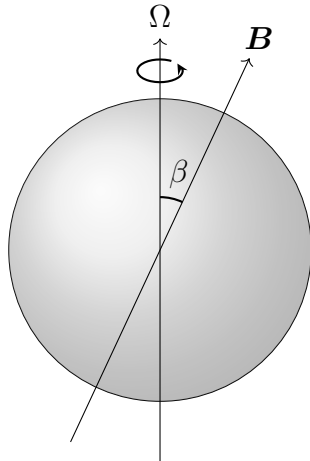


Figure 2.5: Visual representation of the different axes of the two coordinate systems. The rotation axis is given by Ω and the magnetic field axis is given by $\mathbf{B}$. The obliquity angle between the two axes is given by β .

A similar result was reported by Prat et al. (2019), who showed that for low-radial-order modes, the magnetic perturbation follows $\delta\omega/\omega_0 \propto B_0^2/\omega_0^2$, where B_0 is the centre magnetic field strength. The presence of a magnetic field breaks the spherical symmetry of the system and is expected to lift the azimuthal order degeneracy of the modes, similar to rotation. As the magnetic field we will consider does not make a distinction in the azimuthal direction, we expect the effect to be the same for modes with the same $|m|$ -value. This is clearly visible in Fig. 2.4, where the splitting of the azimuthal orders is shown for a dipole mode ($l = 1$). The two azimuthal orders $m = \pm 1$ coincide in frequency.

2.3.3 Combining the perturbation operators

We now have expressions for both the rotation and magnetic field perturbation operators, where both are axisymmetric. Combining these operators is a non-trivial step, as they are not necessarily symmetric around the same axis. As we discussed in Sect. 2.3, we require a basis set of solutions to formulate the perturbations. There are two natural choices for the basis set: one around the rotation axis $\{\xi_{\text{rot}}^{(m)}\}$ and one around the magnetic field axis $\{\xi_{\text{mag}}^{(m')}\}$. Consider ξ_0 to be a solution with fixed angular degree l and radial order n , we can express this solution as a linear combination of the eigenfunctions in the rotational frame and magnetic field frame

$$\xi_0 = \sum_{m=-l}^l r_m \xi_{\text{rot}}^{(m)} = \sum_{m'=-l}^l b_{m'} \xi_{\text{mag}}^{(m')}. \quad (2.46)$$

We note that the eigenfunctions have the same form, but are shifted from the rotational frame (m, θ, ϕ) to the magnetic field frame (m', θ', ϕ') . The transformation between the two frames does not change the radial coordinate and is given by the Wigner d -matrix, which we will represent as $\mathbf{D}$ to comply with our notation scheme. The matrix elements $d_{m'm}^{(l)}$ are described in Appendix B.

Both perturbation operators need to be expressed in the same basis. We choose to write the magnetic field operator in the rotational frame. We note here that the transforma-

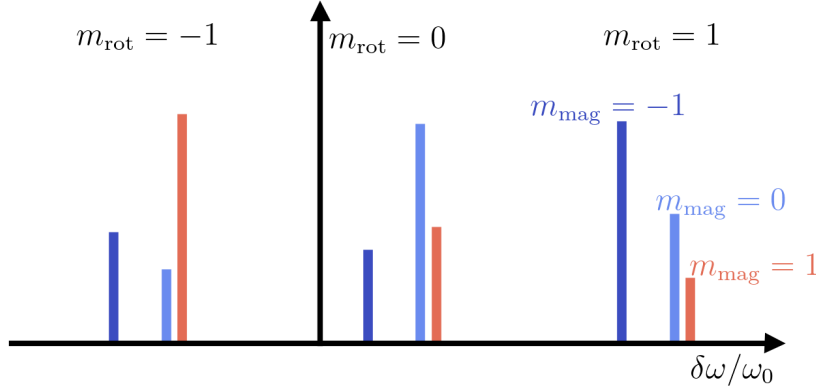


Figure 2.6: Example of the combined effect of misaligned rotation and magnetic field on the eigenfrequencies of a dipole mode ($l = 1$). The different colours give the different independent solutions. The vertical arrow shows the position of the frequency of the mode with no effects of rotation or magnetic field ω_0 .

tion depends on the obliquity angle β between the rotation and magnetic field axis, as visualised in Fig. 2.5. Using the Wigner d -matrix, the basis functions are connected by

$$\xi_{\text{mag}}^{(m')} = \sum_{m'=-l}^l d_{m'm}^{(l)}(\beta) \xi_{\text{rot}}^{(m)}. \quad (2.47)$$

Similarly, the coefficients of Eq. (2.46) can be expressed as $\mathbf{r} = \mathbf{D}\mathbf{b}$, where $\mathbf{r}$ and $\mathbf{b}$ are vectors containing the coefficients r_m and b_m , respectively. Using Eq. (2.31), we can thus write

$$\delta\omega^2 \mathbf{r} = \mathbf{M}_{\text{rot}} \mathbf{r} + \mathbf{D} \mathbf{M}_{\text{mag}} \mathbf{D}^T \mathbf{r}, \quad (2.48)$$

where we have

$$\mathbf{M}_{\text{rot}} = \frac{\text{diag} \left(\langle \delta\mathcal{F}_{\text{rot}}(\xi_{\text{rot}}^{(m)}), \xi_{\text{rot}}^{(m)} \rangle \text{ for } m = -l, \dots, l \right)}{\langle \xi_0, \xi_0 \rangle}, \quad (2.49)$$

$$\mathbf{M}_{\text{mag}} = \frac{\text{diag} \left(\langle \delta\mathcal{F}_{\text{mag}}(\xi_{\text{mag}}^{(m)}), \xi_{\text{mag}}^{(m)} \rangle \text{ for } m = -l, \dots, l \right)}{\langle \xi_0, \xi_0 \rangle}. \quad (2.50)$$

There are a few things to note about this result. Firstly, as we have expressions for the perturbation operators and the Wigner d -matrix, Eq. (2.48) can be used to calculate the perturbation of the eigenvalue. Using the property $\delta\omega^2 = 2\omega_0 \delta\omega$, we can also find the perturbation of the eigenfrequency. Secondly, for the case where the magnetic field is aligned with the rotation axis, the transformation matrix $\mathbf{D}$ is the identity matrix, and the combination of the perturbation matrices is straightforward as they share the same basis. Lastly, the matrices are diagonal and of size $2l + 1$, which means that we can have up to $2l + 1$ different eigenfrequencies for a given l and n . These eigenfrequencies are, however, expressed in the corotating frame of the star. If the magnetic field is misaligned with the rotation axis, the resulting eigenfunctions will consist of multiple m -components and as such, transforming the eigenfrequencies to the inertial frame can lead to $(2l + 1)^2$

different eigenfrequencies. Even though $\mathbf{M}_{\text{rot}}$ and $\mathbf{M}_{\text{mag}}$ are diagonal by definition, the mixing of the modes occurs through the inclusion of the Wigner d -matrix. As such, when the rotational axis is the same as the magnetic axis, we expect only $2l + 1$ different eigenfrequencies for a given l and n .

Figure 2.6 shows the effect of the combined perturbation on the eigenfrequencies of a dipole mode ($l = 1$) for misaligned rotation and magnetic field axes. We obtain 9 different frequencies divided in 3 groups of 3 frequencies, which are the result of the m -components of the eigenfunctions. Each of these coloured groups forms a fixed solution such that relative amplitudes within a solution must hold. Each group of frequencies that forms a solution is as such expected to always be observed together, as they are the components of the same eigenfunction. The relative amplitudes between the different solutions are not fixed, and depend on the driving of the different modes in the star.

We can classify the frequencies by their azimuthal orders as given by the rotational splitting m_{rot} and the magnetic splitting m_{mag} . Each frequency is identified by its unique pair of azimuthal orders $(m_{\text{rot}}, m_{\text{mag}})$. The frequencies which form an independent solution are given by the same magnetic azimuthal order m_{mag} , but different rotational azimuthal orders m_{rot} .

In Fig. 2.6, we see that the rotational splitting is larger than the magnetic splitting, leading to clear groups split by the rotational azimuthal order m_{rot} and subdivided by the magnetic azimuthal order m_{mag} . For pure rotational splitting, not every frequency is guaranteed to be present for each azimuthal order as they are independent solutions. For the combined effect of rotation and magnetic fields, we see that the rotational azimuthal orders m_{rot} of a solution are not independent. This means that in contrast to the pure rotational splitting, there must be a frequency present for each rotational azimuthal order if an independent solution is present.

2.4 Validity regime of the theory

Within the framework presented above, both the Coriolis force and the Lorentz force are considered perturbatively compared to the standard oscillations from pressure and buoyancy. We can make this statement more concrete by considering relevant quantities to these effects in terms of their frequencies. The standard oscillations are given by their unperturbed frequency ω_0 , which will serve as our reference frequency. The natural choice of frequency to express the rotation, is the rotational frequency Ω . As discussed above, we neglect terms on the order of Ω^2 in the derivation of the rotation operator, which means that we require $\Omega \ll \omega_0$ for this assumption to hold. Similarly, a natural frequency expressing the dynamical effect of the magnetic field is the Alfvén frequency ω_A , which is the frequency of a wave obtained by perturbing the magnetic field in a plasma. This frequency is a good measure of the dynamical effects of the magnetic field strength, and we require $\omega_A \ll \omega_0$ for the perturbation theory to hold (Loi, 2021).

Additionally, we made assumptions about the geometry of the system, where we consider the stars to be perfect spheres for all perturbations. While the perturbations break the spherical symmetry, we require them to not break the sphere. This assumption continues to hold for slow rotation and weak magnetic fields.

Lastly, while we did not assume a specific form for the magnetic field, we did make assumptions about the form of the rotation. We will only consider uniform global rotation. An internal magnetic field is expected to efficiently transport angular momentum. In addition, [Mombarg et al. \(2024\)](#) and [Vanlaer et al. \(subm.\)](#) found that the target star of this thesis, HD 192575, has only weak differential rotation, which justifies the assumption of uniform rotation.

2.5 Main takeaways

This chapter discussed the mathematical formalism of stellar oscillations and the effect of rotation and magnetic fields thereupon. We derived the oscillation equations by perturbing equilibrium stellar equations and found that the oscillations can be separated into a radial and horizontal component, the latter of which is expressed by spherical harmonics. Standard oscillation theory only takes the pressure force and buoyancy into account, which leads to a degeneracy in azimuthal order. This degeneracy can be lifted by including effects that break the spherical symmetry of a star, such as rotation and internal magnetic fields. We have derived expressions that include the effects of rotation and internal magnetic fields separately, and have shown how to combine these effects in a single operator, without assuming a specific form for the magnetic field or the angle between the rotation and magnetic field axis. We do not consider differential rotation in this framework and assume the star to rotate as a solid body. The combined effect of rotation and magnetic fields is expressed by Eq. (2.48), which is the central equation to be solved in this thesis. This equation leads to a set of $(2l + 1)^2$ frequencies for a given l and n , which can be split in $2l + 1$ independent solutions. Each frequency is identifiable by its rotational azimuthal order m_{rot} and magnetic azimuthal order m_{mag} , where the magnetic azimuthal order gives the different independent solutions.

Chapter 3

Magnetic field

The internal magnetic fields of stars are an important but elusive component of stellar theory. They lead to additional transport mechanisms, which can heavily influence the stellar interiors (Mestel and Weiss, 1987, Spruit, 2002, Mathis and Zahn, 2005, Fuller et al., 2019, Takahashi and Langer, 2021). However, they are still shrouded in mystery, as only indirect observations of the internal field topologies are possible, with asteroseismology as a powerful tool to do so (Gough and Thompson, 1990, Goode and Thompson, 1992, Takata and Shibahashi, 1994, Bugnet et al., 2021, Bugnet, 2022, Dhouib et al., 2022, Mathis and Bugnet, 2023). To compute the effects of a magnetic field on the stellar oscillations using the framework of Chapter 2, we are required to select a specific field topology, based on the available physical insights. Firstly, the magnetic field needs to satisfy the Maxwell equations inside a self-gravitating, compressible sphere. Secondly, the field needs to be stable over evolutionary timescales. As purely toroidal and purely poloidal fields are unstable our best option is to require a mixed field topology. Additionally, we will require the field to be axisymmetric with independence in the azimuthal angle (ϕ) with respect to the magnetic field axis. We assume the magnetic field to be a *fossil field*, meaning it is a remnant from a magnetic field in the interstellar medium that was captured during the star formation process. As such, we will only consider the magnetic field in the radiative envelope of our target massive star, which is also the region where our oscillations are expected to probe the structure of the star. This chapter presents the model of the analytical magnetic field we will use in our numerical simulations, mainly based on the work presented by Duez and Mathis (2010). They present an analytical description of a magnetic field that adheres to the above conditions and is an extension of the work by Ferraro (1954), Mestel (1956), Prendergast (1956), and Woltjer (1960).

3.1 Theoretical description

The analytical description of the magnetic field as initially derived by Duez and Mathis (2010) requires two quantities to be defined: a scalar function $\Psi(r)$ and a parameter λ . The scalar function Ψ describes the radial flux of the magnetic field and the parameter λ is an inverse length that describes the length scale of the radial flux. The radial flux function Ψ is confined by the radiative envelope of the star between $R_{\min}$ and $R_{\max}$. Given

these two quantities, the magnetic field for $r \in [R_{\min}, R_{\max}]$ is given by

$$\mathbf{B} = \frac{1}{r \sin \theta} \nabla \Psi(r, \theta) \times \hat{\mathbf{e}}_\phi + \frac{1}{r \sin \theta} \Psi(r, \theta) \hat{\mathbf{e}}_\phi \quad (3.1)$$

$$= \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} \hat{\mathbf{e}}_r - \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\lambda}{R_{\max}} \frac{\Psi}{r \sin \theta} \hat{\mathbf{e}}_\phi, \quad (3.2)$$

where $\hat{\mathbf{e}}_i$ denotes the relevant unit vectors in spherical coordinates. Given this formulation, the magnetic field will be axisymmetric. We performed its calculation and indeed found it to be divergence free.

The radial flux function Ψ is constructed as

$$\begin{aligned} \Psi(r, \theta) = \sin^2 \theta \times & \left[K \lambda \frac{r}{R_{\max}} j_1 \left(\lambda \frac{r}{R_{\max}} \right) \right. \\ & - \beta_0 \mu_0 \lambda \frac{r}{R_{\max}} j_1 \left(\lambda \frac{r}{R_{\max}} \right) \int_r^{R_{\max}} y_1 \left(\lambda \frac{\xi}{R_{\max}} \right) \rho(\xi) \xi^3 d\xi \\ & \left. - \beta_0 \mu_0 \lambda \frac{r}{R_{\max}} y_1 \left(\lambda \frac{r}{R_{\max}} \right) \int_{R_{\min}}^r j_1 \left(\lambda \frac{\xi}{R_{\max}} \right) \rho(\xi) \xi^3 d\xi \right], \end{aligned}$$

where K is a constant needed to satisfy the boundary conditions, j_1 (y_1) is the first order spherical Bessel function of the first (second) kind, ρ is the density profile of the star, μ_0 is the vacuum permeability, and β_0 is a scaling constant to get the required magnetic field strength.

The two free parameters K and λ are determined by imposing appropriate boundary conditions on the radial flux function Ψ . As we consider the magnetic field within the radiative zone of the star between $R_{\min}$ and $R_{\max}$, we require the radial flux function to vanish at these boundaries. This does not lead to a zero magnetic field strength at the boundaries as the θ -component of the magnetic field are non-zero. The boundary conditions are expressed as

$$\begin{cases} \Psi(R_{\min}, \theta) = K - \mu_0 \beta_0 \int_{R_{\min}}^{R_{\max}} y_1 \left(\lambda \frac{\xi}{R_{\max}} \right) \rho(\xi) \xi^3 d\xi = 0, \\ \Psi(R_{\max}, \theta) = K j_1(\lambda) - \mu_0 \beta_0 y_1(\lambda) \int_{R_{\min}}^{R_{\max}} j_1 \left(\lambda \frac{\xi}{R_{\max}} \right) \rho(\xi) \xi^3 d\xi = 0. \end{cases} \quad (3.3)$$

Combining the above conditions, we can reduce the expressions to a prescription for determining the inverse length scale λ

$$j_1(\lambda) \int_{R_{\min}}^{R_{\max}} y_1 \left(\lambda \frac{\xi}{R_{\max}} \right) \rho(\xi) \xi^3 d\xi - y_1(\lambda) \int_{R_{\min}}^{R_{\max}} j_1 \left(\lambda \frac{\xi}{R_{\max}} \right) \rho(\xi) \xi^3 d\xi = 0, \quad (3.4)$$

after which K can be determined from the second equation in Eq. (3.3). Equation (3.4) will in general have multiple solutions λ_i for a given density profile. We call those solutions the different *orders of the magnetic field*. As the spherical Bessel functions show oscillatory behaviour, we expect the magnetic field to oscillate in the radial direction with increasing oscillation for higher orders.

3.2 Characteristics of the magnetic field

In the introduction of this chapter, we discussed properties and characteristics we require to hold for the magnetic field. The symmetry for the azimuthal angle ϕ is clear from construction of the radial flux function Ψ . Similarly, the magnetic field we consider indeed contains both a toroidal and a poloidal component. The construction in Eq. (3.1) suggests that the first term is the poloidal component and that the ϕ component is the toroidal field. The curl of poloidal fields are tangential to spheres around the origin, and one can show that

$$\hat{\mathbf{e}}_r \cdot \left[\nabla \times \left(\frac{1}{r \sin \theta} \nabla \Psi \times \hat{\mathbf{e}}_\phi \right) \right] = 0. \quad (3.5)$$

This confirms that the first term is indeed a poloidal field. Similarly, the toroidal field is tangential to spheres around the origin

$$\hat{\mathbf{e}}_r \cdot \left(\frac{1}{r \sin \theta} \Psi \hat{\mathbf{e}}_\phi \right) = 0. \quad (3.6)$$

Figure 3.1 shows the first three orders of the magnetic field model for the density profile of a polytropic model with polytropic index $n = 4.2$. As the magnetic field can be scaled to any required strength, we will consider the maximal field strength to be $|B| = 0.04 \text{ MG}$. The figure shows that the magnetic field is indeed a mixed field with both poloidal and toroidal components. In addition, the increasing oscillatory behaviour of the magnetic field with increasing order is visible, where higher order fields lead to the field topology consisting of multiple twisted tori. The lowest order configuration shows a simple dipole-like field with two poles along the magnetic axis, without containing a singularity. This is clear from the magnetic field strength going to zero in the core. The higher order configurations do show the magnetic field strength being more concentrated in the near-core region. Notably, the magnetic field strength does not go to zero at the upper boundary of the field. This is a consequence of the boundary conditions ensuring the radial flux to vanish, but not its derivatives.

For the field to be a ‘fossil’ magnetic field, it is required that the magnetic field vanishes at the boundaries of the radiative envelope. Figure 3.2 shows that we can limit the extent of the magnetic field, where we limit it between $R_{\min} = 0.2 R_{\text{star}}$ and $R_{\max} = 0.8 R_{\text{star}}$. We see that the toroidal component vanishes smoothly, similar to the radial component. However, the poloidal component does not vanish smoothly at the boundaries.

