

Observational signatures of higher-curvature effects in black hole ringdown

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Preface

Looking back on the past year, it is the moments that were most troubling that I now hold closest. Twice during my thesis, we had a scooping scare (the scientific equivalent of being afraid not to be first at the finish line). Twice was this scare followed by a two-week race. Looking back, it was exactly these hardships which formed me most as a scientist. But this race was never run alone. There are many people I have to thank for their guidance, support, and patience.

First and foremost, to my supervisor Tjonnje, many thanks for the support over the last year. You have been a mentor, navigating me through the mysterious world of gravitational wave science. Many thanks also to Lionel, who taught me how to do research. I really learned a lot from you, and not just about physics. Associated with this I am grateful to all the UK scientists I got to meet during my internship in London for giving me such a welcoming atmosphere.

I want to thank Simon and Tom for always being willing to help me out. They were always there for me when I had another stupid question. I am grateful for the opportunities you gave me, and for Simon to invite me to visit the Niels Bohr Institute in Copenhagen. Also a big thank you to all the collaborators on the papers we wrote in the context of this master thesis: Adrian, Kelvin, Pedro, Maxim, Nico, Thomas and Vitor. Your valuable feedback forced me to really understand my own work.

I am thankful for the welcoming atmosphere of the gravitational wave research group. The seminars, journal clubs and discussions were the place where I really learnt gravitational wave physics. Also a thank you to Daniel, Annelien and everyone of the Einstein Telescope Roadshow for teaching me the craft of science communication, my new favourite hobby. I am also really grateful for my readers Marina and Eva, for taking the time to read my thesis.

To my fellow students and friends, both in Leuven and Antwerp, thank you for making my time as a student wonderful. We made many unforgettable memories. A special thanks to Lander, for all the study sessions together. Without our calls, I might not have kept a sane mind.

To my partner and family, perhaps the biggest thank you of all. Thank you for always being there for me, supporting me throughout the years, and for letting my head rest from physics every once in a while. I am really grateful for that, and I might not have been able to do it without you.

Contribution Statement

Chapters 3 and 4 contain the original work of the student. Chapters 1 and 2 introduce the reader to the literature, motivate and situate the results in the context of state-of-the-art research.

The pseudospectral framework of Chapter 3 was developed in part in [1]. Together with Tom van der Steen and Simon Maenaut, this framework was adapted and also generalised to work for any higher-curvature theory, with the associated equations of motion we generalised. We also calculated the modified boundary conditions. The code was based on a code written earlier by Maxim Jockwer, which could solve for QNMs in GR with the pseudospectral method. This code was adapted, expanded and optimised to calculate the modes of interest. All calculations done and interpretations given in the remainder of this chapter have been done by us [2, 3]. The idea of the phase boundary being the origin of the amplification was mentioned in earlier work, but was expanded and verified here [4]. The results of Chapter 3 have been submitted for publication and are available as a preprint [2, 3].

Chapter 4 is based on the frameworks of [5, 6] and `pyRing` [7]. The results are obtained within this `pyRing` infrastructure. This `pyRing` codebase was extended and adapted to include the scalar EFT QNM corrections calculated in Chapter 3, and with the assumptions that come with the scalar proxy [8]. These adaptations and all the analysis runs have been done by the author. The author acknowledges support from Tom van der Steen and Simon Maenaut in developing the necessary tools.

The author is grateful for the computing resources of the Tycho supercomputer hosted at the SCIENCE HPC center at the University of Copenhagen and the local HPC cluster of the department of physics and astronomy at the KU Leuven.

Summary

In 1916 Albert Einstein published his theory of general relativity (GR), that aimed to describe gravity and the universe at its most fundamental level. It predicted the existence of gravitational waves and black holes, that have since been observationally confirmed. GR describes space and time as a single geometric continuum and gravity as the curvature of this geometry. It has been confirmed in many experiments since it was first described.

Despite this overwhelming experimental evidence for GR, it is not expected to be a completely correct description of our universe. There are multiple theoretical shortcomings of the theory. GR is a classical theory and therefore not reconcilable with quantum mechanics. A complete theory should unify quantum mechanics with GR. No such theory has yet been found, despite many proposals. Apart from this, GR predicts the existence of black holes, which have singularities: points where the curvature becomes infinite. At this point GR itself (which predicted these singularities in the first place) breaks down. Therefore much research is being done to observationally test GR and to look for signatures beyond GR. The recent detection of gravitational waves (GW) has opened a new avenue to do so.

Quasinormal modes, the characteristic oscillations of a perturbed black hole, are one of the best probes to test GR in its high-curvature regime. Measuring these frequencies allows one to test the Kerr hypothesis and constrain deviations from GR. In this study we take the approach of effective field theory (EFT). We build up effective theories that incorporate higher-order corrections that are suppressed by an energy/length scale. We remain within the lowest-order modifications of GR approach.

In this thesis, we calculate the scalar quasinormal mode spectrum of different alternative theories of gravity: scalar Gauss-Bonnet, dynamical Chern-Simons, axi-dilaton gravity, cubic even and cubic odd gravity. In doing so, we expand earlier results obtained for low and intermediate black hole spins. Rapidly rotating black holes are proven to amplify beyond GR effects, originating in a shift in a phase boundary in the phase space of quasinormal modes between zero damped modes and damped modes. This effect is shown to be spin-weight and theory independent.

These theoretical spectra are implemented into a Bayesian statistical pipeline to analyse observational data. We use the scalar quasinormal mode corrections as a proxy for gravitational modes. We apply this pipeline to constrain EFT deviations from GR, finding results that are consistent with GR. We study the accuracy of a scalar proxy, finding that scalar proxies are sufficient for rough estimates and prospect studies only.

Popularising Summary

110 years ago, Einstein gave us a new way to understand gravity: general relativity. Where Newton saw gravity as a force pulling objects together, Einstein saw it as the curvature of space and time itself. This theory revolutionised how we look at the universe, describing gravity from falling apples to the largest cosmic scales, and predicted the existence of black holes and gravitational waves, both of which have since been observed. In the following 110 years, Einstein's theory passed every experimental test thrown at it.

However, despite this overwhelming evidence, Einstein's general relativity is not the end of the story for gravity. It predicts singularities, points in space(time) where physical quantities may become infinite, breaking our understanding. Even more fundamentally, general relativity is not compatible with quantum mechanics: our description of the universe at its smallest scales. A complete theory of everything should reconcile the two, but no such theory has been found yet. We should thus test Einstein's theory with more precision and come up with new ways to test it at its most extreme regimes. Gravitational waves, the stretching and squeezing of space and time, offer such a new test.

One of the best ways to test gravity in such regimes is through quasinormal modes. These are the characteristic oscillations of a perturbed black hole. Like a struck bell, a black hole radiates energy at a set of characteristic frequencies. The frequencies with which a bell rings reveal to us the shape of this bell: you can hear its shape. How you hit the bell does not change its frequencies, only how loud the sound is. Exactly the same happens with black holes: if something strikes the black hole, it sends out a characteristic set of frequencies (tones) that describe its 'shape'. Such ringing black holes develop when two black holes collide, forming a single remnant black hole. We can observe this ringing through gravitational waves: we can quite literally hear the shape of black holes.