3.3 Oscillation sensitivity to magnetic fields

Different types of oscillations probe different layers of the stellar interiors. A good rule of thumb is that the p -modes probe the outer envelope, while the g -modes probe the near-core regions. As such, we expect the oscillations to be sensitive to the magnetic field in different ways, depending on both the wave structure and the field topology. We can express the sensitivity of the oscillations by means of a *sensitivity kernel* $\mathbf{K}$, which is obtained by separating the dependencies on the magnetic field from the perturbation vector $\boldsymbol{\xi}$ in Eq. (2.32):

$$\langle \delta \mathcal{F}(\boldsymbol{\xi}), \boldsymbol{\xi} \rangle = \int_V f(\mathbf{B}) \mathbf{K}(\boldsymbol{\xi}) dV, \quad (3.7)$$

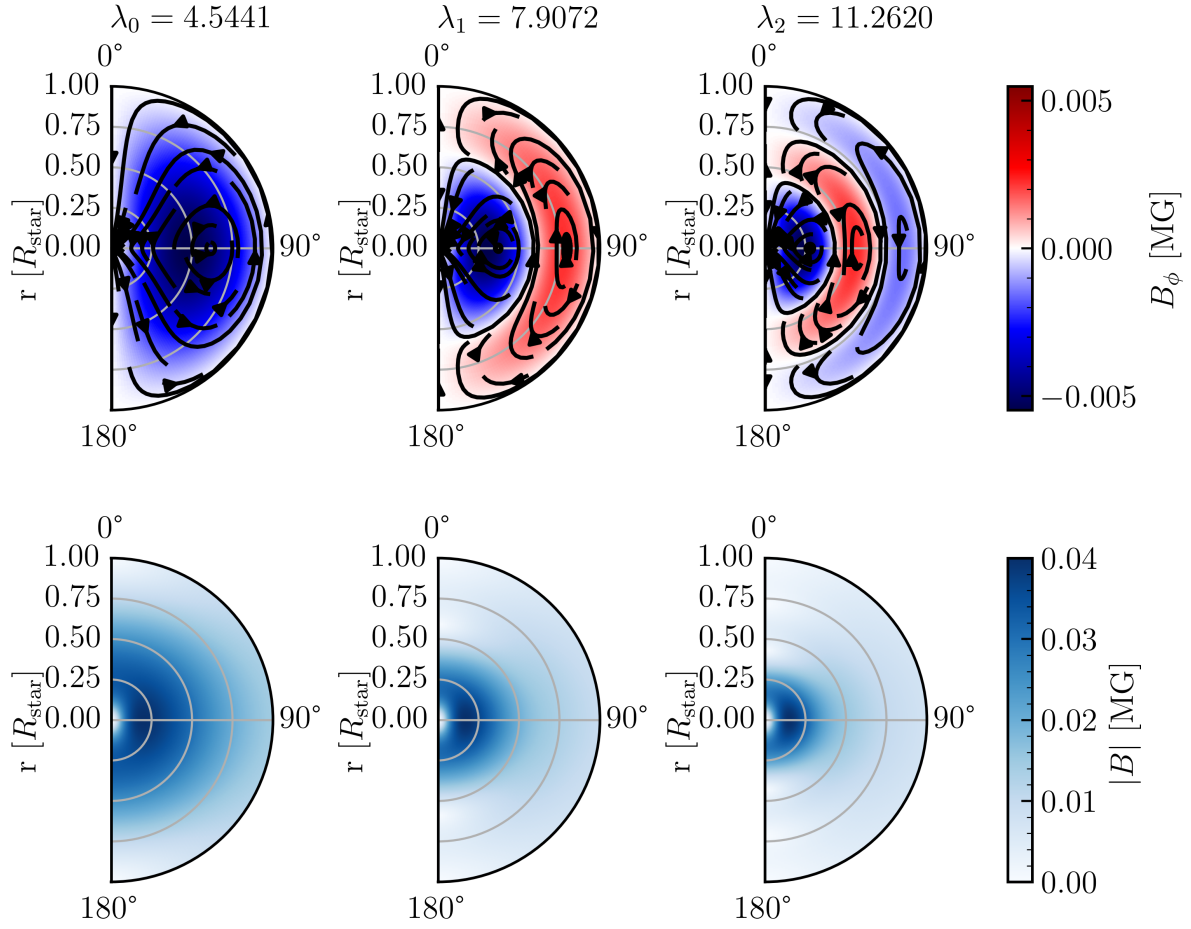


Figure 3.1: First three orders of the magnetic field model of a polytrope with index $n = 4.2$. *Upper*: the magnetic field components: B_r and B_θ (poloidal) are visualised as magnetic field lines while B_ϕ (toroidal) is shown as a colour map. *Lower*: colour maps visualising the total magnetic field strength.

where $f(\mathbf{B})$ contains the magnetic field dependencies and $\mathbf{K}$ is the sensitivity kernel, solely depending on the perturbation vector $\boldsymbol{\xi}$. Das et al. (2020) determined a general expression for the magnetic sensitivity kernel of two modes $\boldsymbol{\xi}_k$ and $\boldsymbol{\xi}_{k'}$ using the magnetic stress tensor $\mathbf{B}\mathbf{B}$

$$\begin{aligned}
 \langle \delta \mathcal{F}(\boldsymbol{\xi}_k), \boldsymbol{\xi}_{k'} \rangle = \int_V \mathbf{B}\mathbf{B} : & \left[\frac{1}{2} \left(\nabla \boldsymbol{\xi}_k \cdot (\nabla \boldsymbol{\xi}_{k'}^*)^\top + \nabla \boldsymbol{\xi}_k^* \cdot (\nabla \boldsymbol{\xi}_{k'})^\top \right) \right. \\
 & + \frac{1}{2} (\nabla \boldsymbol{\xi}_k \cdot \nabla \boldsymbol{\xi}_{k'}^* + \nabla \boldsymbol{\xi}_k^* \cdot \nabla \boldsymbol{\xi}_{k'}) + \mathbf{I} \nabla \cdot \boldsymbol{\xi}_{k'}^* \nabla \cdot \boldsymbol{\xi}_k \\
 & + \frac{1}{2} (\boldsymbol{\xi}_k^* \cdot \nabla \nabla \boldsymbol{\xi}_{k'} + \boldsymbol{\xi}_k \cdot \nabla \nabla \boldsymbol{\xi}_{k'}^*) \\
 & - \frac{1}{2} (\boldsymbol{\xi}_k^* \nabla \nabla \cdot \boldsymbol{\xi}_{k'} + \boldsymbol{\xi}_k \nabla \nabla \cdot \boldsymbol{\xi}_{k'}^*) \\
 & \left. - \frac{3}{2} (\nabla \cdot \boldsymbol{\xi}_k^* \nabla \boldsymbol{\xi}_{k'} + \nabla \cdot \boldsymbol{\xi}_k \nabla \cdot \boldsymbol{\xi}_{k'}^*) \right] dV,
 \end{aligned} \tag{3.8}$$

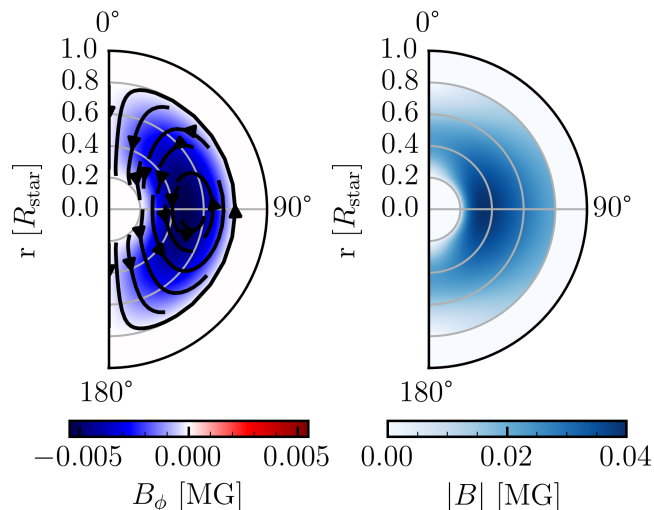


Figure 3.2: Magnetic field components (left) and magnetic field strength (right) for the first order of the magnetic field model of a polytrope with $n = 4.2$ with $R_{\min} = 0.2 R_{\text{star}}$ and $R_{\max} = 0.8 R_{\text{star}}$.

where the double dot designates the tensor product over both indices such that the integration occurs over a scalar field. This formulation indeed splits the components of the magnetic field from the perturbation vector for two modes ξ_k and $\xi_{k'}$. The current formulation does not depend on the coupling of different modes (cf. Eq. (2.50)) such that we only need to consider the case where $k = k'$ for a mode with fixed m , l , and n . The sensitivity kernel will vary strongly between different modes, while the stress tensor solely contains information about the topology of the magnetic field and does not differ between modes.

We will now investigate the expected sensitivity of the oscillations using these sensitivity kernels for a stellar model approximating the structure of the target star HD 192575. We select the best fitting stellar model for HD 192575 as determined by Burssens et al. (2023), who calculated the equilibrium structure using MESA (Modules for Experiments in Stellar Astrophysics Paxton et al., 2011, 2013, 2015, 2018, 2019, Jermyn et al., 2023). The oscillation frequencies and eigenfunctions were determined using the GYRE code (Townsend and Teitler, 2013, Townsend et al., 2018, Goldstein and Townsend, 2020).

The first important factor to consider is where the magnetic field will act on the oscillation modes, as expressed by the stress tensor. The components of stress tensor $\mathbf{B}\mathbf{B}$ contain the mixed products of the magnetic field components (i.e. $\mathbf{B}\mathbf{B}_{ij} = B_i B_j$). As these components are scalar fields defined over the interior of the star, the magnetic stress tensor will be symmetric (i.e. $\mathbf{B}\mathbf{B}_{ij} = \mathbf{B}\mathbf{B}_{ji}$), as can be seen in Fig. 3.3. These components show the expected behaviour of the magnetic field from the description in the previous sections. The radial and ϕ -components both tend to zero for the inner and outer boundary of the interior, as these depend directly on the radial flux function Ψ . The θ -component, however, does not vanish at the upper boundary as it depends on the derivative of the radial flux. All components of the stress tensor tend to increase towards the near-core region.

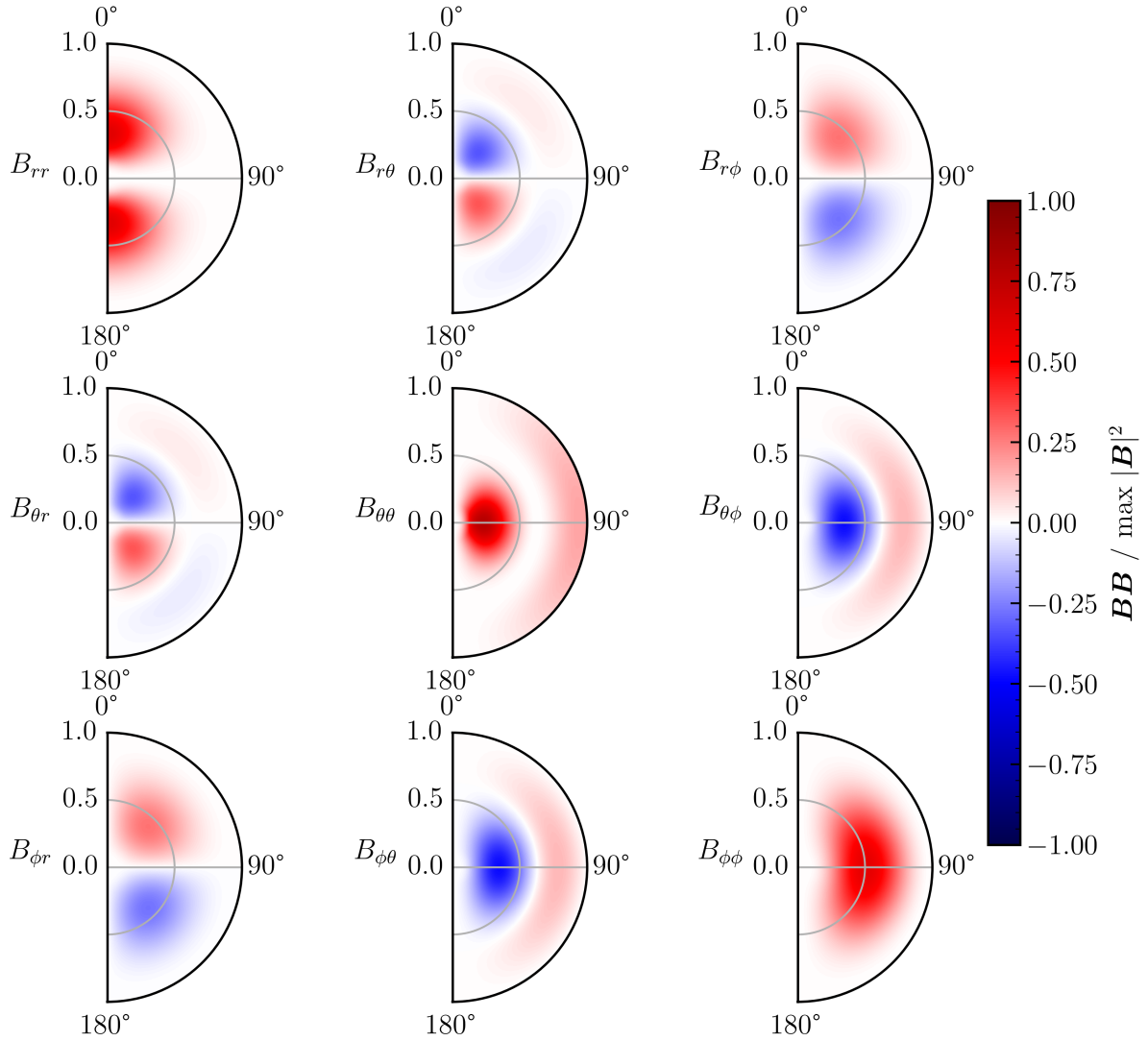


Figure 3.3: Components of the magnetic stress tensor $\mathbf{BB}$ for the magnetic field of order 0 for the best fitting stellar model of HD 192575 as determined by [Burssens et al. \(2023\)](#).

As mentioned above, the sensitivity kernels depend strongly on the mode of the oscillations. We will consider both a p -mode and a g -mode to investigate the expected sensitivity of the oscillations to the magnetic field. We select the mode $n = 2, l = 1, m = 1$ as the p -mode case study we will consider. The absolute value of the components of the magnetic sensitivity kernel are visualised in Fig. 3.4. For all components, the strongest sensitivity is situated near the surface of the star, with the sensitivity generally decreasing towards the core. The sensitivity near the surface is multiple orders of magnitude larger than the sensitivity near the core. This indicates that the p -modes will be mainly sensitive to the B_θ component of the magnetic field, with less sensitivity to the radial and ϕ components, as these are more concentrated near the core such that B_θ is the dominant component near the surface. Notably, the kernel components $K_{\phi\theta}$ and $K_{\theta\phi}$ are zero everywhere. This is a consequence of a symmetry in the expression of the kernel for the eigenfunctions, which will lead to the same behaviour being present for all modes. For $m = 0$, the kernel components $K_{\phi r}$ and $K_{r\phi}$ vanish, as these are dependent on a derivative to respect with

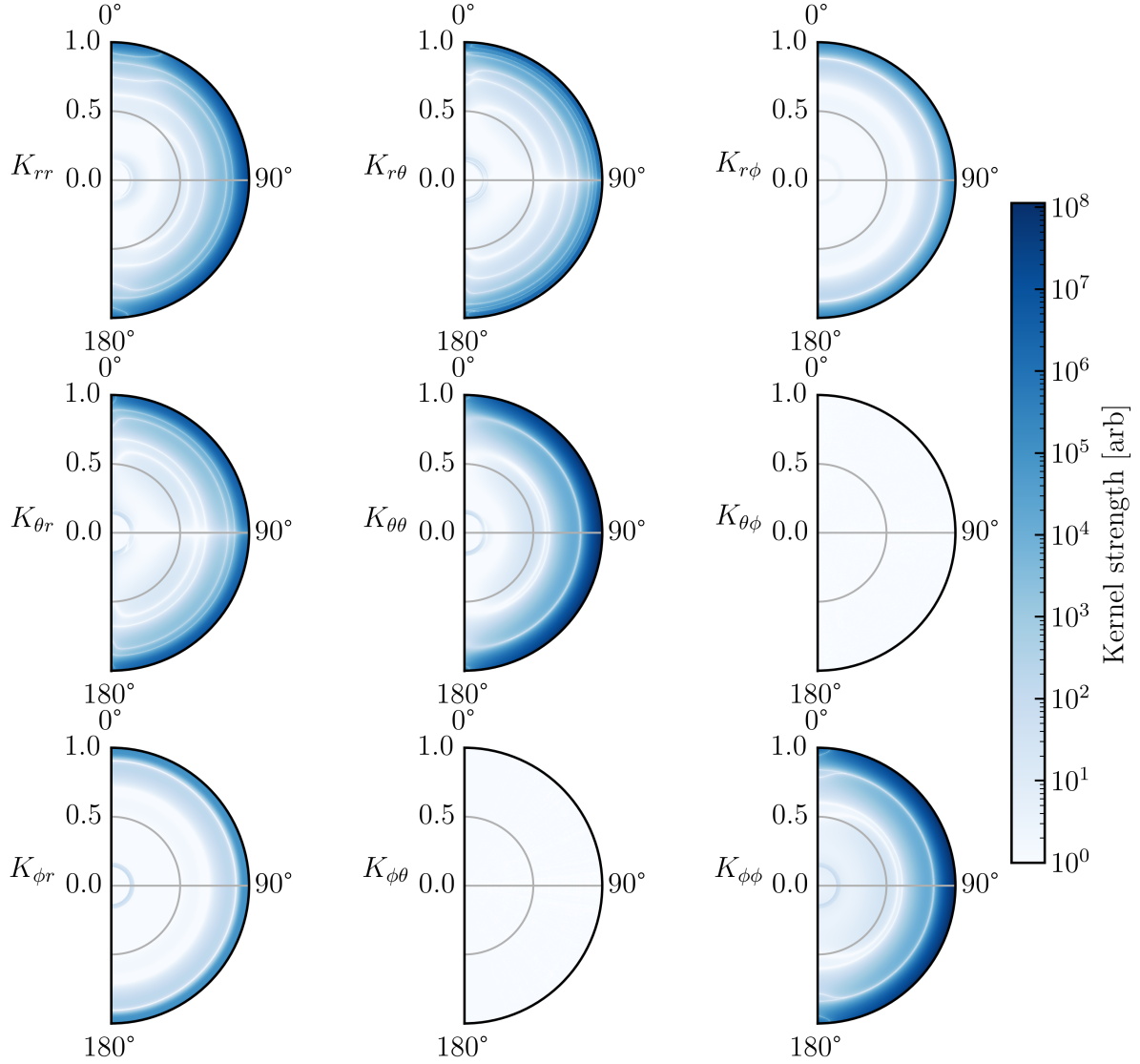


Figure 3.4: Absolute value of the components of the sensitivity kernel for the p -mode $n = 2$, $l = 1$, $m = 1$.

the azimuthal angle. For $m \neq 0$, these are non-zero, but overall weaker than the other components. The kernel for the mode $m = -1$ are identical to those visualised for $m = 1$. This behaviour is indicative of the modes with $m \neq 0$ being more sensitive to an internal magnetic field, where the behaviour is dependent on the absolute value of the azimuthal order $|m|$. All non-zero components of the sensitivity kernel also probe the near-core region, however to a much lesser extent than the near-surface region.

Figure 3.5 shows the sensitivity kernel for the g -mode $n = -2$, $l = 1$, $m = 1$. These kernels are notably different from the p -mode kernels, as the sensitivity is strongly concentrated in the near-core region, with no sensitivity near the surface. The overall sensitivity is much lower than the p -mode kernels, with the surface sensitivity of the p -modes about four orders of magnitude larger than the near-core sensitivity of the g -modes. This effect is mitigated somewhat by the topology of the magnetic field, which is stronger in the near-core region. However, as the total effect is not only dependent on the strength of

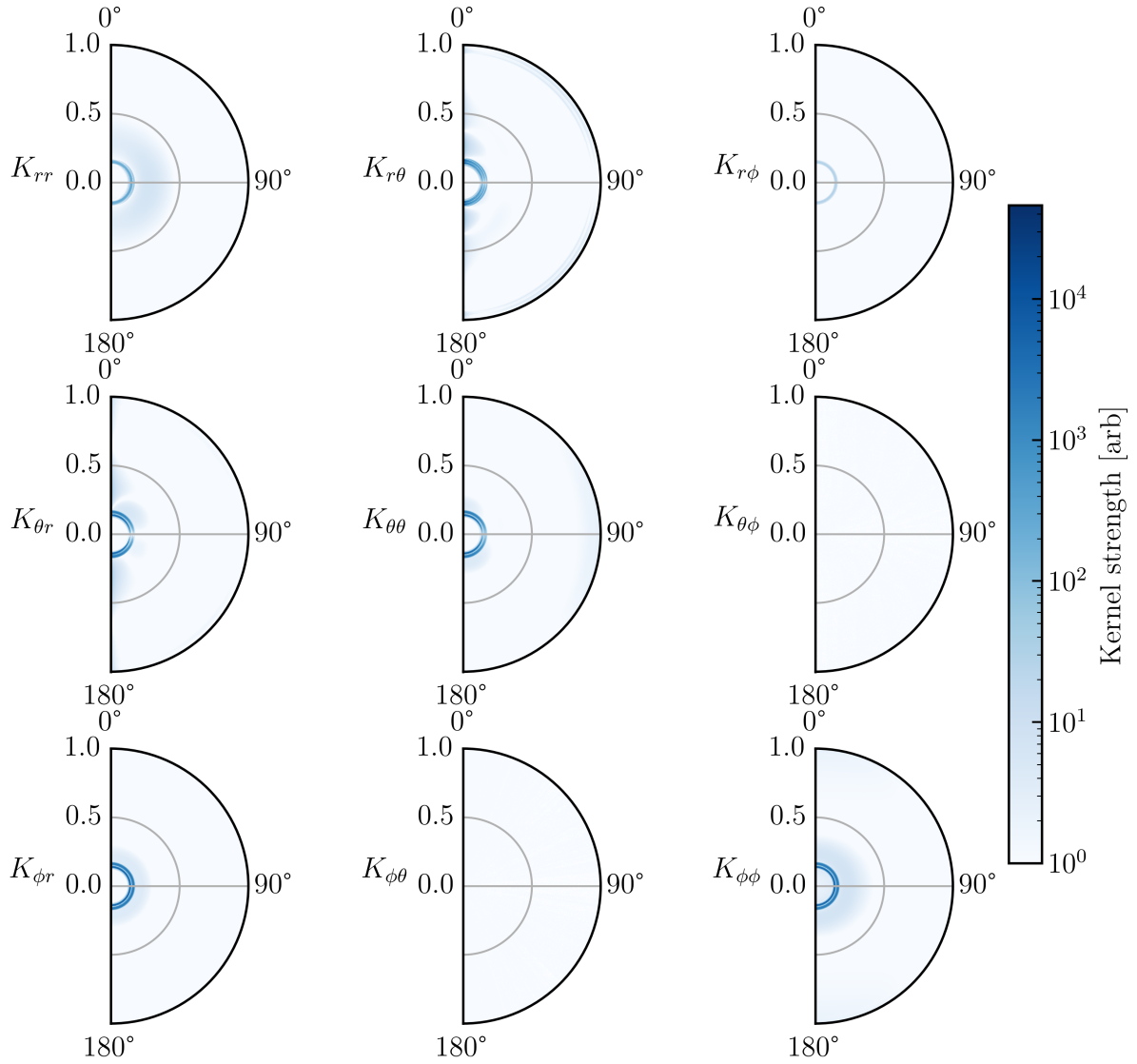


Figure 3.5: Absolute value of the components of the sensitivity kernel for the g -mode $n = -2$, $l = 1$, $m = 1$.

the kernel, but also the integration volume, we expect to see a $10^4 - 10^5$ times stronger effect on the p -modes than on the g -modes. Additionally, the g -mode kernels show the same behaviour with regards to the azimuthal order as the p -modes.

Both kernels show slight asymmetries in the polar angle for $\theta \rightarrow \pi - \theta$. This is a consequence of the numerical implementation and not a physical characteristic of the kernels. These numerical effects are to be expected given the wide range of scales in the sensitivity kernels, which spans over eight orders of magnitude.

3.4 Main takeaways

In this chapter, we described the internal magnetic field model we will use based on [Duez and Mathis \(2010\)](#). The magnetic field depends on the density profile and can be fully

described by the radial flux function Ψ and the inverse length scale λ for a given region $r \in [R_{\min}, R_{\max}]$. We investigated the sensitivity of the oscillation modes to the magnetic field using the sensitivity kernel $\mathbf{K}$, which connects the modes to the magnetic stress tensor $\mathbf{BB}$.

The main takeaways of this chapter are:

- The magnetic field model is a mixed field with both poloidal and toroidal components, which is a requirement for a stable magnetic field. The lowest order of the magnetic field solutions is dipole-like, where no singularities are present in the magnetic field. While the radial and ϕ -component tend to zero at the boundaries, the θ -component does not vanish at the upper boundary. This leaves a residual magnetic field on the stellar surface.
- The sensitivity kernels show that the p -modes are more sensitive to the magnetic field near the surface, while the g -modes are sensitive to the magnetic field in the near-core region. As the increase in sensitivity of the p -modes is much larger than the decrease in magnetic field strength, we expect to see a larger effect on the p -modes than on the g -modes for the same magnetic field of the considered topology.

Chapter 4

Asteroseismic effects of internal magnetic fields

The previous chapters have set up the theoretical background for this thesis. Using the mathematical oscillation framework of Chapter 2 and the magnetic fields described in Chapter 3, we can now investigate the effect of internal magnetic fields on the oscillation frequencies of massive stars. In this chapter, we will discuss the expected perturbations on the oscillation frequencies for a range of massive oscillators. We will start with a description of the implementation of the theoretical framework in Section 4.1. Section 4.2 will discuss the general effects of changing the rotational frequency Ω , magnetic field strength B , and obliquity angle β on the oscillation frequencies. We expand this discussion in Section 4.3 to include different stellar models of different masses and evolutionary stages. The rotational frequency in this chapter will be given as a fraction of the Kepler critical rotational frequency Ω_{crit} as defined by

$$\Omega_{\text{crit}} = \sqrt{\frac{GM_{\text{star}}}{R_{\text{star}}^3}}, \quad (4.1)$$

which is the rotational frequency at which point the centrifugal force equals the force of gravity at the surface.

4.1 Calculating the perturbed modes

The oscillation modes, perturbed by rotation and magnetic fields, are strongly dependent on the equilibrium stellar structure of a star as the oscillation frequencies, displacements, and the topology of the magnetic field are directly dependent on the stellar structure. We used stellar models calculated by PhD student M. Vanrespaille using the stellar structure and evolution code MESA (Paxton et al., 2011, 2013, 2015, 2018, 2019, Jermyn et al., 2023) version r24.08.1. A few deviations from the default settings were implemented. Convection was approximated using the mixing length theory (MLT) prescription following Henyey et al. (1965) with a mixing length parameter of $\alpha_{\text{MLT}} = 2.0$. The convective boundaries were determined following the Ledoux criterion (Ledoux, 1947). The convective boundary mixing (CBM) was calculated using an exponential prescription with an

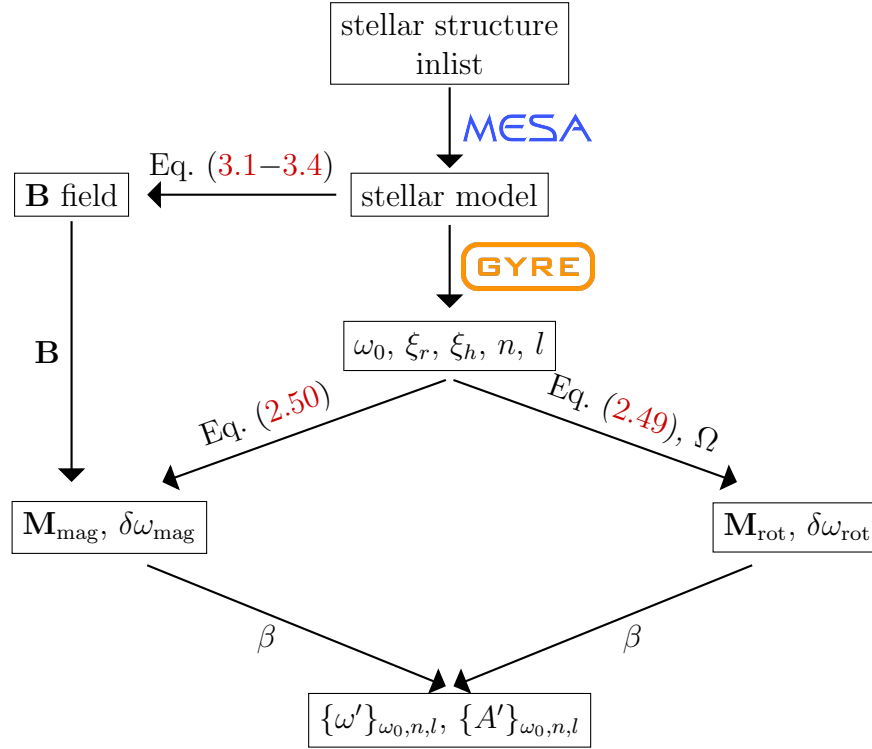


Figure 4.1: Flowchart showing the steps from stellar structure inlist to predicted stellar oscillation frequencies perturbed by both rotation and an internal magnetic field.

overshoot parameter of $f_{\text{CBM}} = 0.02$. Chemical mixing induced by internal gravity waves was allowed following the results of Rogers and McElwaine (2017) with a mixing exponent $\mathcal{A} = -1$. The mass loss was calculated using the prescription of Vink et al. (2001). Lastly, the stellar atmosphere was taken into account using an Eddington gray atmosphere.

The stellar structure models were then used to calculate the stellar oscillation modes with GYRE (Townsend and Teitler, 2013, Townsend et al., 2018, Goldstein and Townsend, 2020) using vacuum boundary conditions. This calculation delivers the unperturbed frequencies ω_0 and the radial and horizontal displacements ξ_r and ξ_h of the oscillation modes for each chosen radial order n and angular degree l .

The above calculations of the stellar structure and oscillation modes deliver the necessary input for the calculations described in Chapter 2 and Chapter 3. The density profile of the star is used to calculate the magnetic field extent and topology following Eq. (3.1–3.4). The magnetic field will be contained within the radiative envelope of the star throughout this thesis. This magnetic field is a required input to calculate the magnetic perturbation on the oscillation modes following Eq. (2.50). The rotational perturbation is calculated by selecting a rotational frequency Ω and calculating the rotational perturbation following Eq. (2.49). Both can be combined after accounting for the obliquity angle β between the rotational and magnetic field axes. This delivers a set of $(2l + 1)^2$ perturbed frequencies $\{\omega'\}$ and amplitudes $\{A'\}$ per unperturbed mode $\omega_0(n, l)$. An overview of the steps in the calculation is given in Fig. 4.1.

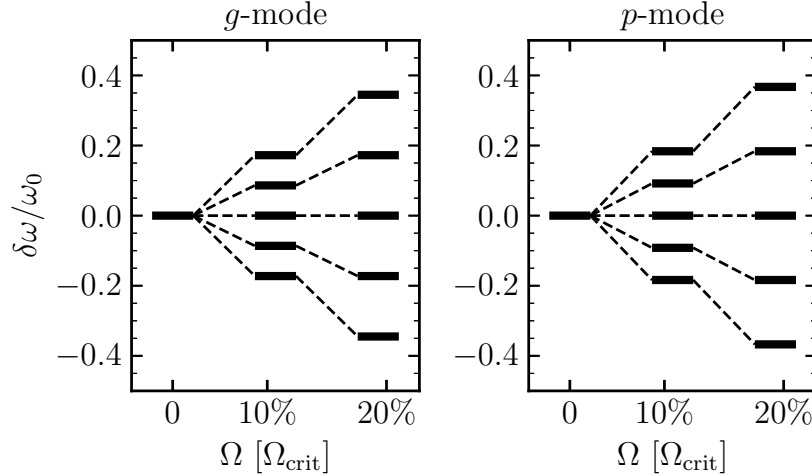


Figure 4.2: Rotational splitting of a g -mode ($n = -3, l = 2$) and a p -mode ($n = 3, l = 2$) for no rotation, 10% of the critical rotation velocity, and 20% of the critical rotation velocity.

4.2 Parameter study

In this section, we will investigate the effect of the different parameters on the oscillation frequencies of a star. The parameters of relevance are the rotational frequency (Ω), the magnetic field strength ($B_{\max}$), and the obliquity angle (β) between the rotational and magnetic field axes. We select a single stellar model of a β Cep star to investigate the effects of these parameters. The stellar model has a mass of $15 M_{\odot}$ and a radius of $7.5 R_{\odot}$ for a core hydrogen mass fraction of $X_c = 0.35$. The convective core has a mass of $4.5 M_{\odot}$ with a radius of $0.16 R_{\odot}$. The magnetic field will be contained within the radiative envelope of the star.

The first parameter we will investigate is the rotational velocity of the star. As described above, we only consider solid body rotation of the star and incorporate the rotational effects to the first order in the oscillation equations. The largest influence on the oscillation frequencies will be the rotational splitting as a consequence of the different inertial frame of the observer to the rotating star. The rotational splitting of the modes can as such give an estimate of the rotational frequency of the star, independent of the inclination angle with the observer. This is in strong contrast to the rotational velocity obtained with spectroscopic measurements, where the Doppler line broadening leads to a $v \sin i$ estimate, which stands for the overall velocity projected on the inclination angle. Usually, this is interpreted as the projected rotational velocity at the equator, ignoring other velocities (Aerts et al., 2014).

Figure 4.2 shows the rotational splittings of a quadrupole ($l = 2$) g -mode and p -mode of radial orders $|n| = 3$ for three different values of the rotational frequency (0%, 10%, and 20% Ω_{crit}). The splitting $\delta\omega$ is calculated relative to the unperturbed mode frequency ω_0 as

$$\delta\omega = \omega - \omega_0. \quad (4.2)$$

Eq. (2.40) gives the expected rotational splitting of the frequencies from our theoretical

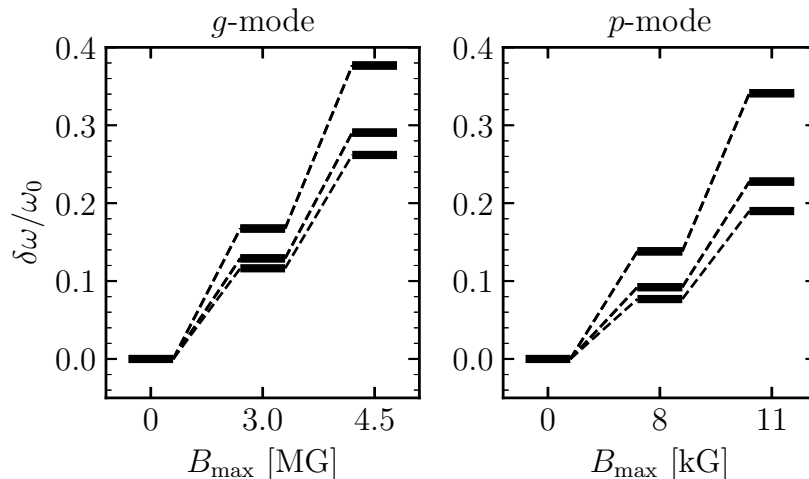


Figure 4.3: Magnetic splitting of a g -mode ($n = -3$, $l = 2$) and a p -mode ($n = 3$, $l = 2$) for no rotation and increasing maximum field strength.

framework. The rotational splittings increase with increasing rotational frequency and both the g -mode and p -mode show a similar behaviour and sensitivity. The dependence on the azimuthal order is clearly visible with the quadrupole splitting into five frequencies, with a similar splitting between consecutive azimuthal orders. Each splitting is on the order of the rotational frequency. All of these aspects are in agreement with the expected behaviour from the theory.

The magnetic splittings are shown in Fig. 4.3 for the same g -mode and p -mode for three values of the maximum magnetic field strength. The magnetic splittings behave noticeably different from the rotational splittings. Firstly, the splittings are solely towards higher frequencies, with the magnetic field causing an increase of the frequency for all azimuthal orders. Secondly, we can distinguish only three different splittings for a quadrupole mode. This indicates that the magnetic splittings depend on the absolute value of the azimuthal order and not on its sign, as predicted by the theory. Thirdly, the p -mode shows a larger asymmetry between the splittings for a similar effect on the modes and lastly, the p -mode shows a larger sensitivity to the magnetic field strength than the g -mode. These last two points are related to the topology of the field and its connection to the probing area of the different modes. As the field strength decreases more slowly in radial coordinate than the sensitivity increases for increasing radial order, we expect a larger sensitivity to the maximum field strength for the p -mode than the g -mode.

The obliquity angle β is the angle between the rotational axis and magnetic field axis. From the theoretical description in Chapter 2, we expect $2l + 1$ frequencies when the axes coincide and $(2l + 1)^2$ frequencies when they are misaligned. The splittings of the dipole g -mode $n = -3$ are shown in Fig. 4.4 for different values of the obliquity angle β . We calculated the splittings for fixed values of rotational frequency $\Omega = 0.22 \Omega_{\text{crit}}$ and magnetic field strength $B_{\text{max}} = 3.5 \text{ MG}$. The amplitude is given such that the square of the amplitude is equal to the expected power distribution between the frequencies. For $\beta = 0$, we indeed have three frequencies with each a relative amplitude of unity. The power distribution spreads for increasing obliquity angle up to $\beta = \pi/2$, where the number

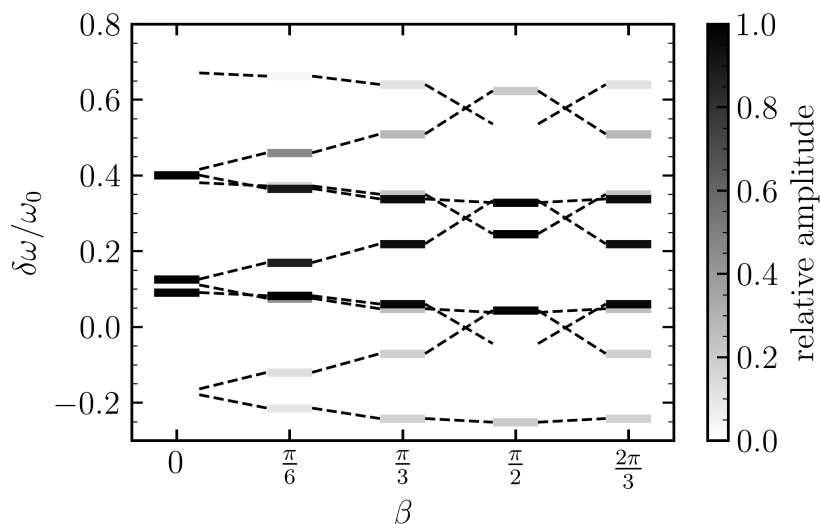


Figure 4.4: Splittings of a g -mode ($n = -3$, $l = 1$) for different values of the obliquity angle β .

of visible frequencies is reduced. This is a consequence of the geometry, where only two spherical harmonics are required to describe a mode. The final splitting for $\beta = 2\pi/3$ shows that indeed the obliquity angle is symmetric around $\pi/2$. From this figure, it is clear that the obliquity angle not only affects the amplitude of the frequencies, but also the frequencies themselves.

4.3 Model study

The previous section showed that the predicted splittings and amplitudes behave as expected from theory. We will now investigate the effect of the magnetic splitting for different stellar models. Specifically, we are interested in the required field strengths for a measurable effect on the oscillation frequencies throughout the main sequence for massive pulsators. We will limit this study to an $8 M_{\odot}$ and $15 M_{\odot}$ star at three different points during their main sequence evolution and consider only the frequency differences. The stage of the model on the main sequence will be denoted by the central hydrogen mass fraction X_c , where we will take $X_c = 0.7$ near the zero age main sequence (ZAMS), $X_c = 0.35$ near the middle age main sequence (MAMS) and $X_c = 2 \cdot 10^{-4}$ near the terminal age main sequence (TAMS).

We show the sensitivity kernels $K_{\theta\theta}(n, l)$ and magnetic field strengths for the $8 M_{\odot}$ star in Fig. 4.5 and for the $15 M_{\odot}$ star in Fig. 4.6. The sensitivity kernels are shown for a p -mode $n = 2$ and $l = 2$ (top) and a g -mode $n = -2$ and $l = 2$ (middle). The magnetic fields are confined to the radiative envelope of the star.

The changes in the magnetic field strength topologies clearly show the shrinking core throughout the main sequence evolution. In addition, the change in the stellar interior has a large impact on the sensitivity kernels, especially of the g -mode, where the sensitivity changes drastically at the TAMS. While for the first part of the main sequence evolution, the g -mode is sensitive to the near-core region, the sensitivity shifts to the outer layers

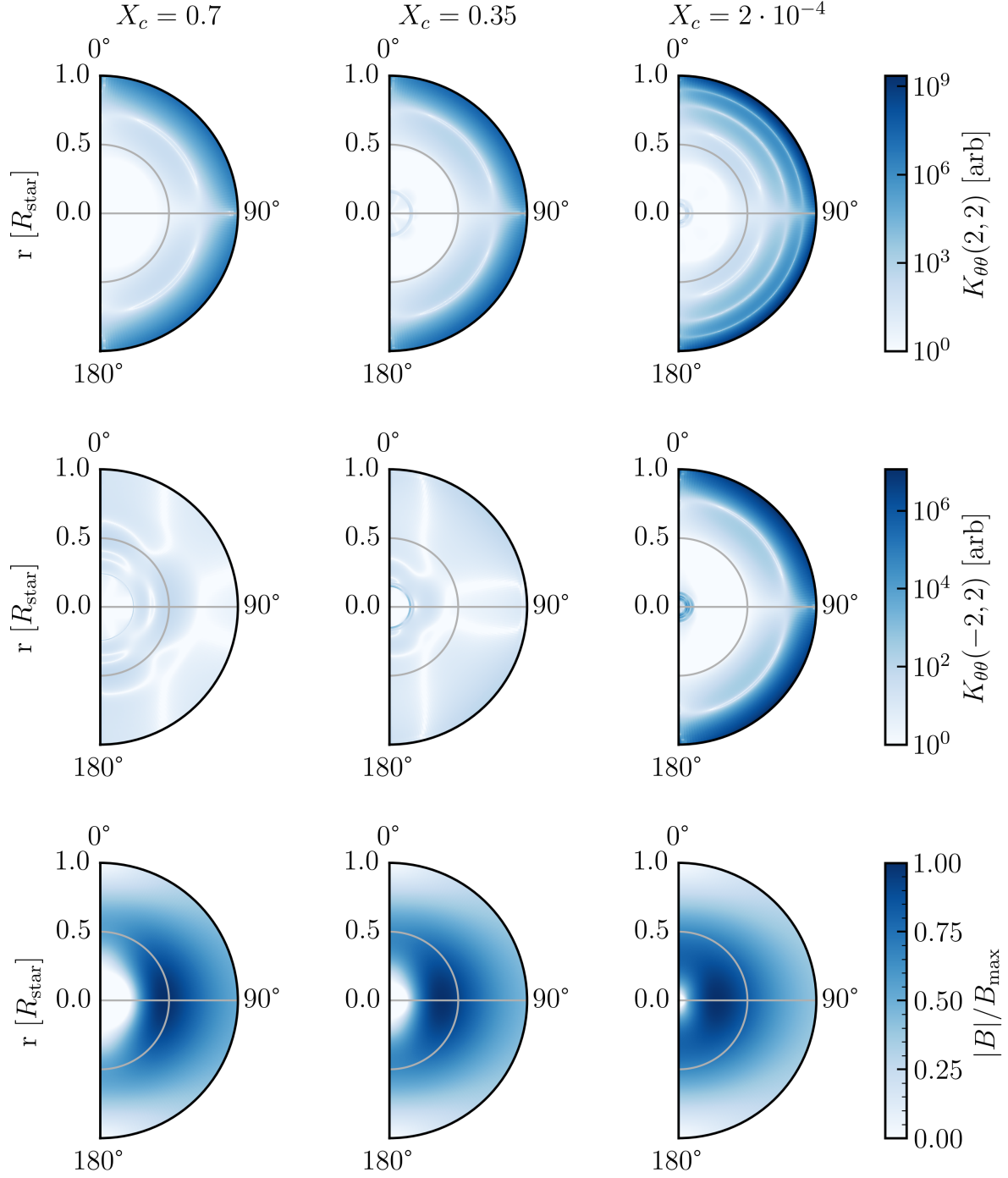


Figure 4.5: Sensitivity kernels and magnetic field strengths for three points in the main sequence of an $8 M_{\odot}$ star. The sensitivity kernels are shown for the p -mode $n = 2$ and $l = 2$ (top) and the g -mode $n = -2$ and $l = 2$ (middle). The bottom row shows the magnetic field strengths. All kernels are shown for magnetic azimuthal order $m = -1$.

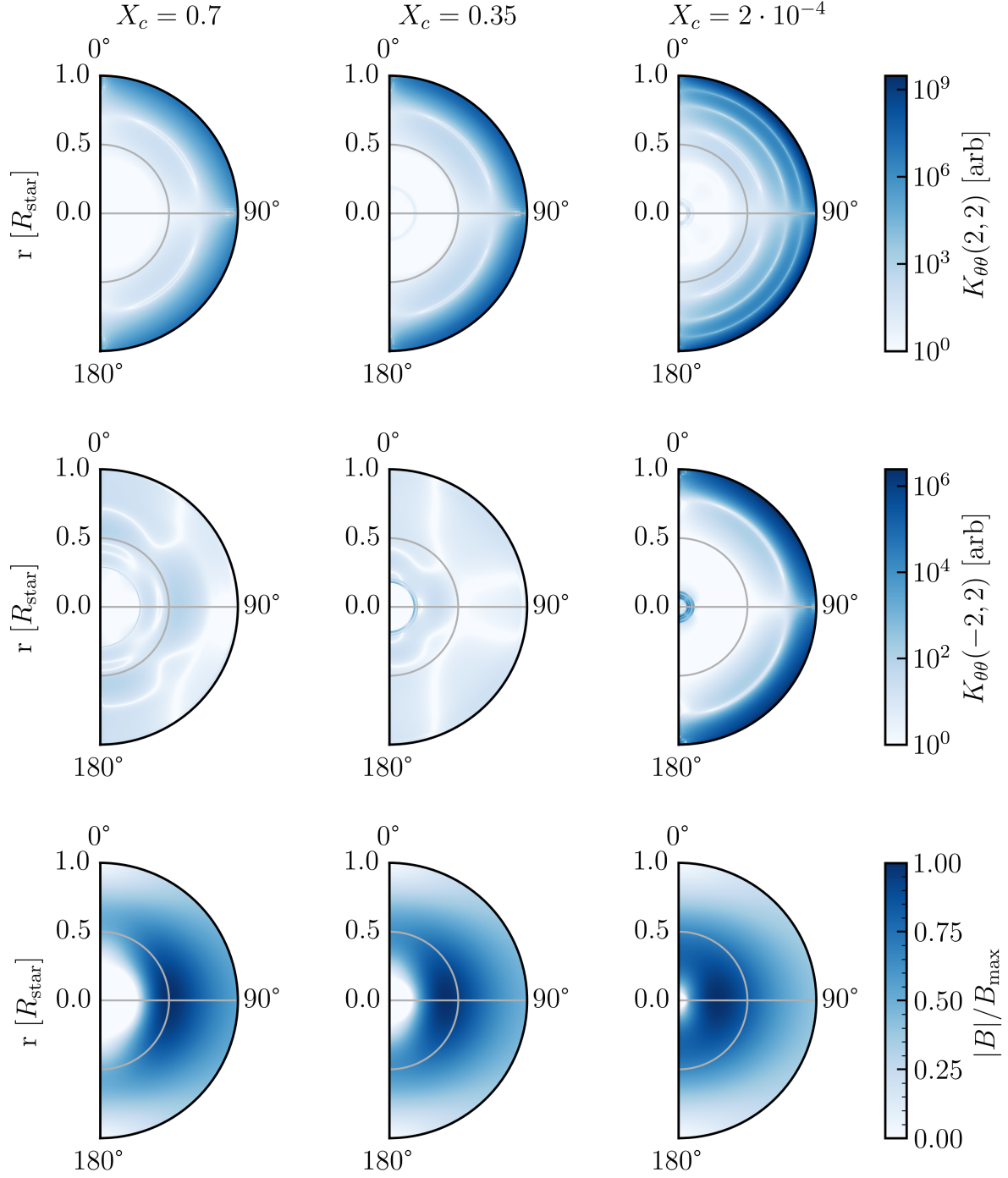


Figure 4.6: Sensitivity kernels and magnetic field strengths for three points in the main sequence of a $15 M_{\odot}$ star. The sensitivity kernels are shown for the p -mode $n = 2$ and $l = 2$ (top) and the g -mode $n = -2$ and $l = 2$ (middle). The bottom row shows the magnetic field strengths. All kernels are shown for magnetic azimuthal order $m = -1$.

of the star for the TAMS and the g -mode becomes a mixed mode. The p -mode shows very similar sensitivity throughout the main sequence, with a slight increase in sensitivity to the near-core and near-surface region. Between the different masses, we see similar behaviour at the same stages of the main sequence.