In this thesis, we calculate these frequencies in alternatives of general relativity, studying how these characteristic modes change in other theories. Rather than looking for a complete theory of everything, which lies far from observations, we take the observationally relevant approach of looking at the first small deviations to general relativity. We see that small black holes that rotate very fast are the best candidates for measuring deviations of Einstein's theory within this setting. We then take these theoretical predictions and put them to test in actual gravitational wave data. For now, general relativity survives yet another test. However, the future generation of gravitational wave detectors may change this. The Einstein Telescope, one of these next-generation detectors, may be built here at the tripoint between Belgium, Germany and the Netherlands. It will observe gravitational waves with unprecedented accuracy. With a little luck, it will be built in our backyard and with it we may finally hear something that Einstein's theory cannot explain.

List of Abbreviations

GR	general relativity
GW	gravitational wave
QNM	quasinormal mode
QNF	quasinormal frequency
EFT	effective field theory
EFE	Einstein field equations
QFT	quantum field theory
BH	black hole
EH	event horizon
EHT	Event Horizon Telescope
PN	post-Newtonian
NR	numerical relativity
NS	neutron star
EM	electromagnetic
LIGO	Laser Interferometer Gravitational-Wave Observatory
ET	Einstein Telescope
PTA	Pulsar Timing Array
LVK	LIGO-Virgo-KAGRA
NP	Newman-Penrose
KG	Klein-Gordon
ODE	ordinary differential equation
WKB	Wentzel-Kramers-Brillouin
SM	Standard Model

UV ultraviolet

sGB scalar Gauss-Bonnet

dCS dynamical Chern-Simons

PSD power spectral density

SNR signal-to-noise ratio

MCMC Markov chain Monte Carlo

ZDM zero damped mode(s)

DM damped mode(s)

List of Symbols

$g_{\mu\nu}$	metric
g	determinant of metric
$\eta_{\mu\nu}$	Minkowski metric
$h_{\mu\nu}$	linearised perturbation around metric
$R_{\mu\nu}$	Ricci tensor
R	Ricci scalar
$R_{\mu\nu\rho\sigma}$	Riemann tensor
$\tilde{R}_{\mu\nu\rho\sigma}$	dual Riemann tensor
$\mathcal{C}$	Kretschmann scalar
$\tilde{\mathcal{C}}$	Chern-Pontryagin scalar
G	gravitational constant (=1)
M	mass (of black hole)
a	spin of the black hole
v	velocity
c	speed of light (=1)
$T_{\mu\nu}$	energy-momentum tensor
S	action
$\mathcal{L}$	Lagrangian
$\mathcal{M}$	(differentiable) manifold
$T_p\mathcal{M}$	tangent space at point p of manifold
$\Gamma_{\alpha\beta}^{\mu}$	Christoffel symbol
ds^2	proper time/metric
n	noise
ω	quasinormal frequency
$\tilde{\omega}$	real part of quasinormal frequency
τ	damping time of frequency
A_{lm}	GR separation constant
A_{lmn}	QNM amplitude
$\omega_{(0)}$	quasinormal frequency in General Relativity (GR)
$\omega_{(1)}$	first-order shift of quasinormal frequency in beyond GR
λ	dimensionless coupling of beyond GR theory
ℓ	dimensionful coupling of beyond GR theory, independent of event, $\lambda = \frac{\ell^4}{M^4}$
H_i	deviations of beyond GR metric from GR metric
$\psi_0, \dots, \psi_4$	Weyl scalars
$\mathcal{T}$	Teukolsky operator
l	orbital angular momentum index
m	azimuthal angular momentum index

s spin weight
 r radial (Boyer-Lindquist) coordinate
 t time coordinate
 θ polar angle
 x cosine of polar angle $\cos \theta$
 ϕ azimuthal angle
 z in Section 2.1.4 the z coordinate, otherwise a hyperboloidally compactified radial coordinate
 $r_{\pm}$ inner $(-)$ or event horizon $(+)$
 ρ radial coordinate in first-order beyond GR Boyer-Lindquist coordinates
 Σ $\rho^2 + a^2 x^2$
 Δ $\rho^2 - 2M\rho + a^2$
 σ standard deviation
 ϑ parameters in Bayesian statistics
 d data in Bayesian statistics
 $\mathcal{B}$ Bayes factor
 C_k covariant noise matrix in detector k
 L likelihood
 $\mathcal{N}$ normalisation of statistical distribution
 π prior
 $\square$ d'Alembertian operator
 $\mathcal{F}$ Fourier transform
 $S_{lm}^{a\omega}(\theta)$ spin-weighted spheroidal harmonic
 ψ perturbation field
 $\psi_{(0)}$ perturbation field solution in GR
 ψ_1 first-order modification of perturbation field beyond GR
 $\mathcal{D}_{m,\omega}^2$ Laplace operator as the Fourier transformed Klein-Gordon operator
 $\mathcal{D}_{(0)}^2$ zeroth-order GR operator $\mathcal{D}^2$
 $\mathcal{D}_{(1)}^2$ first-order beyond GR modification to operator $\mathcal{D}^2$
 α_{ω} regularising function of Teukolsky equation
 α_0 regularising function in GR
 α_1 lowest-order beyond GR modification to regularising function
 $\sim$ denotes that an operator or wavefunction has been regularised
 β ratio between beyond GR modification to regularising function and regularising function in GR
 Θ discretised angular GR (QNM) wavefunction
 R discretised radial (QNM) GR wavefunction
 $\tilde{\psi}_{(0)}$ discretised spatial GR (QNM) wavefunction
 $\tilde{\psi}_{(1)}$ discretised lowest-order beyond GR spatial QNM modification
 N_r number of radial grid points (minus one)
 N_{θ} number of angular grid points (minus one)
 $\bar{\mu}_{cr}$ quantity describing phase boundary between ZDMs and DMs
 $M_{\odot}$ solar mass

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Chapter 1

Motivation

The dawn of gravitational wave (GW) astronomy has opened an entirely new window onto the strong-field regime of gravity [9, 10]. General relativity (GR) (our current description of gravity) has passed every experimental test put to it, so far. Yet there are theoretical reasons to suspect it cannot be the end of the story. Among the many proposed tests of its validity, the ringdown of perturbed black holes, characterised by a discrete spectrum of damped oscillation frequencies, has turned out to be one of the most promising [11–27]. This thesis investigates how these frequencies are modified in theories beyond GR, and how such deviations can be extracted from observational data.

This is done in three steps, corresponding to Chapters 2, 3, and 4. Firstly, one should be familiar with the necessary background, in order to situate and contextualise the research in the state-of-the-art of GW astronomy. Chapter 2 introduces the basic frameworks underlying the research done in this thesis. This includes GR and GW astronomy in Section 2.1, black hole ringdown in Section 2.2, beyond GR in Section 2.3, and statistical data analysis in Section 2.4.