A first relation we investigate is how the overall magnetic field strength affects the mode frequencies. To measure the effect on the oscillation modes, we investigate the asymmetry $\mathcal{A}$ of the splittings, which we define for a given radial order n and angular degree l as

$$\mathcal{A}(n, l) = (\omega_{nl+l} - \omega_{nl0}) - (\omega_{nl0} - \omega_{nl-l}), \quad (4.3)$$

where we denote each frequency as ω_{nlm} , with m the azimuthal order. The asymmetries thus consider the difference in frequency spacing between the zonal mode and the most extreme azimuthal orders. This definition is slightly different than what is commonly used in the literature, but serves our purposes better as it maximizes the measurable effect of the magnetic field asymmetries. For these asymmetries, the effect of rotation cancels to the first order, making the asymmetries a good measure of the effect of the magnetic field within the current framework. We do note that while this definition of the asymmetry maximizes the measurable effect of the magnetic field in the current framework, it makes comparisons between the different angular degrees difficult.

Following the results of [Prat et al. \(2019\)](#) and [Mathis et al. \(2021\)](#), we expect the perturbations of a magnetic field to follow a strict relation with the magnetic field strength given by

$$\left(\frac{\delta\omega}{\omega_0} \right) \propto \frac{B_{\max}^2}{8\pi\omega_0^2}. \quad (4.4)$$

They predicted the perturbations of the oscillation frequencies to be proportional to the square of the magnetic field strength, which directly translates to the asymmetries. We can fit this relation between the asymmetries and the magnetic field strength to obtain a scaling factor c between the two quantities defined following

$$\frac{\mathcal{A}_{n,l}}{10^{-2} \text{ d}^{-1}} = c \left(\frac{B_{\max}}{1 \text{ kG}} \right)^2, \quad (4.5)$$

where the scaling factor c will depend on the radial order n and angular degree l of the mode. We show the scaling factors for a wide range of radial orders in [Fig. 4.7](#) for the $15 M_{\odot}$ star at MAMS in the case of a non-rotating star. The scaling factors are divided between low values ($c \sim 10^{-5}$) for the near-core sensitive modes and large values ($c \sim 10^1$) for near-surface sensitive modes. The transition between the two regimes occurs for low absolute values of the radial order $|n|$ and is very narrow, as only the quadrupole modes with $|n| \leq 1$ and the $n = 1$ dipole mode show clear mixed mode behaviour. The other stellar models show similar behaviour, with shifts in extent and position of the mixed mode regime in radial order. The figures corresponding to the other stellar models for $15 M_{\odot}$ and $8 M_{\odot}$ can be found in [Appendix C](#). The mixed mode regime contains more modes for more evolved main sequence stars.

We will now consider effects of the combined perturbations of rotation and a magnetic field. We are interested in the size of the splittings between the predicted frequencies for misaligned rotation and magnetic field. As we predict $(2l + 1)^2$ frequencies divided

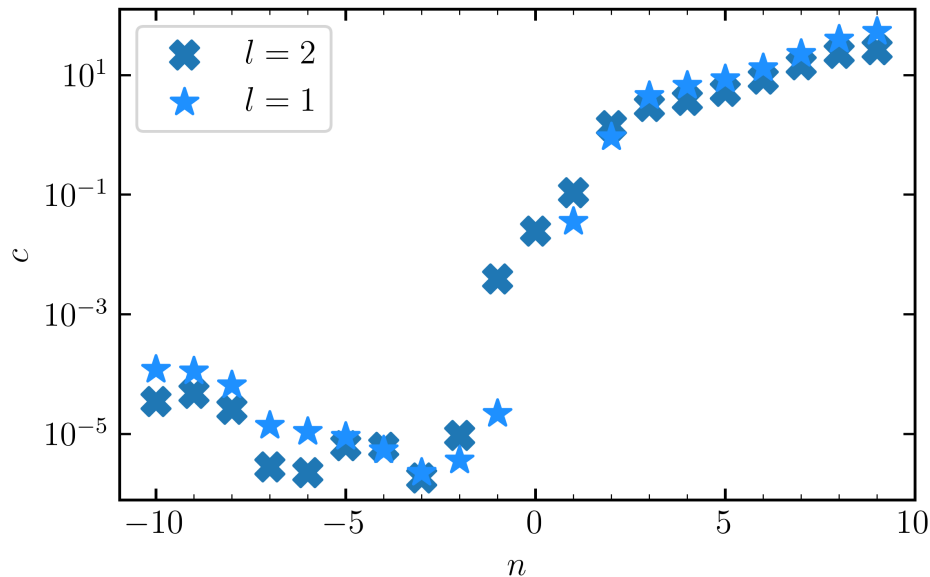


Figure 4.7: Asymmetry scaling factor c for dipole and quadrupole oscillation modes of different radial order n . The stellar model had a mass of $15 M_{\odot}$ with a central hydrogen fraction $X_c = 0.35$.

over $2l + 1$ independent solutions, we can consider multiple types of splittings between the frequencies. We define two different splittings that are of interest to us: the splittings between two consecutive frequencies within an independent solution (Δf) and the splittings between two consecutive frequencies of different solutions (δf). Following the mathematical framework outlined in Chapter 2, we can say that the splittings within a solution are for a given magnetic azimuthal order, but between consecutive rotational azimuthal orders. Similarly, the splittings between the different solutions are for a given rotational azimuthal order, but between consecutive magnetic azimuthal orders. As we will be working within the regime of larger rotational splitting, we will call the splittings within a solution (Δf) the *big splitting* and the splittings between the different solutions (δf) the *small splitting*. The different splittings are visualised in Fig. 4.8 for additional clarity.

The small splittings are constant for the same consecutive magnetic azimuthal orders but different rotational azimuthal orders. Similarly, the big splittings are constant for the same consecutive rotational azimuthal orders but different magnetic azimuthal orders. This is a consequence of the first order approximation of the rotational perturbation. In the rotation dominated regime, the large splitting will be mostly sensitive to the rotational velocity, while the small splitting will be sensitive to the magnetic field strength. We will thus focus on the small splitting for the remainder of this chapter.

Figure 4.9 shows the mean small splittings of both a low order dipole g -mode and a p -mode for a misaligned magnetic field and rotational axes for a massive star ($15 M_{\odot}$) at the MAMS. We show the mean small splitting for different strengths of the magnetic field, where we report on the magnetic field strength at the peak of the magnetic sensitivity kernel of the mode (B_{peak}). We consider three instances of the rotational frequency: no rotation ($\Omega = 0.0$), 10% of the critical rotation frequency ($\Omega = 0.1$) and 20% of the

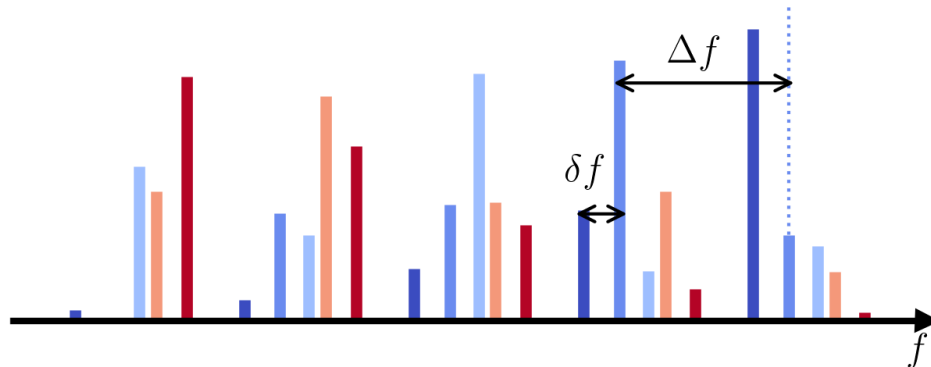


Figure 4.8: Frequency spectrum example for a misaligned magnetic field with the rotational axis. The different colours show the independent solutions. The two different splittings (δf , Δf) are indicated on the figure.

critical rotation frequency ($\Omega = 0.2$). As the small splitting gives a boundary on the frequency resolution necessary to detect the splittings, we can convert the small splitting to a required observational time base. We follow the prescription of [Loumos and Deeming \(1978\)](#) who argued that the required time base T for a given frequency resolution $\Delta\nu$ is given by

$$T = 1.5/\Delta\nu. \quad (4.6)$$

The mean small splittings in Fig. 4.9 shows the expected quadratic increase with the magnetic field strength for no rotation ($\Omega = 0.0$) and for the regime where the magnetic splitting is dominant. For the models with rotation, we see that the rotational effect on the small splitting becomes dominant at large magnetic field strengths for even relatively low rotational velocities. For the g -mode, the first order rotational effect is dominant for a kernel peak magnetic field strength of $B_{\text{peak}} \approx 300$ kG for a star rotating at 10% its critical velocity. As the p -mode is more sensitive to the magnetic field, this turning point occurs at much lower magnetic field strengths of $B_{\text{peak}} \approx 1$ kG. This difference in sensitivity is clearly visible in the required peak strengths of the magnetic field for a given small splitting, with the required field strength for the p -mode being two orders of magnitude lower than the g -mode. Long time base observations are required to detect the small splittings for both the g - and p -modes.

Figure 4.10 shows the mean small splittings for different magnetic field peak strengths for the same modes, but at different evolutionary stages on the main sequence corrected for rotation. For a given peak strength of the magnetic field, we see that the mean small splitting grows multiple orders of magnitude along the main sequence. This is a combined effect of both the changing sensitivity kernels and the change in the magnetic field with the evolution as can be seen in Fig. 4.6. The switch of the $n = -2$ g -mode from a near-core sensitive mode to a near-surface sensitive mixed mode is clearly visible in the figure, as the mode increases by almost four orders of magnitude in mean small splitting at the TAMS compared to the ZAMS and MAMS. Similar figures are available for the $8 M_{\odot}$ star in Appendix D, and show the same behaviour.

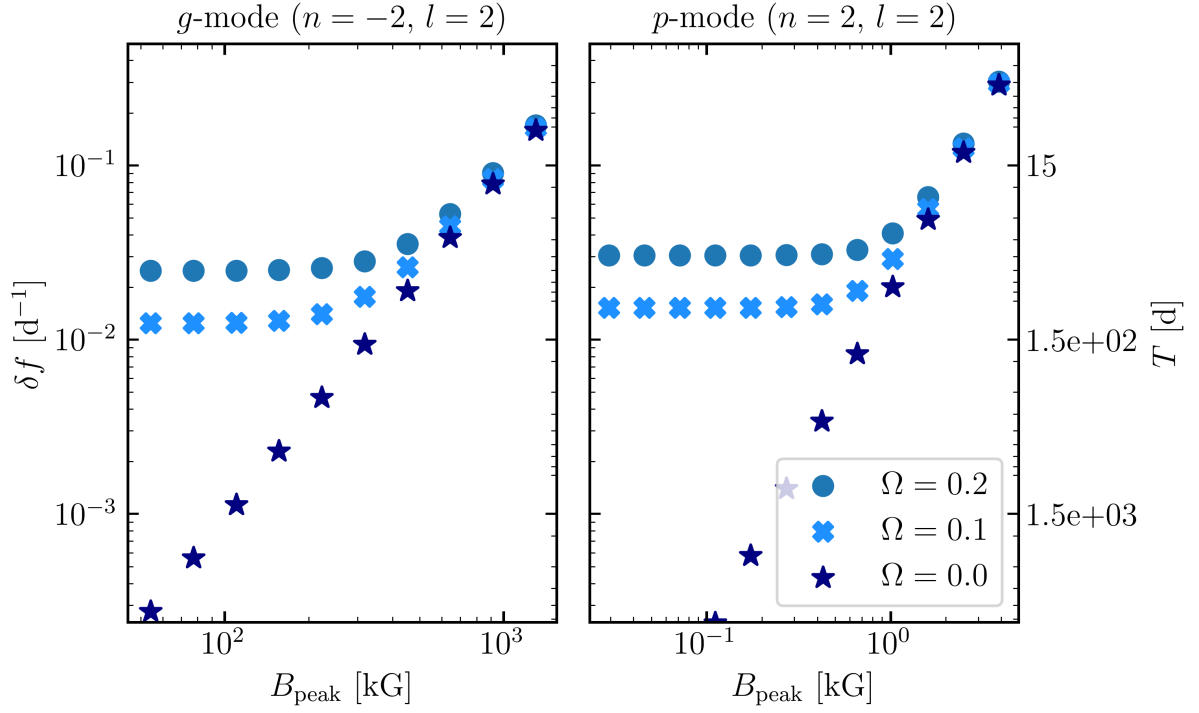


Figure 4.9: Mean small splitting in a misaligned ($\beta = 3\pi/16$) magnetic field for a g -mode ($n = -2$, $l = 2$) and a p -mode ($n = 2$, $l = 2$) of a $15 M_{\odot}$ star with a central hydrogen fraction of $X_c = 0.35$ for three different rotational frequencies.

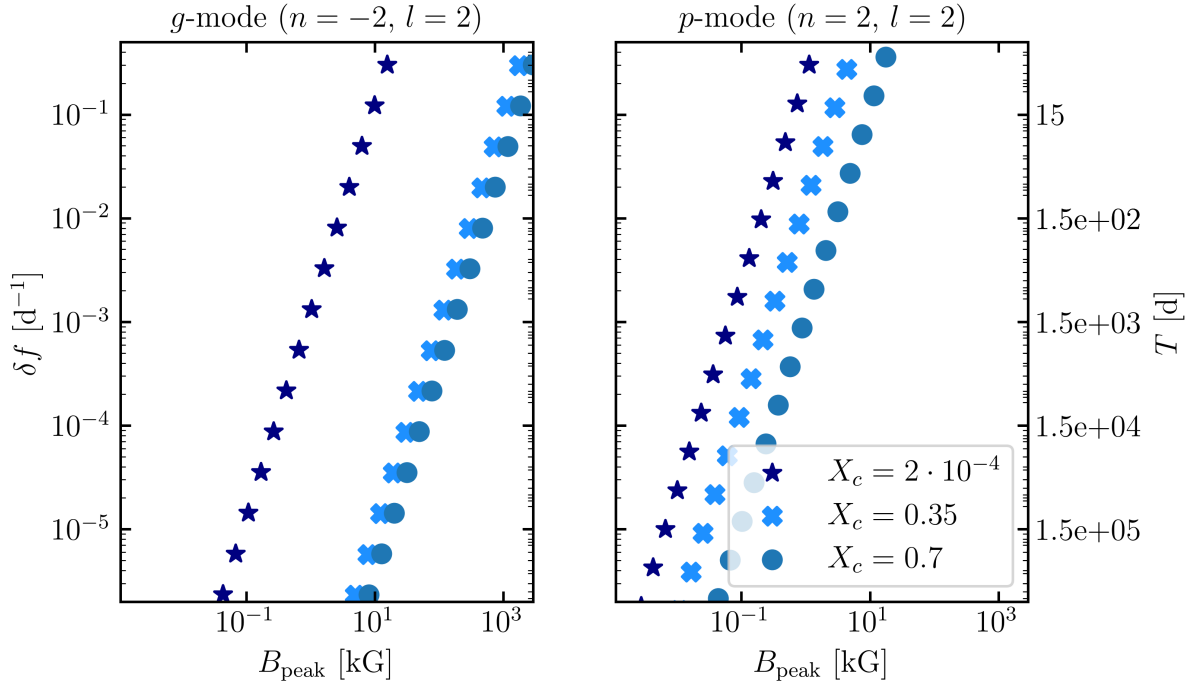


Figure 4.10: Mean small splitting in a misaligned ($\beta = 3\pi/16$) magnetic field for a g -mode ($n = -2$, $l = 2$) and a p -mode ($n = 2$, $l = 2$) of a $15 M_{\odot}$ star for three different instances of the central helium fraction X_c .

4.4 Main takeaways

This chapter discussed the implementation of the theoretical framework starting from stellar structure parameters to the predicted perturbed oscillation frequencies with the combined effects of misaligned rotation and magnetic fields. Using this implementation, we were able to perform a parameter study to investigate the effects of rotation, magnetic fields, and their misaligned combination on the oscillation modes. We extended this investigation to different stellar models of different masses and evolutionary stages. This is the first time such a parameter study has been performed for β Cephei stars using the theoretical framework of Chapter 2. Our main findings are:

- The rotational splittings are symmetric around the zonal mode and depend on the azimuthal order m . This is in contrast with the magnetic splittings, which depends only on the absolute value of the azimuthal order m and perturbs the oscillation mode to higher frequencies. Both results are in agreement with the expected behaviour from the theory.
- When combining the effects of rotation and magnetic field, we can obtain up to $(2l + 1)^2$ frequencies, depending on the obliquity angle β between the rotational and magnetic field axes. The amplitudes of the frequencies are also dependent on the obliquity angle.
- Both the kernel sensitivity and the magnetic field topologies depend on the evolutionary stage of the star. While p -modes are sensitive to the magnetic field near the surface of the star, some low order g -modes have a sensitivity kernel heavily dependent on the evolutionary stage. For massive stars near the ZAMS, the g -modes are sensitive to the near-core region, while at the TAMS, the sensitivity shifts to the outer layers of the star for low order g -modes, as these modes become mixed modes. This shows that the expected values of the magnetic field strength are strongly dependent on the mode identification and the assumed evolutionary stage of the star.
- The perturbative effects of magnetic fields follow a quadratic relation with the magnetic field strength. The measured perturbations must however be corrected for the rotational effects as even a first order rotational perturbation can quickly dominate over the magnetic perturbation. The quadratic relation holds for the asymmetries of the splittings, with a scaling factor strongly dependent on the radial order of the mode. This dependency of the scaling factor shows two regimes, separating the near-core sensitive modes and the near-surface sensitive modes.

Chapter 5

Fitting the asymmetric splittings of HD 192575

The goal of this chapter is to explain the observed g - and p -mode splittings of HD 192575 using our model of combined rotation and magnetic fields. We will fit the frequency differences and amplitudes of the extracted g -mode and p -mode (cf. bottom figure of Fig. 1.6) to a large set of models with different parameters. We only consider the frequency differences of the modes and not the frequencies themselves to minimize the dependency on the internal stellar structure and focus our efforts on the effects of rotation and magnetic fields. Both modes will be fitted independently, allowing us to decouple the results as much as possible from the large-scale magnetic field topology.

To accurately include the amplitudes of a mode, we need to correct for the inclination between the line of sight and the rotation axis. We go over this effect in Section 5.1. The extracted frequencies and amplitudes are described in Section 5.2. We will consider 600 different stellar models for HD 192575 as computed by Burssens et al. (2023) and fitted by Vanlaer et al. (subm) based on different identifications of the oscillation modes. A description of these model parameters can be found in Section 5.3. Section 5.4 describes the fitting algorithm and Section 5.5 discusses the results of the fitting. Similar to the previous chapter, the reported rotational frequencies are reported in units of the critical rotational frequency Ω_{crit} as given by Eq. (4.1).

5.1 Inclination relevation

To accurately fit the amplitudes of the observed frequencies, we must correct our predicted frequencies for the inclination angle between the rotation axis and the line of sight. We will follow the work of Gizon and Solanki (2003) to describe the effect of the inclination angle on the stellar amplitudes. The brightness variations I on the stellar surface as a consequence of stellar oscillations can be approximated as

$$I(t, \theta, \phi) = \sum_{n,l,m} A_{nlm} Y_l^m(\theta, \phi) e^{i\omega_{nlm}t}, \quad (5.1)$$

where A_{nlm} is the amplitude of the oscillation mode of radial order n , angular degree l , and azimuthal order m . While this expression is complex, only the real part describes the

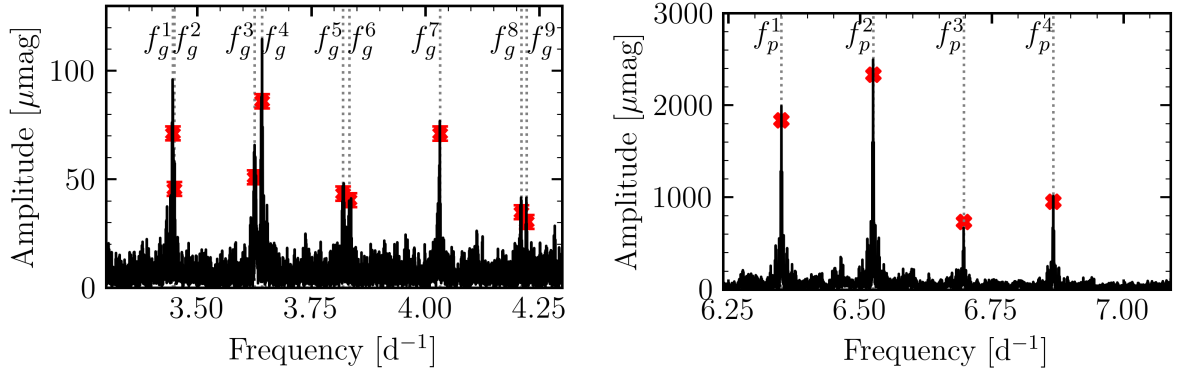


Figure 5.1: Lomb-Scargle periodogram of the g -mode (left) and p -mode (right) of interest. The red crosses indicate the extracted frequencies and amplitudes.

intensity variations of the stellar surface.