Before we can study how these deviations are extracted from observational data, we need a theoretical prediction of how modifications to GR modify the characteristic frequencies of black holes. These theoretical calculations are the second step and will be done in Chapter 3. We call these characteristic modes ‘quasinormal modes’ (QNMs, see Section 2.2). One could do an agnostic test where one simply tests if the observations are consistent with GR. We will use the approach where our search is theory specific [22–25]. This thesis will study the lowest-order correction to these frequencies: as all current tests of GR have measured no deviations, the deviations are expected to be small. As a consequence only the lowest-order correction to GR is taken into account, which is formalised in effective field theory (EFT), see Section 2.3. Some prior work has been done in computing these QNM modifications [28–54]. The calculation of QNMs in EFT-motivated theories of gravity was first limited to non-rotating black holes, after which results were obtained linear or quadratic in spin of the black hole [2, 3]. QNM modifications for rapidly rotating black holes are difficult because no exact solution of the metric (see Section 2.1.1) exists. By solving for the black hole background solution as an expansion in the spin of the black hole, results had been obtained for black holes of spins up to $a \approx 0.85M$ [50, 55–60]. Around the starting point of the thesis, the first numerical solutions to the black hole geometries in EFT theories came out [61–63]. This opened the opportunity to calculate the QNM spectrum for rapidly rotating black holes. Earlier studies of QNMs in rapidly

rotating black holes existed in the eikonal limit ($l \rightarrow \infty$, see Section 2.2.4) [4, 64].

Astrophysical black hole remnants (see Sections 2.1.3 and 2.1.5) are predicted to have rather high spin and furthermore it has been postulated that near-extremal black holes are observationally interesting objects to study, further motivating the need to have QNM spectra beyond GR for such high spins [4, 10, 65–67]. Employing pseudospectral methods [1, 68–71], this thesis calculates scalar QNMs in many EFT beyond GR scenarios for spins up to $a \approx 0.99M$, extending the understood spin range. By comparing the numerical results with the spin expansion results obtained earlier, we find that the spin expansion breaks down around $a \approx 0.7M$ for $\mathcal{O}(a^{14})$.

Strictly speaking we calculate scalar QNMs in this thesis, which are not the gravitational QNMs. Gravitational QNMs are tensorial, with spin weight $s = \pm 2$. As a consequence, they exhibit different dynamics as they backreact and thus have different frequencies. The frequencies of scalar QNMs and gravitational QNMs coincide only in the eikonal limit [25]. However, recent research has shown that scalar QNMs can be a proxy for gravitational QNMs [8]. For current detectors, using the QNM shifts of scalar QNMs was mentioned to provide a sufficiently good approximation for the gravitational QNM shifts, at the very least for rough estimates. This approximation will be revisited in Chapter 4. The QNM modifications obtained in this thesis can thus directly be used as input for the GW data analysis, as done in the last part of the thesis. Gravitational QNMs are generally more difficult to calculate and the physical interpretation to the results given in Section 3.4 is spin-weight independent, justifying restricting ourselves to scalar QNMs.

The third and final step is to test these theoretical predictions on observational data, done in Chapter 4. Data of binary black hole mergers obtained by the LIGO-Virgo-KAGRA (LVK) collaboration gives the opportunity to test beyond GR with unprecedented precision [10–16, 72, 73]. To do this, we implement the theoretical predictions into the existing data analysis pipelines and test the data for deviations from GR. This results in a posterior (statistical distribution, see Section 2.4) for the beyond GR coupling parameter. This third part of the project will be explained in more detail in Chapter 4.

As the number of GW observations increases, alongside the accuracy of these detections, precision tests of GR are becoming more stringent [5, 6, 11, 13, 15, 73–77]. By measuring multiple QNM frequencies of a black hole remnant, one can test the Kerr hypothesis. The recent detection GW250114 was the loudest event to date, being the first one with a statistically confirmed presence of an overtone [11]. Such an observation allows for consistency tests. In this thesis this detection will be used to test the obtained results. We can constrain the beyond GR coupling parameters for the different tested theories. A similar analysis has been done previously on GWTC-3, the then-current LVK catalogue [5, 6]. However, this was done with beyond GR QNM shifts available only for a limited range of spins. Therefore detections with high remnant spin could not reliably be used to probe higher-curvature corrections to the QNM spectrum. An analysis had been done, using these (limited in spin) results, for dynamical Chern-Simons (dCS), scalar Gauss-Bonnet (sGB), cubic and quartic gravity [5, 6]. This analysis is now extended to the most recent noteworthy GW events and the GW events with the highest remnant spin, as these are assumed to have the largest beyond GR corrections and were previously untestable.

Chapter 2

Introduction

“If you wish to make an apple pie from scratch, you must first invent the universe.”

— Carl Sagan

The study of quasinormal modes (QNMs) in theories beyond general relativity (GR) requires a detailed understanding of the theories in which they ‘first’ emerge. In this spirit, it is necessary to start from the beginning, as Carl Sagan remarked [78]. Quasinormal modes are defined in classical GR and gravitational waves (GW) provide the observational context. Any extension of GR inherits and modifies these concepts. For this reason, this thesis starts with a brief overview of general relativity and gravitational wave theory. Only then does it make sense to turn to extensions of the theory and study the impact on observables. This introduction is necessary to understand the motivation towards extensions of GR, QNMs and the observational relevance through detectors like LIGO, Virgo and in the future ET and LISA. After this overview of GR and GWs, the physics and mathematics behind quasinormal modes, characteristic damped oscillations of compact objects like black holes, will be explained. The next part of the introduction aims to explain why the concept of quantum gravity tells us that GR is not the end of the story. Without going into the details of high energy quantum gravity, like e.g. string theory, this thesis will focus on taking an EFT approach to extensions of GR. The last part of the introduction breaks down the framework of studying observations of quasinormal modes. Here the reader becomes familiar with the Bayesian statistical methods that are used in the detection of quasinormal modes and that will play a central role in constraining beyond GR effects. If the reader is already familiar with certain topics, they are free to omit these parts of the introduction.

2.1 General Relativity and Gravitational Waves

2.1.1 The Essence of General Relativity

Ever since Newton it had been known that two masses attract each other. This force was called gravity. His formulation was rather simple and very successful. It was able to predict the orbits of planets and moons. Newton said that the gravitational force was given by $\vec{F} = \frac{GMm}{r^2}\vec{e}_r$. Here G is the gravitational constant, which gives a measure of the strength of gravity. It is remarkable that the attraction is universal. Every two bodies attract each other [79]. Up until the 20th century, this was the general consensus. By

then, some experiments, observations, and conceptual difficulties challenged the theory of Newton and electromagnetism [80]. In 1916 Einstein came up with a solution: General Relativity (GR) [81, 82].

As of today, GR stands as one of the most successful and fundamental theories of physics. The premise of GR is that one should treat space and time as one. It describes gravity as a geometric property of space and time. Space and time combine in four-dimensional spacetime. The curving of spacetime is what we observe as gravity [81]. It is often said that ‘spacetime tells matter how to move, matter tells spacetime how to curve’¹. A common intuitive analogy is the trampoline. If one puts a bowling ball on the trampoline, it is going to curve. If you then put a tennis ball on this trampoline, the tennis ball will spiral down towards the bowling ball: it is the curvature of the trampoline which ‘forces’ the balls together. Therefore one should not think of gravity as a force. Rather the curving of spacetime changes the trajectories of the masses, and this curvature is generated by masses. This purely geometrical insight can be made rigorous through differential geometry [81, 83]. Here, spacetime is a differentiable manifold, which is described through an object called the metric. The link between curvature and mass is encoded in the central equation of GR: the Einstein field equations (EFE):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (2.1)$$

The left-hand side of this equation describes the curvature and metric of spacetime. $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar and $g_{\mu\nu}$ the metric. The metric $g_{\mu\nu}$ is the central object in this theory and indirectly of the entire thesis. Both the Ricci tensor and the Ricci scalar are a non-linear function of the metric and its first and second derivatives [81]. In flat vacuum space, also called Minkowski space, the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-, +, +, +)$. In some textbooks (on e.g. quantum field theory (QFT)) another convention will be used where $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. This change in signature is just a matter of convention and in this thesis we will use the mostly plus convention $(-, +, +, +)$ [81, 83, 84].

The right-hand side of this equation denotes the matter part. G is the gravitational constant, c the speed of light and $T_{\mu\nu}$ the energy-momentum tensor. This tensor describes the presence of matter fields. In this EFE no cosmological constant has been included. It can however be absorbed in the energy-momentum tensor on the right-hand side. In the rest of the thesis we will be working with natural units. This means we will use units that are such that $G = c = 1$ [81]. The energy-momentum tensor is now given by:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \quad (2.2)$$

Here g is the determinant of the metric and S_{matter} the action of the matter fields. The derivative is strictly speaking a functional derivative [81]. The EFE described above are the central part of GR. Note that the equation is highly non-linear. This makes finding general solutions hard. This equation describes the geometric structure of spacetime: it determines the metric. Eq. (2.1) is the embodiment that matter tells spacetime how to curve. The reverse statement that curvature tells matter how to move, is encapsulated

¹This is a quote from J.A. Wheeler [83].

in the geodesic equations [83, 84]. Simply stated one can say that any object will always try to take the shortest path, the particle is in free fall (when no forces act on it). In the absence of forces or gravity, this shortest path is a straight line. This is the first law of Newton. When however gravity is at play, the universe becomes curved and the shortest path may no longer be a straight line. The shortest path is then given by what one calls a geodesic. In the absence of forces, matter will follow geodesics, the generalisation of straight lines. Note that straight lines are also geodesics, but for flat space [81]. These are encoded in the geodesic equation:

$$\frac{d^2 x^\mu}{ds^2} + \sum_{\alpha, \beta} \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \quad (2.3)$$

Here $x^\mu(s)$ is the path of the object, s the variable that parameterises the curve and Γ the Christoffel symbol. This is a function of the metric and its derivative and encodes the curvature of spacetime. This differential equation thus encodes how objects move in curved spacetime. The variable that parameterises the curve is called an affine parameter and is related to ‘proper time’ τ through $s = a\tau + b$, for timelike observers [81]. In GR time is no longer an absolute notion but coordinate dependent. The ‘time’ between two events is thus not a well-defined quantity. The proper time generalises this and is in a sense the ‘time/distance in spacetime’ between two events (time for a timelike observer and distance for a spacelike observer) [81]. The proper time along a timelike curve in spacetime is the time as measured by a clock following that curve. This notion is well-defined and independent of coordinates. It is the central quantity in physics problems like the twin paradox. If the metric in coordinates x^μ is given by $g_{\mu\nu}$, then the square of the infinitesimal proper time $d\tau^2$ between two infinitesimally separated events with coordinates x^μ and $x^\mu + dx^\mu$ is given by: $d\tau^2 = -\sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu$. It is interesting to note the connection between the parameterisation of the motion of an object and the proper time evolution of this object. This infinitesimal interval is important to describe the causal structure of spacetime. If two events are separated by $\Delta s^2 = \sum_{\mu\nu} g_{\mu\nu} \Delta x^\mu \Delta x^\nu$, then they are causally connected if $\Delta s^2 < 0$. If $\Delta s^2 > 0$ they are not causally connected. If $\Delta s^2 = 0$ the events are separated by a time equal to the time needed for light to cross the distance with which they are separated. In other words: light can travel from one point in spacetime to the other. One should read the deltas as integrals over the infinitesimal ones, as the metric can be coordinate dependent. Massive objects however are not fast enough and cannot reach both events, at most one. Paths with $ds^2 = 0$ are called null curves or lightlike curves, because these are the curves light follows on spacetime. Paths with $ds^2 < 0$ are timelike curves and are the curves that massive objects follow. $ds^2 > 0$ are spacelike curves and events on such curves are not necessarily causally connected [81].

In this concise summary of GR no time has been spent to motivate Einstein’s GR or to derive the equations stated above. It is however possible to derive the Einstein equations from more fundamental building blocks [81]. The Ricci scalar R is the simplest (in the sense of lowest-order in derivatives) non-trivial scalar built out of the metric. The simplest Lagrangian one can build to describe curved spacetimes is therefore simply the Ricci scalar $\mathcal{L} = R$. The associated action is

$$S = \int \sqrt{-g} d^4x \mathcal{L} = \int \sqrt{-g} d^4x R. \quad (2.4)$$

The factor $\sqrt{-g}d^4x$ is the 4D-hypervolume element in the integration, with g the determinant of the metric. This action is called the Einstein-Hilbert action and it turns out that this action describes GR. The EFE are simply the Euler-Lagrange equations associated to this action. The observation that the simplest non-trivial action describes GR is a motivation for its low energy validity [81]. At the same time it is a natural stepping stone towards extensions of GR. One can add higher-order terms to the action, like terms with four derivatives (second-order in the Riemann tensor, where the Riemann tensor describes spacetime curvature and is a function of the metric and its first two derivatives). Then this action will no longer describe GR, but some modification. This idea is the starting point of the EFT perspective of modifying GR, adding terms to the action/Lagrangian in a way consistent with imposed symmetries and such that in some low energy limit GR appears [59, 85–88]. This will be explained in more detail in Section 2.3.