We define the observational frame as the inertial frame with the polar axis pointing towards the observer. The inclination angle i is defined between the rotation axis and this polar axis pointing along the line of sight. We can describe the effect of the inclination angle on the observed amplitudes by projecting each spherical harmonic to the line of sight. The amplitude correction factor $\sqrt{\mathcal{E}(i)}$ is given by (Dziembowski, 1977, Toutain and Gouttebroze, 1993)

$$\mathcal{E}(i) = \frac{(l - |m|)!}{(l + |m|)!} \left[P_l^{|m|}(\cos i) \right]^2. \quad (5.2)$$

This correction factor depends on the angular degree l and the absolute value of the azimuthal order m . Any asymmetries between the amplitudes of two modes of opposite rotational azimuthal order m must thus be a consequence of the magnetic field for a single independent solution of the framework above. The inclination angle i defined above differs from the inclination angle ι used in spectroscopic measurements of the rotational velocity $v \sin \iota$, where $i = \pi/2 - \iota$.

The above correction for the inclination angle is purely geometrical in nature and does not account for limb-darkening, nor does it take the coupling of the stellar modes to the atmosphere into account. In the case of pure rotational splitting the inclination correction relies on the assumption of energy equipartition between the different azimuthal orders. This means that the inherent uncorrected amplitudes of the different azimuthal orders are assumed to be equal in strength, such that the difference in observed amplitudes is purely a consequence of the inclination angle. Our current implementation of combined rotation and magnetic field does not require this assumption, as the relative amplitudes are fixed within an independent solution, with only the relative scaling between independent solutions as a free parameter. This means that for each solution of a given magnetic azimuthal order m_{mag} , the relative amplitudes of each rotational azimuthal order m_{rot} is fixed and can be corrected for the inclination angle without the assumption of energy equipartition. This is a major advantage of our framework, as it allows us to fit the inclination angle i without the assumption of energy equipartition.

Table 5.1: Extracted frequencies and amplitudes of the g - and p -mode.

	frequency [d^{-1}]	amplitude [μmag]		frequency [d^{-1}]	amplitude [μmag]
f_g^1	3.445953(6)	71.2 ± 1.6	f_p^1	6.3509313(2)	1835.5 ± 1.6
f_g^2	3.449320(10)	45.5 ± 1.6	f_p^2	6.5250052(2)	2331.8 ± 1.6
f_g^3	3.625411(9)	50.9 ± 1.6	f_p^3	6.6968226(6)	731.8 ± 1.6
f_g^4	3.642091(5)	86.0 ± 1.6	f_p^4	6.8670010(5)	952.8 ± 1.6
f_g^5	3.819262(11)	43.4 ± 1.6			
f_g^6	3.833088(11)	40.4 ± 1.6			
f_g^7	4.032025(6)	71.1 ± 1.6			
f_g^8	4.210734(13)	34.8 ± 1.6			
f_g^9	4.221790(15)	30.3 ± 1.6			

5.2 HD 192575 and its frequencies

The photometric data of HD 192575 obtained by TESS over multiple cycles leads to the Lomb-Scargle periodogram shown in Fig. 1.6. Frequencies and amplitudes are extracted from this spectrum by PhD student M. Vanrespaille with **STAR SHADOW** (IJspeert et al., 2024) as modified following Janssen et al. (2025). **STAR SHADOW** uses *prewhitening techniques* (Vaniček, 1971), where it iteratively determines the most significant frequency (quantified by the largest amplitude in Fourier space) and fits a sinusoidal function to the residual light curve of the fit, starting from the full light curve. At each step in the process, all detected frequencies, amplitudes and phases are refitted to find the optimal solution. This process is repeated until the extracted modes are below a significance threshold.

The prewhitening technique was applied to the TESS lightcurve of HD 192575, with a signal-to-noise threshold of 2.0. This threshold is calculated as the ratio of the extracted amplitude to the mean amplitude in a window of 1 d^{-1} around the frequency of interest. This low threshold was chosen to ensure the extraction of all visually present frequencies, as informed by theoretical considerations, but also to estimate the real noise floor accurately. Table 5.1 lists the numerical values of the extracted frequencies and amplitudes. Each frequency is given an identification number i , increasing from low to high frequency together with an identifier for the mode (f_g^i for the g -mode frequencies and f_p^i for the p -mode frequencies). The extracted frequencies and amplitudes are shown in Fig. 5.1 together with the Lomb-Scargle periodogram. Some extracted frequencies have a lower amplitude than is suggested by the Lomb-Scargle periodogram as a consequence of the prewhitening process.

5.3 Stellar model selection

As outlined in Section 4.1, the calculation of the frequencies and amplitudes requires stellar models, with both the displacements and the magnetic field topology being dependent on the stellar structure. We will work with a grid of **MESA** models initially calculated by Burssens et al. (2023) for the star HD 192575 and later reevaluated by Vanlaer et al. (subm.). Burssens et al. (2023) calculated a grid of stellar models covering the spectroscopic parameters of HD 192575 and used the assumption of an avoided crossing to select the best model. Vanlaer et al. (subm.) reevaluated the determination of the best model

Table 5.2: Best stellar parameters for the different sets of stellar models used in this work, as derived by Vanlaer et al. (subm.).

Parameter	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
$M_{\text{ini}} [M_{\odot}]$	$13.0^{+0.5}_{-2.5}$	$11.5^{+2.0}_{-1.0}$	$11.0^{+1.0}_{-2.0}$	$12.5^{+1.0}_{-2.0}$	$12.0^{+1.5}_{-1.5}$	$12.0^{+1.5}_{-1.5}$
$Z_{\text{ini}} [\text{dex}]$	$0.014^{+0.002}_{-0.002}$	$0.012^{+0.004}_{-0.000}$	$0.016^{+0.000}_{-0.004}$	$0.014^{+0.002}_{-0.002}$	$0.014^{+0.002}_{-0.002}$	$0.012^{+0.004}_{-0.000}$
f_{CBM}	$0.035^{+0.000}_{-0.030}$	$0.035^{+0.000}_{-0.030}$	$0.025^{+0.010}_{-0.020}$	$0.010^{+0.025}_{-0.005}$	$0.10^{+0.025}_{-0.005}$	$0.015^{+0.020}_{-0.010}$
X_c	$0.236^{+0.025}_{-0.220}$	$0.136^{+0.175}_{-0.120}$	$0.251^{+0.160}_{-0.235}$	$0.016^{+0.285}_{-0.000}$	$0.026^{+0.225}_{-0.010}$	$0.016^{+0.395}_{-0.000}$
g -mode	(2, -1)	(2, -1)	(2, -2)	(2, -2)	(2, -2)	(2, -2)
p -mode	(2, +2)	(2, +3)	(2, +0)	(2, +1)	(2, +2)	(2, +3)

for the same grid of stellar models, opting to consider the p -mode frequencies and only 5 frequencies in the g -mode regime of interest (f_p^1 , f_p^4 , f_p^6 , f_p^7 , and f_p^9). Together with two additional p -mode multiplets, they derived six sets of stellar models with different identifications for the g -mode and the p -mode. We will use these six sets of stellar models to fit the observed splittings and amplitudes of the g -mode. Each set contains 100 individual models spread over the 95% confidence interval on the derived stellar properties. The stellar parameters of the best fitting models per set are listed in Table 5.2 together with the radial order n and angular degree l assumed for the modes. HD 192575 is identified as a $12 M_{\odot}$ star between the middle and the end of the main sequence. The g -mode is identified by Vanlaer et al. as a quadrupole mode $l = 2$ with radial order $n = -1$ or $n = -2$. Similarly the p -mode is identified as a quadrupole mode $l = 2$ with a radial order ranging between $n = 0$ and $n = 3$. Note that for the classification $n = 0$, this mode would be a fundamental f -mode and not a p -mode. We will however continue to refer to this mode as a p -mode.

The stellar model grid has an initial mass resolution of $\Delta M = 0.5 M_{\odot}$ between 10.5 and $13.5 M_{\odot}$ and an initial metallicity resolution of $\Delta Z = 0.002 \text{ dex}$ between $Z = 0.014 \text{ dex}$ and $Z = 0.016 \text{ dex}$. The convective boundary mixing parameter f_{CBM} is varied between 0.005 and 0.035 , with a resolution of $\Delta f_{\text{CBM}} = 0.005$. The core hydrogen mass fraction X_c is varied between 0.351 and 0.01 , with a resolution of $\Delta X_c = 0.005$ between $X_c = 0.013$ and $X_c = 0.25$ and a resolution of $\Delta X_c = 0.01$ above $X_c = 0.25$. Fig. 5.2 shows the different stellar parameters for each set of models. A slight vertical offset is applied to the corner plots to improve readability. The figure additionally shows the Hertzsprung-Russell diagram of the different models, with the spectroscopic parameters of HD 192575 as determined by Burssens et al. (2023).

5.4 Fitting algorithm

For each stellar model, we calculated the oscillation frequencies and displacements using GYRE with vacuum boundary conditions. Together with the stellar parameters, this delivered the required input to calculate the perturbed frequencies and relative amplitudes of the g -mode. In addition to the stellar parameters, we fit four additional parameters: the rotational frequency (Ω), the magnetic field strength (B_{max}), the obliquity angle between the rotation and magnetic field axes (β), and the inclination angle (i) between the rotation axis and the line of sight.

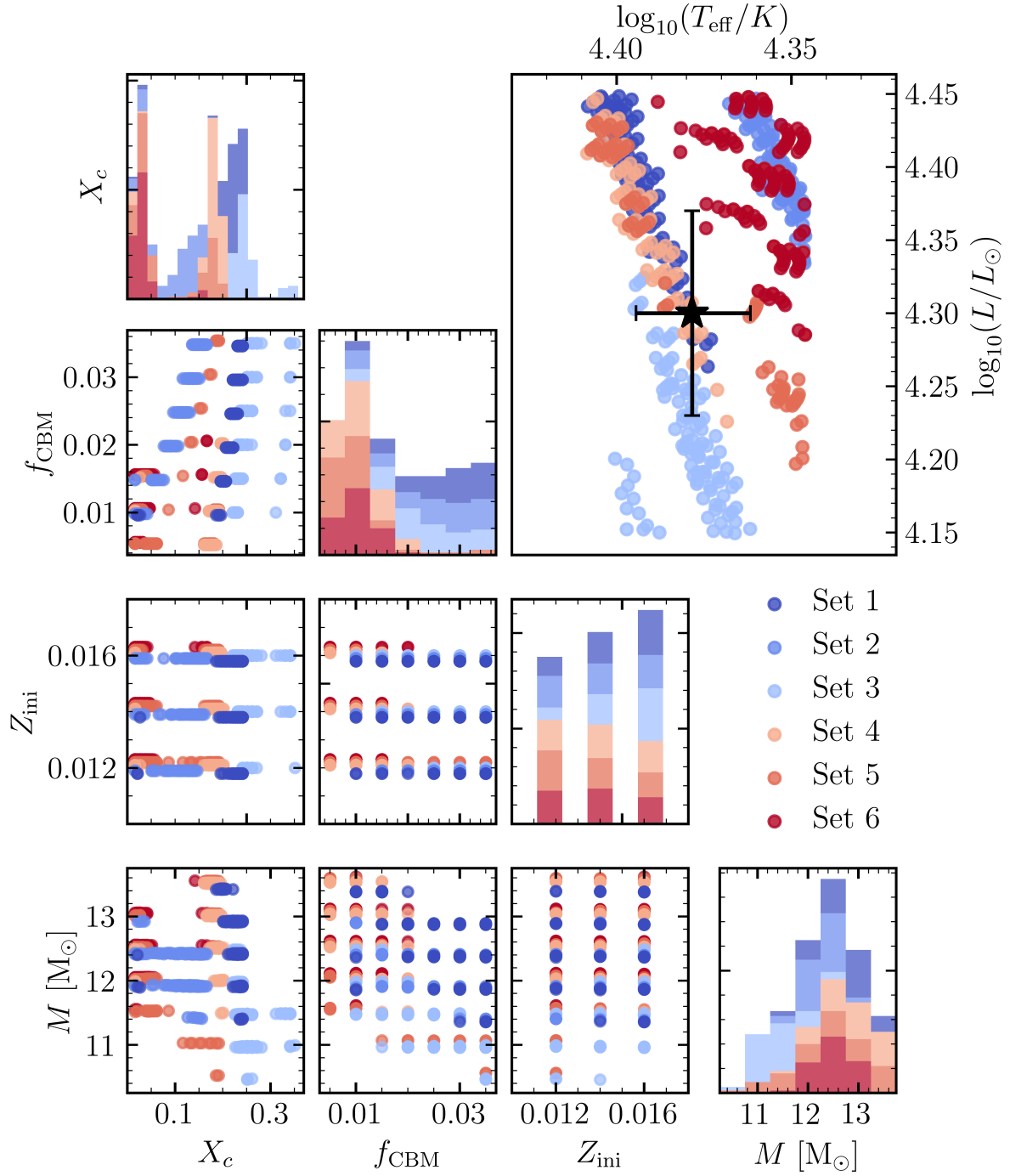


Figure 5.2: Corner plot showing the different stellar model parameters for each set. The histograms on the diagonal are stacked. The upper right plot shows a corresponding Hertzsprung-Russell diagram. The black star shows the spectroscopic parameters of HD 192575 and their one standard deviation as determined by [Burssens et al. \(2023\)](#).

We fit the observed frequency splittings rather than the absolute frequencies, focussing our efforts on the additional perturbations rather than determining the best stellar model. To fit the observed splittings, we identify a ‘central’ frequency f_{cen} as having a rotational azimuthal order $m_{\text{rot}} = 0$ around which we will calculate the frequency splittings $\Delta f_i = f_i - f_{\text{cen}}$. Identifying the rotational azimuthal order of the central frequency limits the possible combinations and speeds up the fitting process, while leaving the magnetic azimuthal order of this frequency free.

Starting from a stellar model, we calculated a grid of perturbed frequencies and relative amplitudes for a range of rotational frequencies Ω , magnetic field strengths B_{max} , obliquity angles β , and inclination angles i . An initial estimate of the rotational frequency Ω is available from the previous modelling efforts of [Burssens et al. \(2023\)](#) and [Vanlaer et al. \(subm.\)](#), who estimated the rotational frequency to be between $\Omega = 0.16 - 0.18 \Omega_{\text{crit}}$. The obliquity angle and the inclination angle are limited between 0 and $\pi/2$, leaving only the magnetic field strength to require an initial guess. This initial guess is obtained from the quadratic relation of the magnetic field strength and the asymmetries as discussed in [Section 4.3](#). By fitting the quadratic relation of theoretical predicted asymmetries for different strengths of the magnetic field for a chosen obliquity angle ($\beta = \pi/3$) and rotational frequency ($\Omega = 0.18$), we can obtain an initial guess for the magnetic field strength B_{max} by interpolating the required field strength to the observed asymmetries. As to not be limited to a single identification, we repeated this process for the maximum and minimum values of the observed asymmetries that can be obtained through different identifications. This leads to two initial guesses for the magnetic field strength $B_{\text{ini,min}}$ and $B_{\text{ini,max}}$ separated by $\Delta B_{\text{ini}} = B_{\text{ini,max}} - B_{\text{ini,min}}$. The initial range of magnetic field strengths is then extended between $\max(B_{\text{ini,min}} - \Delta B_{\text{ini}}, 0)$ and $B_{\text{ini,max}} + \Delta B_{\text{ini}}$.

For each set of frequencies and amplitudes we calculate the distance to the observed frequencies and amplitudes using a modified χ^2 -metric:

$$d^2 = \sum_{i=0}^n \left(\frac{\Delta f_{i,\text{theo}} - \Delta f_{i,\text{obs}}}{\sigma_{\Delta f_{i,\text{obs}}}} \right)^2 + c \cdot \sum_{i=0}^N \left(\frac{A_{i,\text{theo}} - A_{i,\text{obs}}}{\sigma_{A_{i,\text{obs}}}} \right)^2, \quad (5.3)$$

where we sum of the n observed frequencies and the N predicted frequencies of all necessary independent solutions to explain the observed frequencies. We require that $n \leq N$, such that every selected observed frequency is considered in the fitting. The weight c is a free parameter that we can adjust to fit the observed frequencies and amplitudes. The difference in summation between the frequencies and amplitudes is to penalize the fit for predicted frequencies that are not observed. For the predicted amplitudes with no observed counterpart, we set $A_{i,\text{obs}} = 0 \mu\text{mag}$ and $\sigma_{A_{i,\text{obs}}} = 1.6 \mu\text{mag}$, which is the uncertainty on the extracted amplitudes.

In general, there will be more predicted frequencies than observed frequencies within this framework, necessitating the selection of a subset of predicted frequencies to fit the observed frequencies. This *preselection* is based on the frequencies of the observed mode. As each set of $(2l+1)^2$ frequencies consists of $2l+1$ independent solutions, all combinations of independent solutions that can explain the observed frequencies within a limit Δf_{lim} are selected. This means that for each observed frequency f_{obs} , there is at least one unique predicted frequency f_{theo} such that $|f_{\text{obs}} - f_{\text{theo}}| \leq \Delta f_{\text{lim}}$. Within each combination, the observed frequencies are matched with their closest theoretical counterpart. The predicted

relative amplitudes are then rescaled to approximate the matched observed amplitudes as closely as possible. After the matching and rescaling, the distance d^2 can be calculated for the selected combination of frequencies. All allowed combinations are considered to find the combination with the lowest distance, which is selected as the best fit for the given set of parameters. This process is repeated for all combinations of the parameters Ω , $B_{\max}$, β , and i for a given stellar model.

To ensure that the fitting process reaches a minimum in the parameter space of a single stellar model, the process is repeated for a few iterations of the parameter grid. A first iteration is calculated with a coarse grid of parameters around the initial guesses as derived above. The obliquity angle β and the inclination angle i are allowed to vary between 0 and $\pi/2$. This grid considers only two rotational frequencies ($\Omega = 0.18$ and 0.20) to speed up the fitting process and converge to an appropriate magnetic field strength. The best fit of this iteration is then used for a second iteration with a finer grid of parameters, where all parameters are free to vary. Three additional iterations are calculated over such a semi-coarse grid to find the local minimum of the parameter space. The final iteration is a fine grid around the best fit of the previous iteration. We limit the iterations to only four semi-coarse grids before the final grid to ensure an acceptable computation time of the fit. Extensive testing showed this to be sufficient to find a minimum of the parameter space.

The fitting algorithm contains five free parameters that need to be selected for the fitting process. These are: a central frequency f_{cen} identified with rotational azimuthal order $m = 0$, one (or more) initial guesses for the rotational frequency, the obliquity angle for which to calculate the initial guess for the magnetic field strength, the weight parameter c in the distance metric, and the maximum accepted frequency deviation from the observed frequency Δf_{lim} .

For each combination of parameters, we obtain a distance d^2 between the observed and predicted frequencies and amplitudes. The best fitting model minimizes this distance to $d_{\min}^2$. For each model, we can calculate the $\mathcal{F}$ statistic to compare with the best fit. This statistic is a measure of the difference in the explained variance expressed by the distance and is given by

$$\mathcal{F}(d^2) = \frac{d^2 - d_{\min}^2}{d_{\min}^2} \frac{N - K}{K} \sim F_{K, N-K}, \quad (5.4)$$

for a number of observables N , and a number of free parameters K . The $\mathcal{F}$ statistic is distributed as an F -distribution with K and $N - K$ degrees of freedom. This leads to an asymptotic $100(1 - \alpha)\%$ confidence region of the parameters $\boldsymbol{\theta}$, given by

$$\left\{ \boldsymbol{\theta} \left| d^2(\boldsymbol{\theta}) \leq d_{\min}^2 \left(1 + \frac{K}{N - K} F_{\alpha} \right) \right. \right\}. \quad (5.5)$$

Equation 5.4 gives the expected distribution of the $\mathcal{F}$ statistic based on the metric d^2 . We can use this distribution to define a likelihood function for a set of parameters $\boldsymbol{\theta}$, given by

$$\mathcal{L}(\boldsymbol{\theta}) = F_{K, N-K} \left(\frac{d^2(\boldsymbol{\theta}) - d_{\min}^2}{d_{\min}^2} \frac{N - K}{K} \right). \quad (5.6)$$

Table 5.3: Best model parameters and 95% confidence intervals for the asteroseismic modelling of the g -mode.