2.1.2 The Geometry of GR

Geometry is the language of GR. Therefore an introduction into GR is not complete without also giving a crash course on differential geometry. Spacetime is a manifold. A manifold is a space that locally resembles an n -dimensional Euclidean space [89]. It is thus a space that has the same dimension locally everywhere. Examples include a torus (donut), a sphere, a plane and any Lie Group. The property that the manifold is locally n -dimensional everywhere is defined more mathematically using charts. A set $\mathcal{M}$ is an n -dimensional manifold if there exists an atlas $\mathcal{A}$, which contains charts ϕ_U with $\phi_U : U \subset \mathcal{M} \rightarrow \mathbb{R}^n : x \mapsto \phi_U(x) \in \mathbb{R}^n$ [89]. Here $U \subset \mathcal{M}$ is an open subset of $\mathcal{M}$ and ϕ_U is a continuous bijection between U and a subset of $\mathbb{R}^n$. Each of these charts can be seen as a local coordinate system for the manifold. Some manifolds can be covered by a single chart, like e.g. the plane. Some manifolds cannot be covered by a single chart and need multiple charts, like e.g. the circle [89]. Often one will only mention a single coordinate system in the context of GR. This coordinate system is a chart. It is crucial to realise that sometimes this coordinate system used in GR solutions does not cover the entire manifold, despite being the only chart given. An example is the typical Schwarzschild metric given in Section 2.1.3. Strictly speaking the chart is only for the exterior of the black hole spacetime [81]. There does however exist a chart covering the entire Schwarzschild manifold: Kruskal coordinates [81]. The lesson to be learned is that effects that one sees may be coordinate artefacts and not physical. This will be elaborated upon in Section 2.1.3. Another lesson learned is that manifolds can have different charts. The definition of manifolds does not restrict the form of the chart. Therefore the coordinates one uses to describe the manifold are arbitrary. Physics does not depend on the choice of coordinates and every smooth coordinate change is allowed. This mathematical insight is central in GR. In Newtonian physics, only inertial frames with velocity $v \ll c$ are allowed. In special relativity this changes to inertial frames with arbitrary velocity. In GR any coordinate frame is allowed, because one can choose arbitrary charts to map the manifold [81, 83, 84]. Strictly speaking only diffeomorphisms are allowed because we demand the manifold to be a differentiable manifold: one must be able to take derivatives (which comes back in the EFE). This symmetry of diffeomorphisms will come back later.

It is possible to give extra structure to manifolds. In this thesis we will restrict our attention to four-dimensional Lorentzian differentiable manifolds, because these are the

manifolds of GR. A differentiable manifold is a topological² manifold of which one can take any two charts and the map between these charts on the intersection is smooth [89]. This is a function from one chart to the other and is thus a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. The meaning of smooth is the standard one from multivariate calculus. An n -dimensional Lorentzian manifold is a manifold with a metric tensor defined on it, where the metric tensor has signature $(n-1, 1)$ [81]. This signature was mentioned earlier with respect to the Minkowski metric. It is *(number of positive eigenvalues, number of negative eigenvalues)* of the metric tensor in matrix form. It is mostly plus by our convention. Demanding that the metric has this signature makes sure there exists a timelike and a spacelike direction: There exist notions of what a local observer would call time and space [81]. A metric tensor is a structure defined on a manifold that allows one to define lengths and angles in the tangent space to the manifold $\mathcal{M}$ at any given point p . This metric tensor must be non-degenerate (no eigenvalues are zero), smooth and symmetric [89].

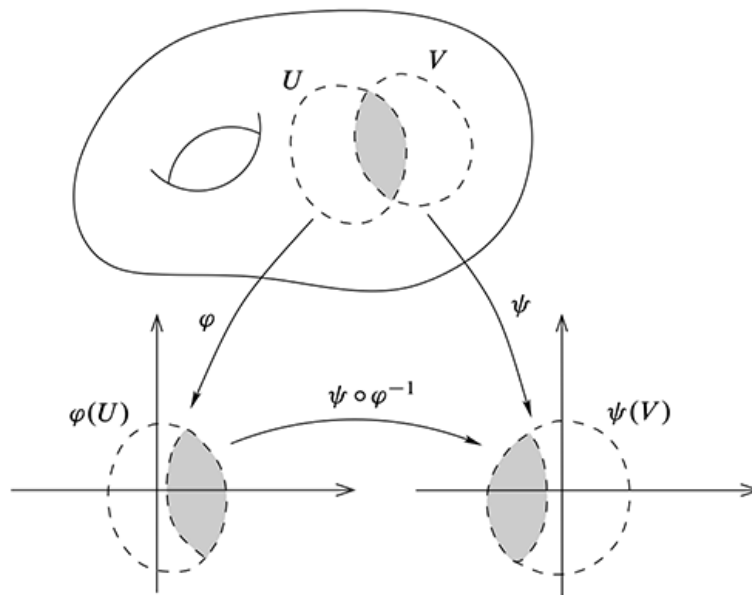


Figure 2.1: Visualisation of a smooth manifold with two charts. Figure from [89].

The notion of a tangent space has not been defined yet, but intuitively its meaning is clear: it is the tangent space to a manifold at a given point. Like a tangent line on a circle or a tangent plane on a sphere. Mathematically the definition is given by the vector space of the equivalence classes of curves on the manifold through a point with the same derivative at this point [89]. This vector space of derivatives can be seen as the vector space of the vectors tangent to the manifold and has the same dimension as the dimension of the manifold. Every point of the manifold has a tangent space and the collection of all these tangent spaces is the tangent bundle. A tangent bundle is in itself a manifold [89]. The metric tensor acts on objects in this tangent space. It can calculate the distance or ‘angles’ between the vectors in the tangent space at a point p . Note how the fact that the GR metric has a negative eigenvalue implies that the square of distances need not be positive. It is on the tangent bundle that most of the objects of interest live or act upon. These objects are tensors. Tensors are a generalisation of vectors and matrices. On a

²a manifold is topological if its topology is Hausdorff and second countable [89].

manifold they are defined as follows. A type- (p, q) tensor T on a manifold $\mathcal{M}$ is defined as being a smooth multilinear map $T_p : \underbrace{T_p M^* \times \dots \times T_p M^*}_{p\text{-times}} \times \underbrace{T_p M \times \dots \times T_p M}_{q\text{-times}} \rightarrow \mathbb{R}$ at

each point p on the manifold [89]. It is smooth in the sense of changing the point p on $\mathcal{M}$. This definition is abstract. It means that a tensor takes p vectors from the dual of the tangent space and q vectors from the tangent space and results in a number. This formal definition shows that a tensor is an object that is independent of coordinates. In almost all cases however the tensor will be introduced/characterised in a certain set of coordinates/chart. An example of a tensor is the metric (tensor). It is typically shown as a matrix $g_{\mu\nu}$, but this is in a coordinate dependent setting. A metric is a $(0, 2)$ -tensor since it takes two vectors (from the tangent space) and gives out a real number: the scalar product between these vectors [81].

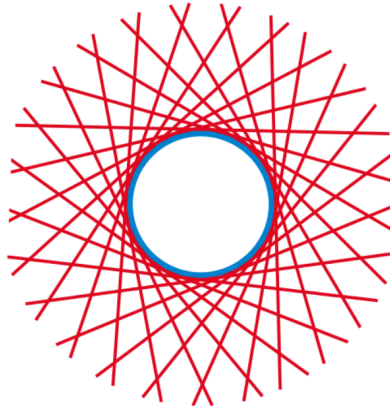


Figure 2.2: Visualisation of Tangent bundle of circle. Figure from [90].