Parameter	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
$M_{\text{ini}} [M_{\odot}]$	$11.5^{+2.0}_{-0.0}$	$12.0^{+1.0}_{-0.5}$	$11.5^{+1.0}_{-1.0}$	$13.0^{+0.5}_{-1.5}$	$13.0^{+0.5}_{-2.5}$	$12.0^{+1.5}_{-0.5}$
$Z_{\text{ini}} [\text{dex}]$	$0.012^{+0.004}_{-0.000}$	$0.016^{+0.000}_{-0.004}$	$0.014^{+0.002}_{-0.002}$	$0.016^{+0.000}_{-0.004}$	$0.016^{+0.000}_{-0.004}$	$0.012^{+0.004}_{-0.000}$
f_{CBM}	$0.035^{+0.000}_{-0.025}$	$0.035^{+0.000}_{-0.025}$	$0.030^{+0.005}_{-0.020}$	$0.005^{+0.015}_{-0.000}$	$0.005^{+0.030}_{-0.000}$	$0.010^{+0.010}_{-0.005}$
X_c	$0.236^{+0.005}_{-0.230}$	$0.151^{+0.015}_{-0.145}$	$0.261^{+0.020}_{-0.040}$	$0.166^{+0.035}_{-0.150}$	$0.166^{+0.025}_{-0.150}$	$0.031^{+0.135}_{-0.015}$
$B_{\text{max}} [\text{kG}]$	17^{+5}_{-14}	$3.6^{+14.5}_{-0.7}$	570^{+600}_{-250}	159^{+622}_{-154}	159^{+520}_{-150}	37^{+59}_{-34}
$\Omega [\Omega_{\text{crit}}]$	$0.16^{+0.06}_{-0.01}$	$0.21^{+0.02}_{-0.02}$	$0.14^{+0.01}_{-0.01}$	$0.15^{+0.08}_{-0.02}$	$0.15^{+0.09}_{-0.01}$	$0.19^{+0.05}_{-0.03}$
$\beta [^{\circ}]$	69^{+28}_{-23}	59^{+28}_{-14}	63^{+23}_{-17}	59^{+29}_{-15}	59^{+28}_{-15}	57^{+30}_{-13}
$i [^{\circ}]$	66^{+18}_{-22}	66^{+8}_{-22}	66^{+7}_{-6}	66^{+5}_{-7}	66^{+8}_{-23}	66^{+6}_{-22}
m_{mag}	(0, 1)	(1, 2)	(1, 2)	(1, 2)	(1, 2)	(1, 2)
$R_{\text{peak}} [R_{\text{star}}]$	0.991	0.994	0.151	0.124	0.124	0.994
$B_{\text{peak}} [\text{kG}]$	4.81	0.99	230	33	34	10
$\Delta f_{\text{shift}} [\text{d}^{-1}]$	0.15	0.19	0.11	0.26	0.26	0.11
$v \sin \iota [\text{km/s}]$	33^{+58}_{-29}	40^{+39}_{-37}	30^{+28}_{-20}	33^{+32}_{-24}	33^{+48}_{-30}	38^{+47}_{-34}
$\langle \Delta f / \sigma_{f, \text{obs}} \rangle$	581	689	744	596	596	661
$\langle \Delta A / \sigma_{A, \text{obs}} \rangle$	1.75	1.05	1.03	1.18	1.18	1.40

5.5 Fitting results

The fitting algorithm can be applied to both the observed g -mode and p -mode frequencies and amplitudes, using the stellar models and identifications as determined Vanlaer et al. (subm.). We fit both modes independently, allowing full freedom of the parameters for both modes. This reduces the dependence of the fit on the chosen magnetic field topology.

5.5.1 Fitting the g -mode

All identifications of the g -mode reported by Vanlaer et al. (subm.) are quadrupole modes. The nine frequencies of the g -mode can be explained by a minimum of two independent solutions which have been split by both rotation and the magnetic field. This leads to a minimum of 10 predicted frequencies for the g -mode.

We apply the fitting process to find an alternative explanation for the observed frequency splittings and amplitudes of the g -mode of HD 192575. We select f_g^5 as the central frequency around which we will fit the frequency differences. We take the initial guess for the rotational frequency at $\Omega = 0.18$. The initial magnetic field strength range is derived for an obliquity angle of $\beta = \pi/3$. The weight parameter c is set to 10^4 and we set the maximum frequency deviation to $\Delta f_{\text{lim}} = 0.1 \text{ d}^{-1}$. We have $N = 18$ observables (9 frequencies and amplitudes) and $K = 8$ free parameters, four of which are for the stellar model, while the other four are to determine the magnetic field, rotation, and inclination.

Figure 5.3 shows the best fitting frequencies and amplitudes for the different sets of stellar models. The 95% confidence interval on the predicted frequency differences are smaller than the size of the markers. The predicted frequencies show a good agreement with the observed frequencies for all sets, with similar predicted amplitudes and uncertainties between the sets. We require each observed frequency to be matched with a predicted

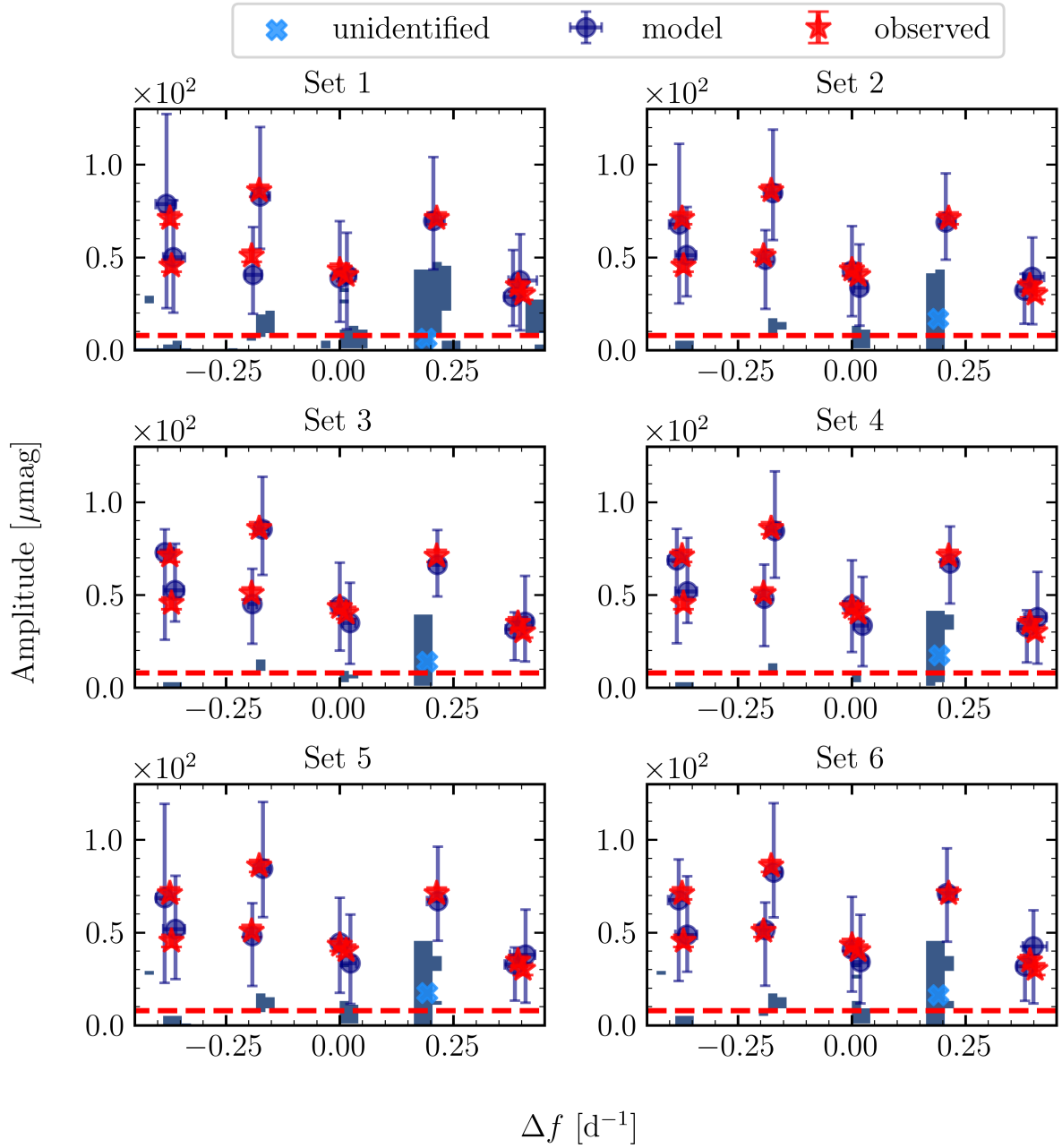


Figure 5.3: Predicted and observed frequencies and amplitudes of the g -mode within a 95% confidence interval per set. The red horizontal dashed line shows the average noise level of the periodogram. Not all predicted frequencies are observed. The light blue cross shows the unidentified predicted frequency of the best fitting model. The blue areas show the 95% coverage region of predicted frequencies that are not observed.

frequency, which leads to predicted frequencies that are not seen in the data as we have 9 observed frequencies and a minimum of 10 predicted frequencies. We classify these as *unidentified frequencies*. The light blue cross shows the unidentified frequency for the best fitting model. The blue shaded areas show the region covered by all unidentified frequencies and corresponding amplitudes of fits within the 95% confidence region. These are mostly low in amplitude and close to the mean noise level of the periodogram (red dashed line).

The best fitting model parameters are listed in Table 5.3 with the 95% confidence intervals per model set. The derived stellar parameters are, as expected, consistent with those determined by Vanlaer et al. (subm.) as we only considered models within the 95% confidence interval of their derived parameters. The derived parameters for the additional perturbations are of more interest. The required maximum magnetic field strengths $B_{\max}$ show a large range of values, depending on the set of stellar models. We see that set 1, 2, and 6 show remarkably low magnetic field strengths compared to the other sets. This is due to the probing depth of the g -mode at these stellar models, as near-surface mixed mode probing requires a lower maximum magnetic field strengths to explain the observed splittings. The probing depth of the best fitting model can be approximately expressed as the radius (R_{peak}) at which the $K_{\theta\theta}$ kernel peaks for azimuthal order $m = 1$. The differences in this estimate show indeed that the low field strengths are related to the probing depth of the mode. All sets identify the g -mode for low radial orders $n = -1$ or $n = -2$, making their probing regime very sensitive to the stellar structure. This sensitivity is also reflected in the marginal likelihoods of the magnetic field strength, as shown in Fig. 5.4. For set 5, the field strength likelihood is bimodal for the same identification, which is likely a signature of the changing probing depth for different stellar models within the same set, where the g -mode is a mixed mode for some of the stellar models. The derived rotational frequencies are consistent with the previous estimates of Burssens et al. (2023), but still show a difference between the model set as a result of a change in the Ledoux constant between stellar models.

The obliquity angle β and the inclination angle i are consistent between the different sets. The inclination angle has no dependence on the radial order of the mode, and only depends on the rotational azimuthal order m_{rot} of the frequency and the angular degree l of the mode. A consistent inclination angle estimate of $i \approx 66^\circ$ for all sets is as such expected. The value of the obliquity angle β is likely dominated by the small splitting and relative amplitudes, leading to a consistent value between sets.

In addition to the derived parameters, Table 5.3 also lists the magnetic azimuthal orders m_{mag} of the best fitting model. For all sets, the best model only requires two independent solutions to explain all observed frequencies of the g -mode. This connects to the blue shaded areas in Fig. 5.3, showing the unobserved frequencies that our model does predict. These are mostly close to f_g^7 , as an independent solution has a frequency for each rotational azimuthal order.

The additional magnetic shift Δf_{shift} is the difference with the unperturbed frequency ω_0 and the perturbed frequency for rotational and magnetic azimuthal order $m = 0$ of the best fitting model. For each set, this shift is well below the maximum frequency difference between the unperturbed frequencies determined of the different stellar models, indicating that no large deviation in the determination of the stellar parameters is expected for the

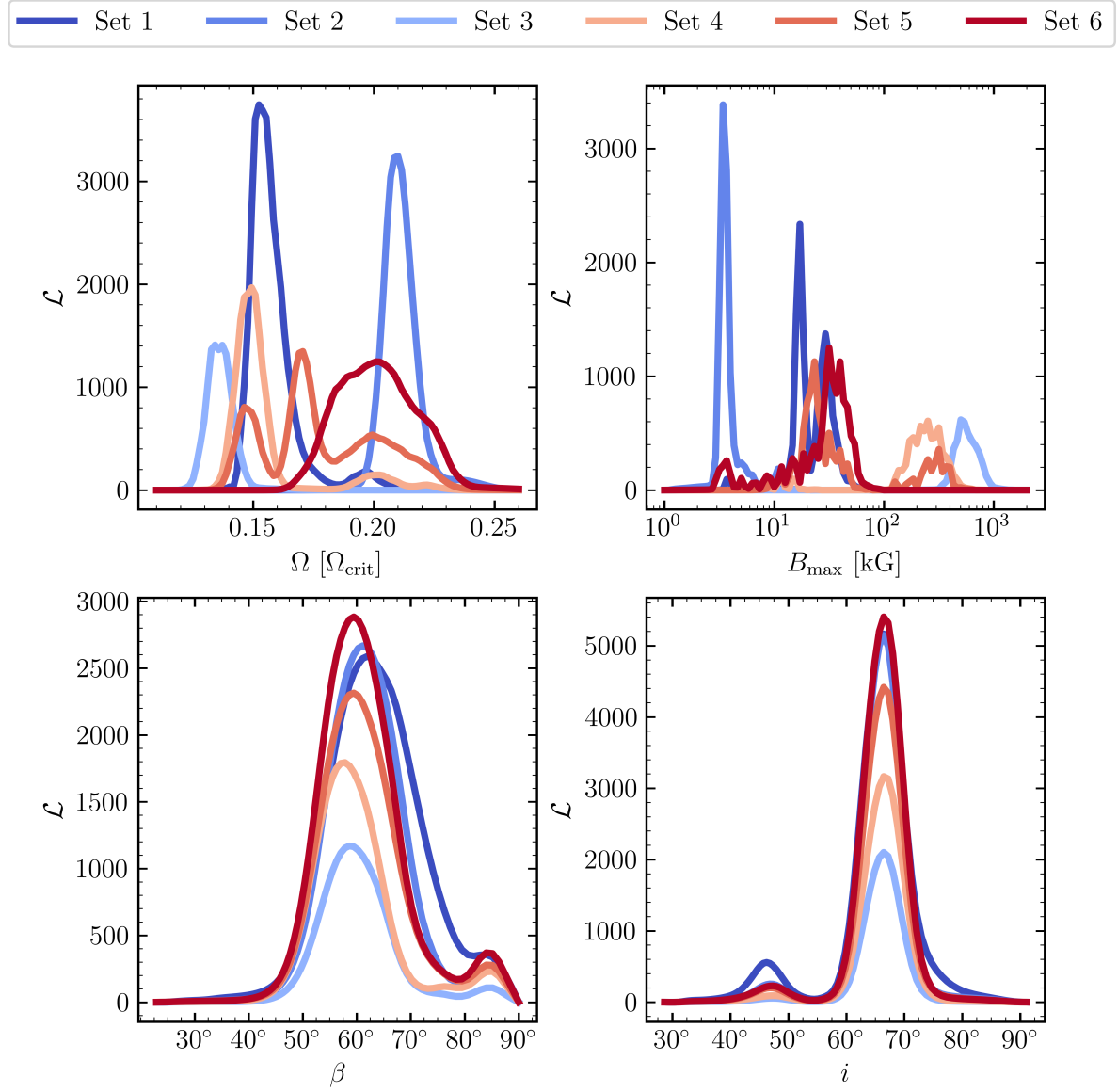


Figure 5.4: Marginal likelihoods of the g -mode fitting of the parameters required for the rotational and magnetic perturbations. The likelihoods are shown for the different sets of stellar models.

Table 5.4: Best model parameters and 95% confidence intervals for the asteroseismic modelling of the p -mode.

Parameter	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
$B_{\max}$ [kG]	$3.1^{+20.4}_{-1.7}$	$3.4^{+4.2}_{-2.7}$	26^{+53}_{-22}	$9.7^{+15.9}_{-8.6}$	$1.6^{+24.9}_{-0.5}$	$2.3^{+5.3}_{-1.5}$
Ω [Ω_{crit}]	$0.139^{+0.043}_{-0.007}$	$0.195^{+0.010}_{-0.016}$	$0.128^{+0.003}_{-0.033}$	$0.139^{+0.067}_{-0.017}$	$0.139^{+0.070}_{-0.014}$	$0.155^{+0.054}_{-0.003}$
β [$^\circ$]	75^{+11}_{-45}	67^{+21}_{-41}	68^{+21}_{-47}	67^{+22}_{-44}	74^{+16}_{-53}	74^{+15}_{-41}
i [$^\circ$]	62^{+26}_{-31}	64^{+24}_{-36}	63^{+25}_{-32}	64^{+24}_{-37}	71^{+17}_{-45}	63^{+15}_{-29}
m_{mag}	0	-1	-1	-1	1	0
R_{peak} [R_{star}]	0.996	0.995	0.995	0.995	0.996	0.996
B_{peak} [kG]	0.88	0.92	7.51	2.79	0.48	0.64
Δf_{shift} [d^{-1}]	0.24	0.54	0.91	0.95	0.06	0.24
$v \sin \iota$ [km/s]	34^{+38}_{-32}	40^{+41}_{-38}	31^{+28}_{-30}	33^{+44}_{-31}	24^{+58}_{-22}	43^{+36}_{-40}
$\langle \Delta f / \sigma_{f, \text{obs}} \rangle$	4748	6510	9398	8327	8327	3978
$\langle \Delta A / \sigma_{A, \text{obs}} \rangle$	10	14	7	11	10	9

inclusion of the magnetic field in the asteroseismic fit.

In addition to the rotational frequency estimation, we can also estimate the projected rotational velocity $v \sin \iota$ of the star. This includes the uncertainties on the inclination angle i , the rotational frequency Ω , but also the stellar parameters. For all model sets, the predicted projected rotational velocity is consistent with the previous estimates of [Burssens et al. \(2023\)](#), who reported a spectroscopically derived value of $v \sin \iota = 27^{+6}_{-8}$ km/s.

Table 5.3 also reports on the average distance between the observed and predicted frequencies and amplitudes for the best fitting model in units of the observed uncertainties, denoted as $\langle \Delta f / \sigma_{f, \text{obs}} \rangle$ and $\langle \Delta A / \sigma_{A, \text{obs}} \rangle$ respectively. This mean was calculated over each matched set of observed and predicted value. We exclude the central frequency f_{cen} from the frequency mean, as these are perfectly matched by definition. The amplitude distance of the central frequency is included in the mean amplitude distance. The large distance in frequencies is expected, as this is a consequence of large theoretical uncertainties. The amplitudes match well between observed values and best fitting model with average distances around a single standard deviation of the observed amplitudes.

5.5.2 Fitting the p -mode

The fitting process of the p -mode differs slightly from the g -mode, as we only include 4 observed frequencies. We identify f_p^3 as the central frequency f_{cen} with rotational azimuthal order $m = 0$. This differs from the azimuthal identification of [Vanlaer et al. \(subm.\)](#), who identified this frequency as azimuthal order $m = 1$. We require the predicted set of frequencies to consist of a single independent solution, as no clear splittings are available. The weight parameter is set to $c = 10^3$ and the maximum frequency deviation is set to $\Delta f_{\text{lim}} = 0.5 \text{ d}^{-1}$. Due to the low number of included observations, we limit the fitting to only predict the rotational and magnetic parameters, while including all stellar models. We have a total of $N = 8$ observables and $K = 4$ free parameters, including only the parameters for the rotational and magnetic perturbations. This limited amount of frequencies allows for only a single estimate of the asymmetry $\mathcal{A}$ necessary to calculate the initial magnetic field strength range. We calculate the asymmetry between f_p^2 and

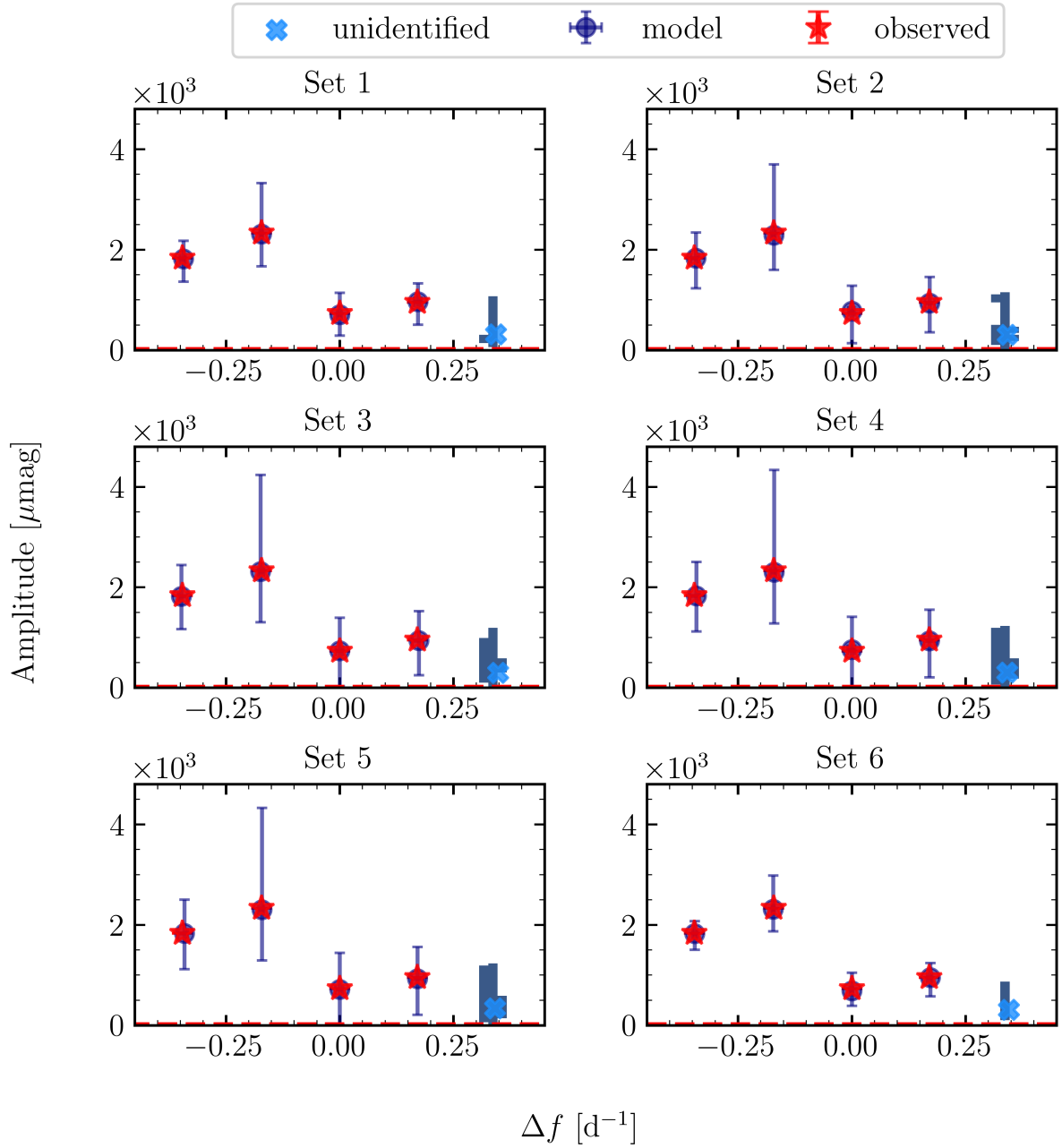


Figure 5.5: Predicted and observed frequencies and amplitudes of the p -mode within a 95% confidence interval per set. The red horizontal dashed line shows the average noise level of the periodogram. Not all predicted frequencies are observed. The light blue cross shows the unidentified predicted frequency of the best fitting model. The blue areas show the 95% coverage region of predicted frequencies that are not observed.

f_p^4 around f_p^3 as to obtain the minimum estimate $B_{\text{ini,min}}$ and use 10 times this value to obtain the maximum estimate $B_{\text{ini,max}}$.

Figure 5.5 shows the best fitting frequencies and amplitudes for the different sets of stellar models for the p -mode. There is a good agreement between the observed and predicted frequencies for all model sets. Each model set does predict an additional frequency that is not identified in the observed frequencies, but has low amplitude. The blue shaded areas show the region where the unidentified frequencies can be located within the 95% confidence interval and shows low variability. The narrow frequency confidence interval on the predicted frequencies can be immediately connected to the low uncertainty on the predicted rotational frequency reported in Table 5.4, as this factor has the largest and most direct impact on the predicted frequencies. All predicted parameters are consistent with the values derived from the g -mode fittings.

The estimated obliquity angles β and inclination angles i show a larger uncertainty than the g -mode, as we force the fit to be a single independent solution and include only four observed frequencies. The obliquity angle is in first instance dominated by the small splittings between the different solutions, which are not available for the p -mode. The inclination angle is less constrained as not all rotational azimuthal orders are observed. The derived parameters and best estimates are still closely related to the g -mode fitting, giving confidence in the derived parameters and the identification of the central frequency.

All probing depths are near the surface of the star. The probing depths are determined for the kernel $K_{\theta\theta}$ for azimuthal order of the identified mode. The exact identification of the radial order of the mode does not have a large impact on the estimated probing depth. However, the identifications of higher radial order $n = 2$ or $n = 3$ of set 1, 2, 5, and 6 do show a smaller estimated magnetic field at the probing depth compared to the other sets. This connects to the increase in sensitivity of the p -mode with radial order, as discussed in Section 4.3.

The induced frequency shift Δf_{shift} is for the p -mode similar to the unperturbed frequency variability present in the 95% confidence set of stellar models. This indicates that slight changes in the stellar parameters might be present when consistently including the effects of the magnetic field in the fit of the stellar parameters when including this p -mode. The predicted projected rotational velocity $v \sin i$ is consistent with the previous estimates of Burssens et al. (2023), with a larger uncertainty due to the large uncertainty on the inclination angle i .

The average distance between the observed and predicted frequencies and amplitudes in units of the observational uncertainties are larger than for the g -mode by an order of magnitude. While this indicates a less accurate fit than the g -mode, these values are expected given the theoretical model uncertainties. As discussed in Aerts et al. (2018), theoretical model uncertainties can dominate the uncertainties for long time base observations. This suggest that a fitting algorithm that estimates these unknown theoretical model uncertainties could be better suited.

The marginal likelihoods of the p -mode in Fig. 5.6 show similar trends as the g -mode, with multimodal distributions for most parameters and sets. The larger confidence intervals are reflected in the wider distributions of the marginal likelihoods. The likelihood distribution of the magnetic field strength shows a lower spread and mulitimodality for the p -mode as

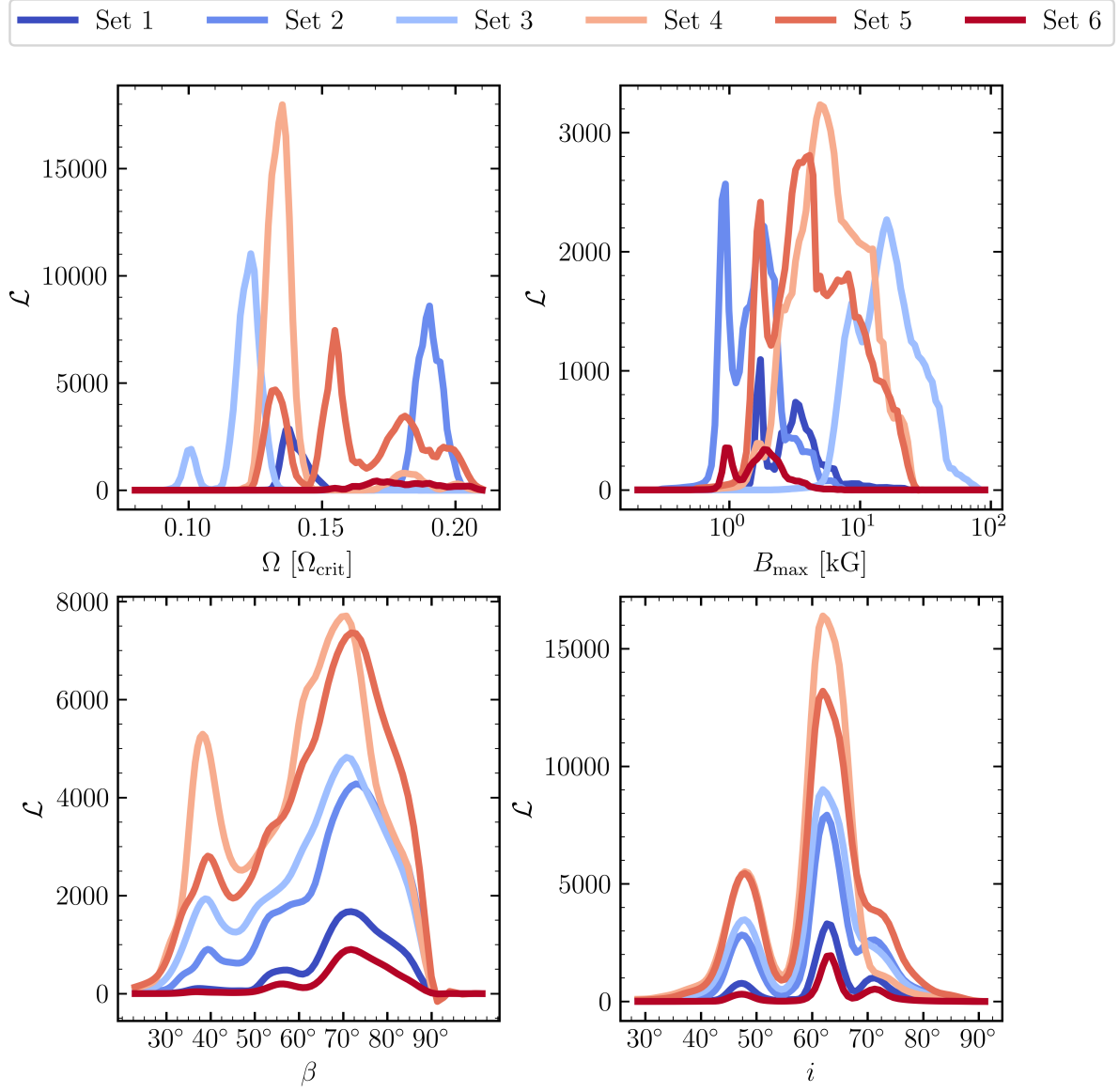


Figure 5.6: Marginal likelihoods of the p -mode fitting of the parameters required for the rotational and magnetic perturbations. The likelihoods are shown for the different sets of stellar models.

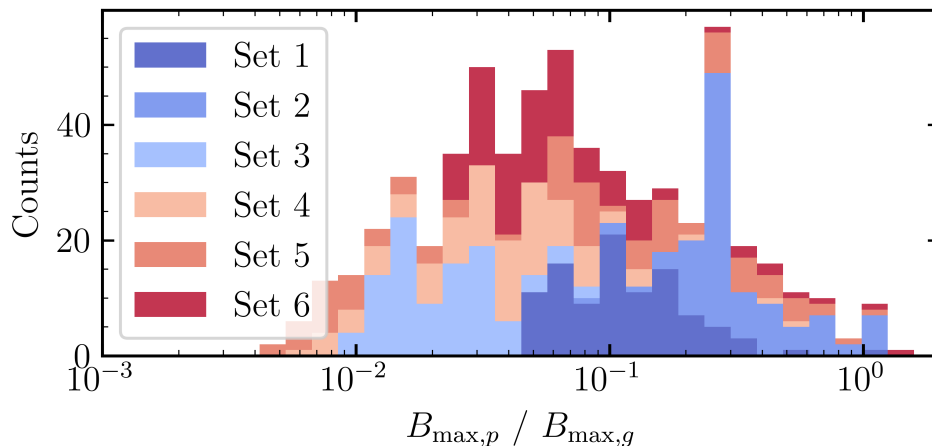


Figure 5.7: Stacked histogram of the magnetic field strength ratio $B_{\max,p}/B_{\max,g}$ for the best fitting parameters of each stellar model.

the probing depths do not drastically change between the models as was the case for the g -mode.

The differences in the maximum magnetic field strength $B_{\max}$ derived from the p -mode and the g -mode differ for each set of stellar models. While this difference is not statistically significant due to the large uncertainties, it is an indication that the magnetic field topology implemented in the model does not fully capture the possible internal magnetic field structure of HD 192575. For each stellar model, we selected the best fitting models for both the g -mode and the p -mode and calculate the ratio of the maximum magnetic field strengths $B_{\max,p}/B_{\max,g}$. We show the stacked histogram of these ratios for all sets in Fig. 5.7. While a large spread is visible, the ratios peak strongly for $B_{\max,p}$ between one and two orders of magnitude smaller than $B_{\max,g}$. This indicates that the current topology of the magnetic field does not fully capture the possible internal magnetic field structure of HD 192575. This would require a magnetic field that decreases in strength with radius more strongly than the current model. To investigate the topology further, more accurate measurements with a clear identification are required at different probing depths.

5.6 Main takeaways

This chapter presented the results of the asteroseismic modelling of the frequency splittings of HD 192575 including the effects of rotation and a magnetic field. The fitting process started from six different sets of stellar models derived by [Burssens et al. \(2023\)](#) and Vanlaer et al. (subm.), which were perturbed to calculate the expected frequencies and amplitudes of the identified g -mode and p -mode. The fitting process was done independently for both modes, allowing the parameters to be free to vary between the two cases. The main conclusions are as follows:

- Both modes can be explained within reasonable range by including the effects of rotation and a magnetic field. This forgoes the assumption of an avoided crossing, which heavily limited the parameter space of the stellar models. The derived

rotational frequency, obliquity angle, and inclination angle are consistent between model sets and modes.

- The g -mode is best explained by two independent solutions of misaligned magnetic field and rotation. Depending on the identification and the stellar model, the mode shows a g -mode near-core sensitivity or a mixed mode sensitivity. This variability leads to a wide range in derived maximum magnetic field strengths and limits our ability to constrain the internal magnetic field topology.
- The p -mode is assumed to be a single independent solution and leads to fitting parameters that are consistent with the g -mode. The derived magnetic field strengths are still strongly dependent on the stellar structure model, but show a smaller range of values than the g -mode.
- The derived rotational frequency and projected rotational velocity are consistent with previous estimates of both asteroseismic and spectroscopic approaches. Additionally, the derived inclination angle is consistent with the previous estimate of [Burssens et al. \(2023\)](#), who relied on the assumption of energy equipartition in the rotational splitting to estimate the inclination angle.

Chapter 6

Conclusions and outlook

This thesis presents an alternative explanation for the observed frequency splittings of the massive β Cep star HD 192575 than the one presented in [Burssens et al. \(2023\)](#). The interpretation is based on the combination of a misaligned magnetic field with the axis of rotation. Chapter 1 presented the context of the work, expanding upon the previous work done on HD 192575 and the current state of asteroseismic modelling of massive stars. The observed frequency splittings of the g -modes were previously interpreted as an avoided crossing, putting a tight constraint on the possible internal structure models of the star. In this thesis, we have let go of this assumption and expanded the analysis to include the effects of rotation and an internal magnetic field.

Chapter 2 presented the necessary theoretical background of the model. We showed that the combination of rotation and magnetic field can lead to $(2l + 1)^2$ different frequencies for a single mode, where l is the angular degree of the mode. The framework allowed in addition to include the relative amplitudes of the modes, as the set of frequencies consists of different independent solutions in which the relative amplitudes are fixed.

In Chapter 3, we presented the magnetic field topology used in this thesis. The topology is chosen such that the magnetic field is contained within the radiative envelope of the star. The sensitivity of the oscillation modes to the magnetic field can be expressed by the magnetic kernel, which connects the different modes to the magnetic field tensor $\mathbf{BB}$.

We studied the effect of a misaligned magnetic field on the oscillation frequencies and amplitudes in Chapter 4. The sensitivity of the oscillation modes to the magnetic field is strongly dependent on the stellar model for low radial orders, with low order g -modes showing mixed mode behaviour for models close to the TAMS. The induced magnetic splittings increase with the square of the magnetic field strength, and are easily dominated by the effects of rotation for even weak rotational effects.

We finally fitted our model to the observed frequencies of HD 192575 in Chapter 5 for a grid of 600 stellar models, showing that the observed splittings can be explained by a misaligned magnetic field and rotation axis with properties similar to previous estimates and theoretical expectations.

6.1 Conclusions

The main conclusion of this thesis is the new interpretation of the observed frequency splittings of the g -mode of HD 192575, which we can explain as the combined effects of rotation and a misaligned magnetic field. In addition to the close fit of the g -mode, we can explain the observed frequencies and amplitudes of a p -mode as well. An independent analysis of the p -mode frequencies and amplitudes leads to similar parameter estimates as those of the g -mode. The magnetically split multiplet interpretation is utterly novel, with no previous case available in the literature for any main-sequence star.

Other explanation for the observed g -mode splittings would include very specific configuration of higher angular degree modes. Given the consistent explanation between different modes for the inclusion of a magnetic field, we claim that an internal magnetic field is the most plausible cause of the observed modes. This leads to this thesis presenting the first detection of an internal magnetic field in a main-sequence star.

The uncertainty of the identification of the g -mode leads to a predicted magnetic field sensitivity of either near-core or near-surface for mixed mode behaviour. This limits the mode for investigating the magnetic field topology. The derived magnetic field strengths necessary to explain the observed splittings are in the range of a few kG near the surface and tens of kG to hundreds of kG near the core. The derived rotation rates are similar to those found in previous studies. Considering the best fit for each stellar model, we find evidence that the implemented magnetic field topology decreases too weakly in strength with radius to simultaneously explain the g -mode and p -mode. Other topologies could be explored, such as the higher order fields presented in Chapter 3. These are, however, not dipole-like and increase the complexity of the topology.

This thesis describes the first asteroseismic modelling of an internal magnetic field in a massive star. The used theoretical framework allows for the inclusion of the relative amplitudes of the oscillation modes, which is normally excluded in asteroseismic modelling.

6.2 Outlook

The results in this thesis present many interesting avenues for future work. The most immediate follow-up would be to include additional oscillation modes of HD 192575 in the analysis. Especially additional g -modes of higher radial order would be of interest, as these are less likely to be possible mixed modes, leading to a more confident estimation of the near-core magnetic field strength. Fig. 6.1 shows a possible g -mode of HD 192575 that shows similar frequency splittings as the g -mode used in this thesis. These frequencies have not yet been identified or included in the modelling of HD 192575 and have as such been excluded from this study. The inclusion of additional g -modes with higher radial order would allow for an investigation in the internal magnetic field topology present in massive stars like HD 192575. Probing the internal magnetic field topology of HD 192575 would in addition benefit heavily from measurements of the surface magnetic field strength through spectropolarimetry, as this would give a boundary condition on the required magnetic field.

In addition to a higher order g -mode, more p -modes can be included in the analysis. This

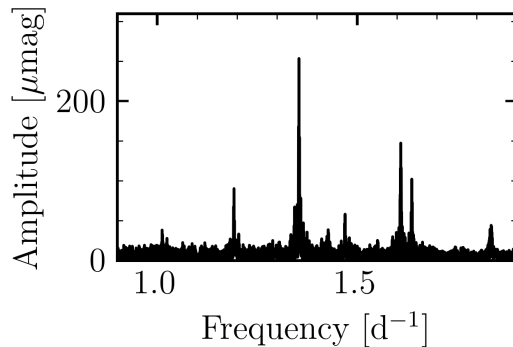


Figure 6.1: Higher order g -mode of HD 192575 showing interesting frequency splittings.

would narrow the range of both possible stellar parameters and magnetic field strengths. A simultaneous fit of different modes with similar probing regions would allow for a more precise estimation of the internal magnetic field strength and rotation rate, while still limiting the effect of the exact field topology on the splittings. The accuracy of the estimated obliquity and inclination angles would in addition also be improved with such a combined fit.

As internal magnetic fields are underexplored in massive stars, this work presents a method to study these internal magnetic fields. The model and methods presented in this thesis can be directly applied to other stars, even outside the massive star regime. While the model in its current form is flexible with respect to magnetic field topology and widely applicable, extensions are possible. For example, including the effects of rotational deformation will allow to consider stars with larger rotational frequencies. Similarly, the inclusion of limb-darkening on the amplitudes of the oscillation modes for the fitting could improve the accuracy of the derived parameters, especially the obliquity and inclination angle.

The two-step fitting procedure of first fitting the stellar models on the observed frequencies and using those derived parameters to fit the magnetic field and rotation parameters seems to be the most reasonable for current asteroseismic modelling techniques. The additional frequency shift induced by the magnetic field on the unperturbed frequencies is close to the theoretical uncertainties of the used stellar model grids. For stars with evidence of a strong magnetic field, a mitigation step of including a frequency shift on the unperturbed frequencies could be considered in the first step. However, this would require an additional parameter to be included in grids which are currently already computationally expensive to calculate and fit observations to.

This thesis has expanded the state-of-the-art by reporting on the first asteroseismic modelling of a magnetic field in a massive star. We have found strong evidence for an internal magnetic field and derived estimates of the magnetic field strength for different stellar models of the star. The magnetically split multiplet is a novel interpretation of the observed frequency splittings, which has not been reported in the literature before for any main-sequence star. The results presented in this thesis open the way for future asteroseismic studies of massive stars, which can include the effects of magnetic fields in addition to rotation.

Appendix A

Detailed matrix calculations

We calculate the matrix elements of the perturbation framework in more detail for Eq. (2.49). We will consider the normalization term which is present for both the rotation and magnetic field matrix, and the rotation matrix elements. Both can be reduced to a one-dimensional integral over the radius of the star, which is computationally more efficient than the full three-dimensional integral. A similar approach cannot be taken for the magnetic field matrix, as the θ -dependence cannot be separated.

A.1 Normalization term

We start by analyzing the normalization of the perturbation matrices

$$\langle \xi_0, \xi_0 \rangle = \int \rho \xi_0^* \cdot \xi_0 dV, \quad (\text{A.1})$$

where ξ_0 is the unperturbed eigenvector associated with the eigenfrequency ω_0 for chosen l and n . We have

$$\xi_0 = (\xi_r(r) Y_l^m(\theta, \phi)) \hat{\mathbf{e}}_r + \left(\xi_h(r) \frac{\partial Y_l^m}{\partial \theta} \right) \hat{\mathbf{e}}_\theta + \left(\frac{\xi_h(r)}{\sin \theta} \frac{\partial Y_l^m}{\partial \phi} \right) \hat{\mathbf{e}}_\phi,$$

as ξ_r and ξ_h depend on r and l only, and where the spherical harmonics can be written as $Y_l^m = c_{l,m} P_l^m(\cos \theta) e^{im\phi}$ with P_l^m the associated Legendre polynomials and $c_{l,m}$ a constant depending on l and m .

We can now calculate the norm of the eigenvector.

$$\begin{aligned} \langle \xi_0, \xi_0 \rangle &= \int \rho \left[\xi_r^2 Y_l^m (Y_l^m)^* + \xi_h^2 \left(\frac{\partial Y_l^m}{\partial \theta} \frac{\partial (Y_l^m)^*}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial (Y_l^m)^*}{\partial \phi} \right) \right] dV \\ &= \int \rho \left[\xi_r^2 Y_l^m (Y_l^m)^* + \xi_h^2 \left(\frac{\partial Y_l^m}{\partial \theta} \frac{\partial (Y_l^m)^*}{\partial \theta} + \frac{m^2}{\sin^2 \theta} Y_l^m (Y_l^m)^* \right) \right] dV \end{aligned}$$

As each spherical harmonic is multiplied with its complex conjugate, the ϕ -dependence vanishes and the integral over ϕ gives a factor 2π . We will now consider the integrals over θ separately.

First, we consider

$$\begin{aligned} \int_0^\pi (Y_l^m)^* Y_l^m \sin \theta d\theta &= \int_0^\pi c_{l,m}^2 (P_l^m(\cos \theta))^2 \sin \theta d\theta \\ &= c_{l,m}^2 \int_{-1}^1 (P_l^m(x))^2 dx \\ &= \frac{1}{2\pi}. \end{aligned}$$

Second, we have the integral

$$\begin{aligned} \int_0^\pi \left(\frac{\partial Y_l^m}{\partial \theta} \frac{\partial (Y_l^m)^*}{\partial \theta} + \frac{m^2}{\sin^2 \theta} Y_l^m (Y_l^m)^* \right) \sin \theta d\theta \\ &= c_{l,m}^2 \int_0^\pi \left(\left(\frac{dP_l^m(\cos \theta)}{d\theta} \right)^2 + \frac{m^2}{\sin^2 \theta} P_l^m(\cos \theta)^2 \right) d\theta \\ &= -c_{l,m}^2 \int_0^\pi P_l^m(\cos \theta) \left[\frac{d}{d\theta} \left(\sin \theta \frac{dP_l^m(\cos \theta)}{d\theta} \right) - \frac{m^2}{\sin \theta} P_l^m(\cos \theta) \right] d\theta \\ &= c_{l,m}^2 l(l+1) \int_{-1}^1 P_l^m(x)^2 dx \\ &= \frac{l(l+1)}{2\pi}. \end{aligned}$$

Combining this with the expression above, we find

$$\langle \xi_0, \xi_0 \rangle = \int_0^{R_{\text{star}}} \rho (\xi_r^2 + l(l+1) \xi_h^2) r^2 dr, \quad (\text{A.2})$$

where R_{star} is the radius of the star.