More intuitively, a tensor really is just a generalisation of vectors and matrices. A (p, q) -type tensor has p upper indices and q lower indices. Compare this to a vector, which has one index, and a matrix, which has two. The elements of such a tensor are given by its action on the basis of the vector spaces. Upper indices for basis elements in the dual space, lower indices for elements in the vector space. A tensor is then a generalised matrix with $p + q$ indices which transforms covariantly under coordinate transformations [89]. Transforming covariantly just means that when changing the coordinate system (and with it the basis of the vector spaces), the elements of the tensor change in such a way that the action of the tensor on its arguments remains unchanged. Tensors and vectors are abstract objects independent of coordinates, therefore acting with a tensor on a vector should not depend on the choice of coordinates. This is completely similar to doing a change of basis in linear algebra. The matrix elements may differ in another basis (and may be more convenient, e.g. diagonal), but the result it has on vectors remains the same physically.

Using the metric, there is a way to change the indices of a tensor from upper to lower or vice versa. To change an index from upper to lower, one multiplies with the metric as follows:

$$\sum_{\rho_1} T_{\mu_1 \dots \mu_n}{}^{\rho_1 \dots \rho_m} g_{\rho_1 \sigma} = T_{\mu_1 \dots \mu_n \sigma}{}^{\rho_2 \dots \rho_m}. \quad (2.5)$$

With T a tensor [81]. Thus one can change indices by multiplying with the metric and summing over the relevant index. The inverse happens when changing an index from lower to upper. For this one uses the inverse metric. The inverse metric is constructed by inverting the matrix representation of the metric. If the metric is the matrix $g_{\mu\nu}$ in a certain coordinate chart, the inverse metric $g^{\mu\nu}$ is simply the inverse matrix [81]. Using this, one changes the indices of a tensor from lower to upper as follows:

$$\sum_{\mu_1} T_{\mu_1 \dots \mu_n}{}^{\rho_1 \dots \rho_m} g^{\mu_1 \sigma} = T^{\sigma}{}_{\mu_2 \dots \mu_n}{}^{\rho_1 \dots \rho_m}. \quad (2.6)$$

Note how the indices remain in the same position. General indices will always be written with Greek letters. In some cases one will see roman indices. This is for the spatial coordinates. 0 is the index for the time coordinate. Last but not least, in the remainder of this thesis the Einstein summation convention will be used. This means that any repeated index will be summed over, without explicitly writing down the summation. One of the indices will be upper, one will be lower. This follows from the demand that results need to be coordinate chart independent [81]. This means one will write the following simplification in Einstein summation index:

$$\sum_{\mu\rho\sigma} A_{\mu\nu\rho} B^{\mu\sigma} C_{\sigma\tau}{}^{\rho} \equiv A_{\mu\nu\rho} B^{\mu\sigma} C_{\sigma\tau}{}^{\rho}. \quad (2.7)$$

Although this section may have been quite technical and its results and jargon may not be used much in the remainder of the thesis, it is useful to keep it in mind, because much of the formal structure underlying GR and its extensions is built upon these concepts and implicitly manipulates these structures.

2.1.3 Black Holes

Now one is familiar with the basic physics and mathematics of spacetime, one can study the objects that arise in GR. The most intriguing of these objects are black holes. These are objects that are so compact that its gravity prevents anything, including light, from escaping [81, 83, 84]. These objects are the densest and most compact objects in our universe and have been observed through both electromagnetic and gravitational signatures [9, 91]. Black holes are no longer objects in spacetime, but are really spacetime itself, as their energy-momentum tensor vanishes [81]. A black hole is a classical solution to the full EFE which possesses an event horizon (EH). An event horizon (EH) is a null hypersurface that causally separates the interior of the black hole from the exterior [81]. The EH formalises the statement that no information can come out of a black hole. The EH is a point of no return: once an object has passed the EH, there is no way back. Within the EH, all future-directed geodesics point inwards towards the singularity. This EH is often said to be the boundary of the black hole. Black holes have strong gravitational fields and a non-trivial causal structure [81].

In GR black holes have a singularity. This means that the curvature of spacetime is infinite somewhere inside the black hole (in the centre). These singularities are somewhat problematic. GR is based on spacetime being a differentiable manifold and at the same time predicts the existence of black holes. The first statement is not reconcilable with the second statement as black holes have singularities: GR predicts its own demise [81].

Solving these singularities, or removing them, is one of the main reasons to look at modifications of GR [83, 85, 92, 93]. Extensions to GR will be explored in Section 2.3.

The simplest black hole is the Schwarzschild black hole. This is a static (time-independent) spherically symmetric black hole. It is a vacuum configuration, meaning that the energy-momentum tensor vanishes identically. The metric is given by:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (2.8)$$

Here t is the time coordinate. It is the time according to an observer at infinity, r is the radius, with the centre of the black hole (BH) corresponding to $r = 0$, θ is the polar angle and ϕ the azimuthal angle [81]. The EH, the surface of no return, is given by $r = 2M$. At this point in these spherical coordinates the metric becomes ill-defined and time and space switch roles (time gets a positive signature and space a negative signature instead of vice versa). This means that inside of the black hole someone is no longer obligated to move forwards in time but is obligated to move in space: towards the singularity. There is a singularity at $r = 0$ where the curvature becomes infinite [83]. Despite the metric having a singularity at $r = 2M$ the EH, this is not a singularity, the spacetime is well behaved here. It was already mentioned in Section 2.1.2 that this is a coordinate effect, and thus chart dependent. There exist coordinates where the EH is regular, e.g. Eddington-Finkelstein coordinates [81]. The causal structure of the EH being a point of no return is however physical and present in every choice of coordinates [81].

The Schwarzschild BH is not just ‘a’ spherically symmetric spacetime. It can be proven that it is the only spherically symmetric vacuum spacetime, with Minkowski space being captured in some limit of $M \rightarrow 0$. Birkhoff’s theorem states thus that any spherically symmetric spacetime that is vacuum must have the Schwarzschild metric [81]. This means that e.g. the vacuum spacetime outside of a spherically symmetric star must be described by the Schwarzschild metric. Because the radius of the star R is a lot larger than $R \gg 2M$, there is no EH present in such cases. However, physical objects are rarely, if ever, spherically symmetric objects. In reality objects rotate. A rotating black hole is described by the Kerr metric. This is a stationary (symmetric in time, but can be rotating), axisymmetric vacuum spacetime that describes a rotating black hole with axial symmetry [84]. The axial symmetry is around its axis of rotation. The Kerr metric is given by:

$$ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2}dr^2 + (r^2 + a^2 \cos^2 \theta)d\theta^2 + (r^2 + a^2)\sin^2 \theta d\phi^2 \\ + \frac{2Mr}{r^2 + a^2 \cos^2 \theta}(a \sin^2 \theta d\phi - dt)^2. \quad (2.9)$$

Here t is the time coordinate for an observer at infinity, r the radial coordinate, θ the polar angle and ϕ the azimuthal angle. a is the spin of the black hole and given by $a = \frac{J}{M}$ with J the angular momentum of the black hole [81]. In GR the spin can take values between $a = 0$ and $a = M$. $a = 0$ corresponds to Schwarzschild black holes (no angular momentum, and thus not rotating) and $a = M$ corresponds to an extremal black hole, which is maximally spinning [81]. A Kerr black hole has a singularity at $r = 0$, but it is no

longer a point in space but a ring singularity [83]. A Kerr black hole has two horizons, an inner horizon and an outer (event) horizon. This outer horizon is the point of no return. The horizons are the solutions to $r^2 - 2Mr + a^2 = 0$, as once again the signature of space and time flips at this point. For an extremal black hole the inner and the outer horizon overlap and are the same [81]. Extremal black holes are exotic objects and studying them requires great care. However, they are interesting to study for many (beyond GR) reasons [81, 83, 84, 94]. In this thesis we will be studying black holes with arbitrary spin, but also focussing on rapidly rotating black holes. These are black holes that are spinning fast, close to the maximally spinning extremal BHs, but not exactly maximally spinning, e.g. a spin of $a = 0.99M$. They are also sometimes called near-extremal black holes, but this term is typically reserved for when one examines spins even closer to extremality [2, 95, 96]. This provides insights into extremal black holes and observationally interesting phenomena. From a(n) (astro)physical point of view, there is an upper bound on the spin of a black hole: the Thorne bound $a \lesssim 0.998M$ [97].

Kerr black holes are the physical black holes of our universe, as far as we know [14–16]. Therefore studying their properties is essential in understanding real-life black holes and GWs. The fact that this metric was fully characterised by two parameters: the mass M and spin a of the BH, hides a deep simplicity. This simplicity is called the no-hair theorem [98–100]. This simplicity is in stark contrast with everything else. It is impossible to fully characterise a star, planet or human with two parameters. Strictly speaking there are four parameters that make up a black hole: mass, spin, electric charge, and magnetic charge, but the latter two are almost systematically neglected [101].

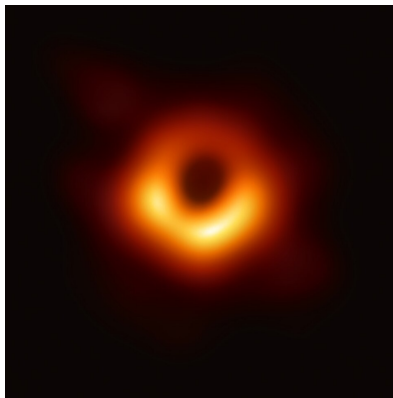


Figure 2.3: Image of supermassive black hole in centre of M87 galaxy, by EHT. One sees the light ring as the coloured region. Figure from [91].

Black holes are key objects in understanding the universe and the laws that dictate what happens within this universe. Black holes are interesting to astrophysicists because they are one of the possible end points of stellar evolution [102]. Furthermore supermassive black holes (in the centre of galaxies) are crucial in our understanding of the formation and evolution of galaxies [103]. For theoretical physicists black holes are interesting because they are among the most extreme objects in the universe [81]. They are a strong field configuration, the curvature becomes infinite at the singularity. This means they are one of the best ways to probe deviations from GR in the strong field regime, where classical GR is expected to become incomplete, as explained in Section 2.3 [104, 105]. Further-

more, they are an object where both QFT and GR play a role [106]. This means they are the natural playing ground for quantum gravity effects. A more comprehensive overview of why we need quantum gravity and the role it plays in this thesis is found in Section 2.3.

Black holes have several observables. Most notable among these are their GW signatures. Black holes can collide and merge with one another, or with other compact objects [9]. These ‘binary black hole mergers’ are, at the moment of writing, the prime source of GWs and give invaluable information about spacetime and black holes [10]. Furthermore black holes can be perturbed by e.g. the merger of two black holes or matter falling in. These perturbations lead to GWs being emitted with frequencies characteristic of the black hole. These are called quasinormal modes (QNM) and will be explained elaborately in Section 2.2 and are the core of this thesis [25].

Another clear observable is the light ring. A black hole is compact enough to support bound null curves. This means there are geodesics/trajectories around the black hole for massless matter (e.g. photons) that are bound in space [81]. The lightlike geodesics at a fixed r around a Kerr black hole are the light ring. For Schwarzschild it is located at $r = 3M$. These geodesics are unstable, meaning that nearby paths diverge from the geodesic [81]. This set of bound null geodesics is observable, it is in a sense what one sees when looking at pictures of black holes. Light on geodesics close to the light ring may circulate several times around the black hole before propagating to infinity. These form a ring of light. Inside this ring there is no light, because inside light does not escape to infinity [91, 107–109]. There is a link between the light ring and QNMs through what is called the eikonal limit [110]. This link will be explained in Section 2.2.4.

2.1.4 Linearised Gravity and Gravitational Waves

After the hard work of Einstein to build a complete non-linear theory of gravity, it is time to go back to a linear form and linearise the EFE. One can linearise the EFE around different backgrounds. The most trivial one is to linearise the EFE around a flat background, this means only looking at first-order terms in the perturbation [111]. In doing so in the lowest-order in c , one can reconstruct Newtonian gravity. In an intuitive sense, Newtonian gravity is linearised general relativity where additionally one sets $c \rightarrow \infty$, or where the velocity of the objects of interest is a lot lower than c [81]. Linearising around a flat background gives $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. One can then rewrite the EFE as a function of the linearised deviation from flatness $h_{\mu\nu}$. Note that it may not be possible to find a global coordinate chart throughout the spacetime where one can do such a decomposition. Locally however this is always possible, through Riemann normal coordinates [81]. In the framework of linearised gravity it is assumed the linearisation requirement is realised in a sufficiently large region of spacetime. To model GR at terrestrial conditions, a linearisation of gravity is possible and natural [111].

Before linearising the equations, it is important to understand the gauge freedom of linearised gravity. In GR, one can take into account arbitrary coordinate transformations $x^\mu \mapsto x'^\mu$. This freedom breaks down in linearised gravity because here $|h_{\mu\nu}| \ll 1$ and one can thus not change to coordinates where this fails. What remains is a smaller set

of coordinate transformations called gauge transformations [111]. As long as $\partial_\rho \xi^\mu$ is at most of the same order of magnitude as $|h_{\mu\nu}|$ one can change $x^\mu \mapsto x^\mu + \xi^\mu$. The metric changes as $h_{\mu\nu} \mapsto h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$. The second class of coordinate changes that are allowed are global Lorentz transformations. These are rotations and boosts independent of coordinates. The boosts must be constrained as not to spoil $|h_{\mu\nu}| \ll 1$ [111].