A.2 Rotation matrix

We now consider the rotation matrix elements. To this end, we use the expression of the rotation perturbation as given in Eq. (2.37) to calculate

$$\langle \mathcal{F}_{\text{rot}} \xi_{0,\mathbf{m}}, \xi_{0,\mathbf{m}} \rangle = -2i\omega_0 \int \rho (\boldsymbol{\Omega} \times \xi_{0,\mathbf{m}})^* \cdot \xi_{0,\mathbf{m}} dV, \quad (\text{A.3})$$

where we have that

$$\boldsymbol{\Omega} \times \xi_{0,\mathbf{m}} = im\Omega \left[-\xi_h Y_l^m \mathbf{e}_r - \xi_h \frac{\cos \theta}{\sin \theta} Y_l^m \mathbf{e}_\theta + \left(\xi_r Y_l^m \sin \theta + \xi_h \frac{\partial Y_l^m}{\partial \theta} \cos \theta \right) \mathbf{e}_\phi \right]. \quad (\text{A.4})$$

As such, we find that

$$\begin{aligned}
\langle \mathcal{F}_{\text{rot}} \xi_{\mathbf{0}, \mathbf{m}}, \xi_{\mathbf{0}, \mathbf{m}} \rangle &= -2i\omega_0 \Omega \int 2im\rho \left[\xi_h \xi_r (Y_l^m)^* Y_l^m + (\xi_h)^2 \frac{\cos \theta}{\sin \theta} (Y_l^m)^* \frac{\partial Y_l^m}{\partial \theta} \right] dV \\
&= 8\pi m \omega_0 \Omega \int \rho \left[\xi_h \xi_r (Y_l^m)^* Y_l^m + (\xi_h)^2 \frac{\cos \theta}{\sin \theta} (Y_l^m)^* \frac{\partial Y_l^m}{\partial \theta} \right] r^2 \sin \theta dr d\theta,
\end{aligned}$$

where we integrated out the ϕ -dependence. This can be done as the ϕ -dependence of the spherical harmonics is proportional to $e^{im\phi}$. As each spherical harmonic is multiplied by its complex conjugate, the ϕ -dependence vanishes and the integral over ϕ gives a factor 2π . The remaining terms in the integral are such that they have sole dependence on r or θ , with no term having a dependence on both. This allows us to split the integral as follows

$$\begin{aligned}
\langle \mathcal{F}_{\text{rot}} \xi_{\mathbf{0}, \mathbf{m}}, \xi_{\mathbf{0}, \mathbf{m}} \rangle &= 8\pi m \omega_0 \Omega \left[\int_0^{R_{\text{star}}} \rho \xi_h \xi_r r^2 dr \int_0^\pi (Y_l^m)^* Y_l^m \sin \theta d\theta + \right. \\
&\quad \left. \int_0^{R_{\text{star}}} \rho (\xi_h)^2 r^2 dr \int_0^\pi \frac{\cos \theta}{\sin \theta} (Y_l^m)^* \frac{\partial Y_l^m}{\partial \theta} \sin \theta d\theta \right],
\end{aligned}$$

where R_{star} is the radius of the star.

Both integrals over θ can be solved analytically. To ease the notation, we rewrite the spherical harmonics as their associated Legendre polynomials with a constant factor $c_{l,m}$ depending on l and m . This gives $Y_l^m(\theta) = c_{l,m} P_l^m(\cos \theta)$, where we have dropped the dependence on the azimuthal angle. The first integral is already solved above, and we approach the second integral similarly.

$$\begin{aligned}
\int_0^\pi \frac{\cos \theta}{\sin \theta} (Y_l^m)^* \frac{\partial Y_l^m}{\partial \theta} \sin \theta d\theta &= -c_{l,m}^2 \int_{-1}^1 P_l^m(x) \frac{dP_l^m}{dx} x dx \\
&= -c_{l,m}^2 \frac{1}{2} \int_{-1}^1 \frac{d(P_l^m)^2}{dx} x dx \\
&= -c_{l,m}^2 \left[\frac{1}{2} x P_l^m(x)^2 \right]_{-1}^1 + c_{l,m}^2 \frac{1}{2} \int_{-1}^1 P_l^m(x)^2 dx \\
&= \frac{1}{4\pi},
\end{aligned}$$

where we integrated by parts in the third line. As the associated Legendre polynomials are all symmetric or antisymmetric, the square is guaranteed to be symmetric. The first term on the third line thus vanishes.

We can now combine both results and obtain

$$\langle \mathcal{F}_{\text{rot}} \xi_{\mathbf{0}, \mathbf{m}}, \xi_{\mathbf{0}, \mathbf{m}} \rangle = 2m\omega_0 \Omega \int_0^{R_{\text{star}}} \rho (2\xi_h \xi_r + (\xi_h)^2) r^2 dr. \quad (\text{A.5})$$

Appendix B

Wigner d -matrix

The Wigner d -matrix $\mathbf{D}$ is used to transform between bases of spherical harmonics. We consider it for the transformation between $(m, \theta, \phi) \rightarrow (m', \theta', \phi')$, which are separated by an obliquity angle β . Its elements are given by (following the notation of [Loi, 2021](#)):

$$d_{mm'}^{(l)}(\beta) = [(l+m)!(l-m)!(l+m')!(l-m')!]^{1/2} \times \sum_s \left[\frac{(-1)^{m-m'+s} \left(\cos \frac{\beta}{2}\right)^{2l+m'-m-2s} \left(\sin \frac{\beta}{2}\right)^{m-m'+2s}}{(l+m'-s)!s!(m-m'+s)!(l-m-s)!} \right], \quad (\text{B.1})$$

where s is such that the factorials are positive. The matrix is such that the transformation of spherical harmonics between two reference frames can be written as

$$Y_l^m(\theta, \phi) = \sum_{m'} d_{mm'}^{(l)}(\beta) Y_l^{m'}(\theta', \phi'). \quad (\text{B.2})$$

Appendix C

Asymmerty scaling factors

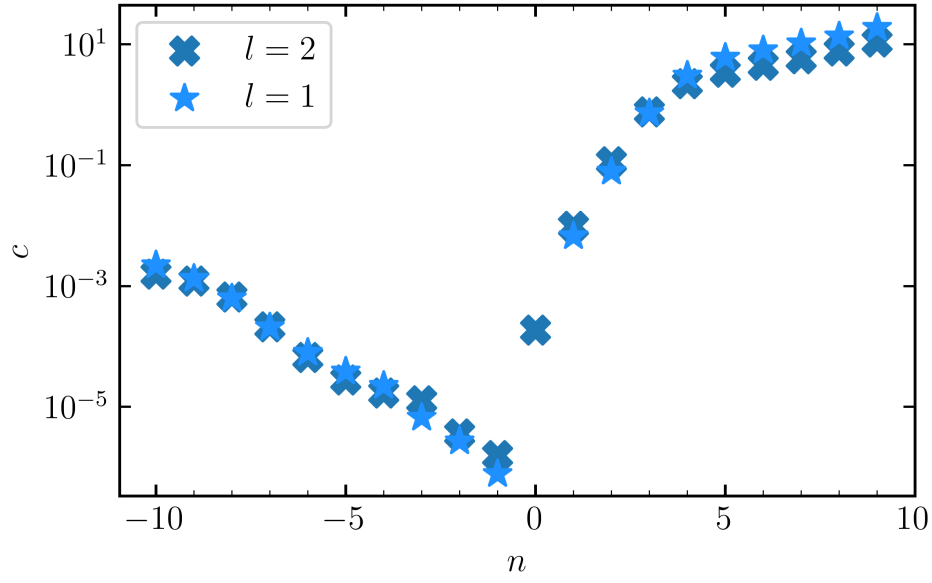


Figure C.1: Asymmetry scaling factor c for dipole and quadrupole oscillation modes of different radial order n for the stellar model of $15 M_{\odot}$ and a central hydrogen fraction $X_c = 0.7$.

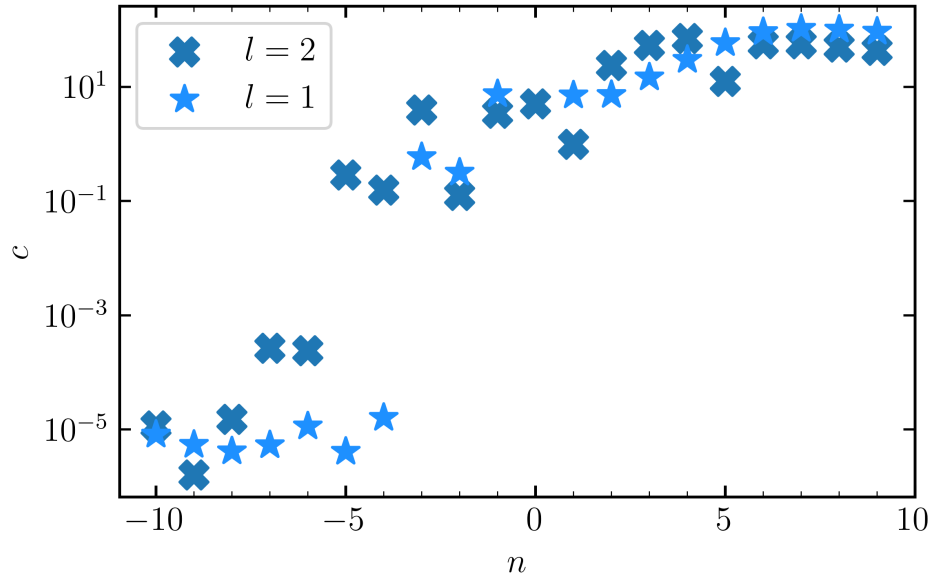


Figure C.2: Asymmetry scaling factor c for dipole and quadrupole oscillation modes of different radial order n for the stellar model of $15 M_{\odot}$ and a central hydrogen fraction $X_c = 2 \cdot 10^{-4}$.

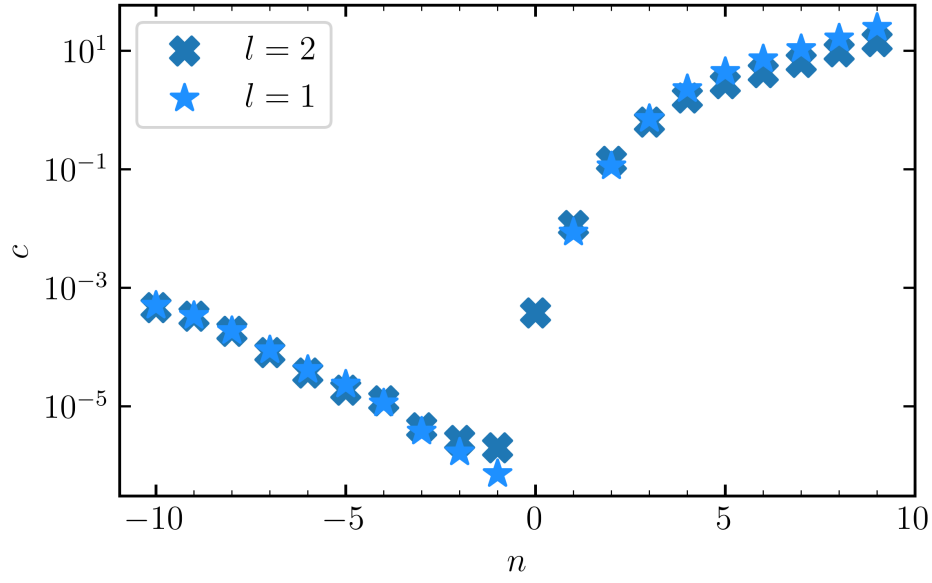


Figure C.3: Asymmetry scaling factor c for dipole and quadrupole oscillation modes of different radial order n for the stellar model of $8 M_{\odot}$ and a central hydrogen fraction $X_c = 0.7$.

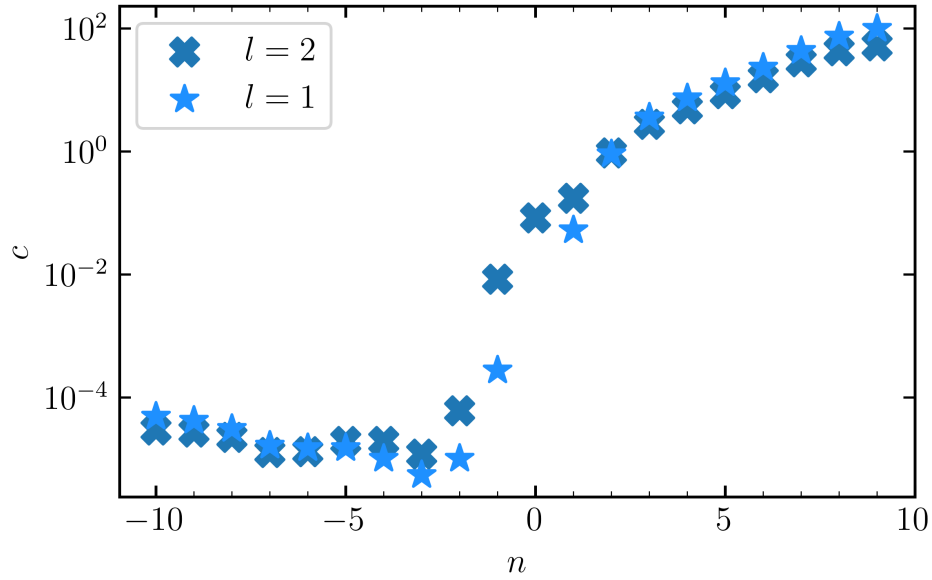


Figure C.4: Asymmetry scaling factor c for dipole and quadrupole oscillation modes of different radial order n for the stellar model of $8 M_{\odot}$ and a central hydrogen fraction $X_c = 0.35$.

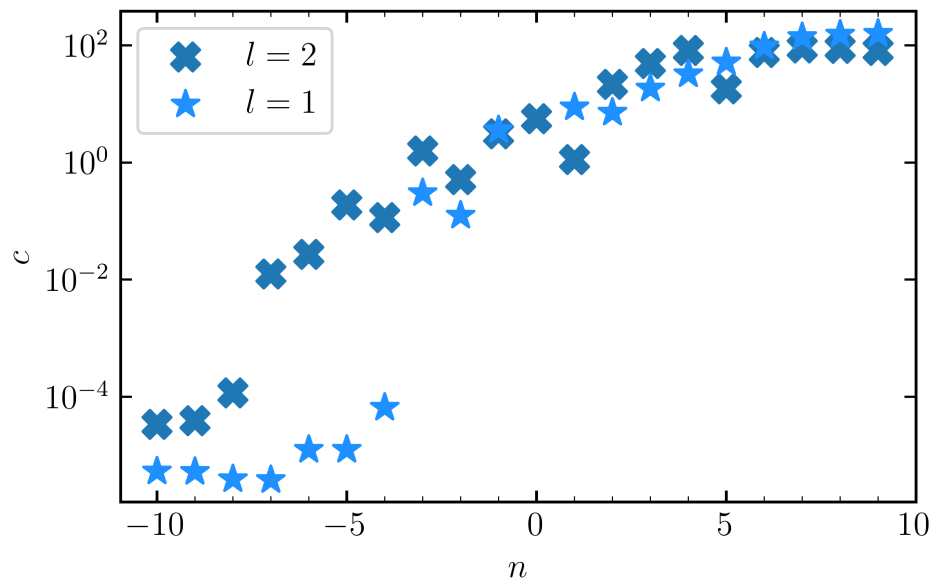


Figure C.5: Asymmetry scaling factor c for dipole and quadrupole oscillation modes of different radial order n for the stellar model of $8 M_{\odot}$ and a central hydrogen fraction $X_c = 2 \cdot 10^{-4}$.

Appendix D

$8\text{ M}_{\odot}$ small splittings

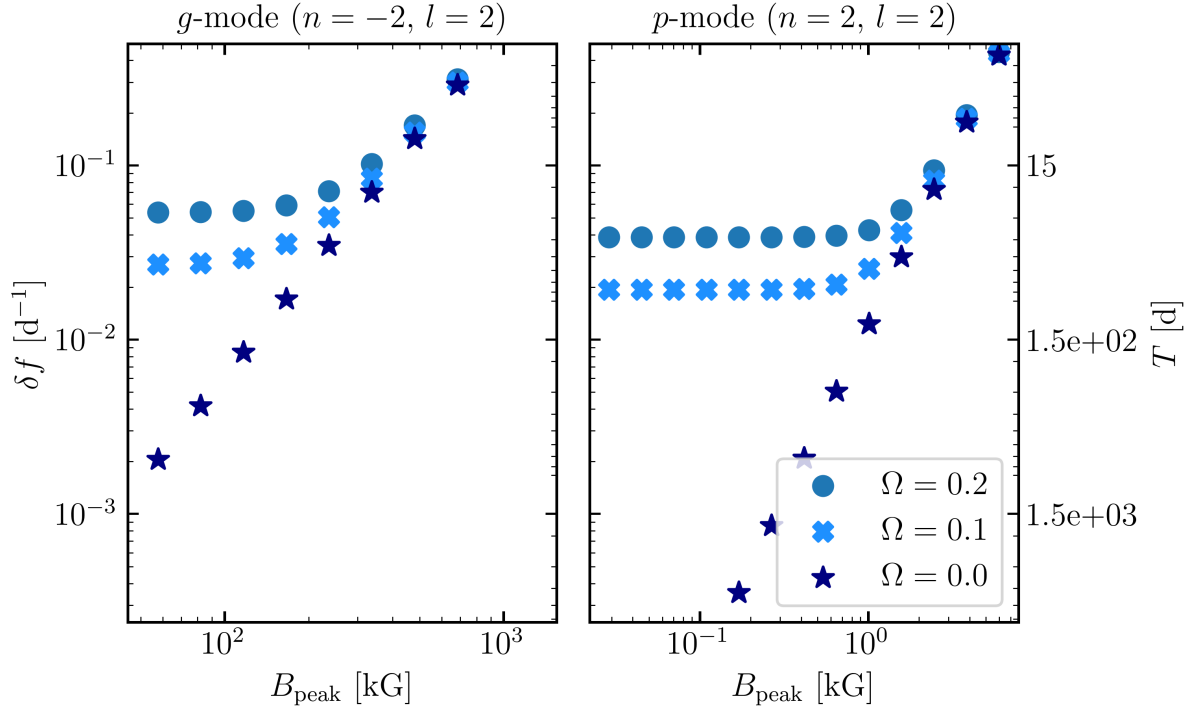


Figure D.1: Mean small splitting in a misaligned ($\beta = 3\pi/16$) magnetic field for a g -mode ($n = -2, l = 2$) and a p -mode ($n = 2, l = 2$) of an $8 M_{\odot}$ star with a central hydrogen fraction of $X_c = 0.3$ for three different rotational frequencies.

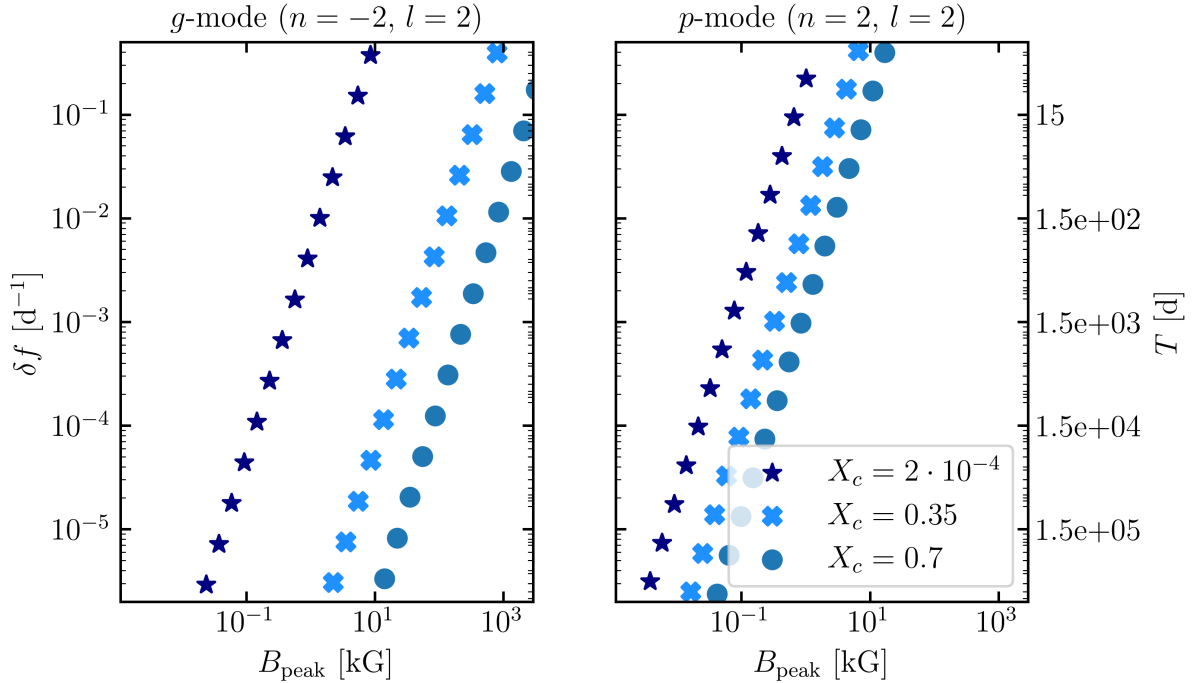


Figure D.2: Mean small splitting in a misaligned ($\beta = 3\pi/16$) magnetic field for a g -mode ($n = -2, l = 2$) and a p -mode ($n = 2, l = 2$) of an $8 M_{\odot}$ star for three different instances of the central helium fraction X_c .

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DEPARTMENT OF PHYSICS AND ASTRONOMY

Celestijnenlaan 200d bus 2412

3001 LEUVEN, BELGIË

tel. + 32 16 32 71 24

fax + 32 16 32 71 24

www.kuleuven.be