Now we know the gauge freedom of linearised gravity, we can linearise the EFE around Minkowski (flat) spacetime, which results in the following equation:

$$\square \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \quad (2.10)$$

Here $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, with $h = \eta^{\mu\nu}h_{\mu\nu}$ the trace of the perturbation. $T_{\mu\nu}$ is the (linearised) energy-momentum tensor. To arrive at this equation, some gauge freedom has been used up. This equation is in the Harmonic gauge where one sets $\partial^\nu \bar{h}_{\mu\nu} = 0$ [111]. What is remarkable, is that this is a sourced wave equation. If $T_{\mu\nu}$ is zero, one gets $\square \bar{h}_{\mu\nu} = 0$. Reinstating factors of c one gets [111]:

$$\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) \bar{h}_{\mu\nu} = 0. \quad (2.11)$$

This is a wave equation for waves travelling at the speed of light: gravitational waves move at the speed of light. If one includes a source, one uses the Green's function method to find the metric perturbation, resulting in [111] (not in natural units):

$$\bar{h}_{\mu\nu}(t, \vec{x}) = \frac{4G}{c^2} \int \frac{T_{\mu\nu}(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 \vec{x}'. \quad (2.12)$$

The integral is over the spatial region where $T_{\mu\nu} \neq 0$, which is assumed to be a finite region. These waves, sourced by $T_{\mu\nu}$, are called gravitational waves (GWs). In this context the metric is a wave on top of a flat background. More generally it does not have to be on top of a flat background, but it is (approximately) in this context that we measure them. Gravitational waves are thus natural in the context of linearised gravity [111]. They are waves of spacetime perturbation/curvature and these ‘ripples in spacetime’ move at the speed of light. From this equation it is unclear if this is a coordinate artefact or not but it can be proven that these gravitational waves can carry energy and are therefore physical [81, 83, 111]. A question that remains, is how many degrees of freedom a gravitational wave has. The answer turns out to be 2. A symmetric tensor like $h_{\mu\nu}$ naturally has 10 d.o.f. in 4 dimensions. The harmonic gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$ cancels 4 degrees of freedom. There is however residual gauge freedom. If $\square \xi_\mu = 0$, then the gauge transformation $h_{\mu\nu} \mapsto h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$ does not spoil the harmonic gauge, that is $\partial^\nu \bar{h}_{\mu\nu} = 0$ still holds. With such a gauge transformation one has 4 extra degrees of freedom one can annihilate (using the 4 ξ_μ). One is indeed left with 2 d.o.f. [111]. This is typically shown in the transverse-traceless gauge, where:

$$h_{0\mu} = 0, \quad h^i_i = 0, \quad \partial^j h_{ij} = 0. \quad (2.13)$$

The two degrees of freedom are the two possible polarisations of the waves. If one takes a monochromatic wave in the z-direction, this results in the following metric in natural units [111]:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + h_+ \cos(\omega(t-z)) & h_\times \cos(\omega(t-z)) & 0 \\ 0 & h_\times \cos(\omega(t-z)) & 1 - h_+ \cos(\omega(t-z)) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.14)$$

Notice how the wave is transversal. One has a $+$ and a $\times$ polarisation. Their names come from how they act on matter. To understand the action of gravitational waves, consider the relative motion of nearby test particles in free fall. Imagine one starts with a ring of particles. When a gravitational wave passes by, it will stretch and squeeze the ring of test particles. The $+$ polarization squeezes and stretches it in the form of a $+$, the $\times$ polarisation in the form of a $\times$, as visualised in Figure 2.4 [111]. Thus gravitational waves that pass by the earth will stretch and squeeze us in two directions, depending on the polarisations present. This change is ‘extremely’ small, but can be measured in gravitational wave detectors, see Section 2.1.5. In this linearised regime the energy-momentum tensor of gravitational waves is given by [111]:

$$T_{\mu\nu}^{GW} = \frac{1}{32\pi} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle. \quad (2.15)$$

In this section, we linearised gravity around a flat background. If one linearises gravity around a (Kerr) black hole one gets black hole perturbation theory [25]. This is the essence of the Teukolsky Equation and QNMs that will be explained in Section 2.2. This linearisation around a black hole is not straightforward and is the playground of this thesis [112].

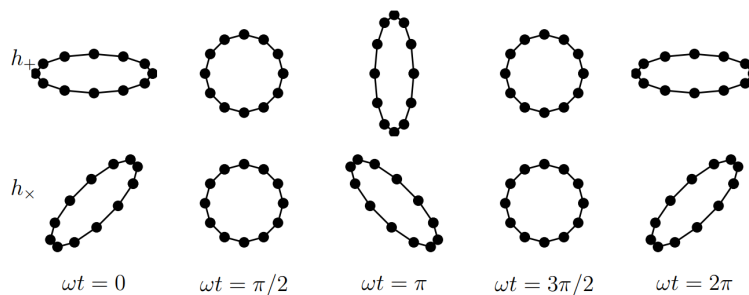


Figure 2.4: Deformation of a ring of test particles due to the $+$ and $\times$ polarisations. Figure from [113].

2.1.5 Gravitational Waves in Practice

Gravitational waves (GWs) are omnipresent in the universe. Even swinging your arm generates GWs. However, these are too faint to ever be measured. Only the heaviest, most violent events in the universe can be measured with our current detectors. GWs are incredibly weak. GWs may be produced in very violent, strong gravity environments, but by the time they arrive on earth they have decayed so much their amplitude is typically of the size $|h| \sim 10^{-21}$. Every meter stretches and squeezes $10^{-21}m$ [111, 114]. This is a million times smaller than the size of an atomic nucleus. However, this effect has been measured [9]. Later in this section the basics of the detectors will be explained. First the different GW sources will be explained, with some digression on the observationally most

important one: the collision/merger of two black holes.

The sources that have been measured thus far are the collisions of compact objects [10]. Two compact objects get close to each other, spiral into each other and merge into one remnant object. Most typically one observes two black holes merging. This collision can be separated into three steps, as shown in Figure 2.5 [111, 114]. The first part is called the inspiral. The black holes spiral around each other and slowly get closer together. They get closer together because they lose energy through GWs. This produces GWs that slowly increase in amplitude and frequency, as the BHs get closer and closer together. The inspiral is modelled through e.g. a post-Newtonian (PN) approximation [111]. A PN expansion is an expansion of the metric tensor, the energy-momentum tensor, and the EFE in orders of v/c , with v the relevant velocity and c the speed of light. It is an extension of linearised theory for weakly self-gravitating systems (like BHs) for which the motion of the interior of the body (not relevant for a BH) and the body as a whole are small [111]. The result is that one approaches the EFE with increasing complexity. The expressions in PN theory are notoriously large. However, it gives accurate gravitational waveforms for the inspiral part of the binary black hole merger [111, 114]. The second part of the collision is the merger. In this part the two black holes actually merge into one final black hole. This is a strong gravity regime and one needs the full non-linear EFE to model the merging of two black holes. The merger is modelled through numerical relativity (NR). Numerical relativity is the art of solving the EFE numerically [114]. It searches for solutions of the full non-linear EFE through a 3+1 decomposition. After Einstein put so much work into uniting space and time into one four-dimensional manifold, NR-specialists artificially reverse this unification. They formulate the EFE as a Cauchy problem. One has a three-dimensional spacelike slice that evolves in time. The exact formulation is non-trivial as it turns out that naively decomposing space and time gives an ill-conditioned problem [115]. Specific gauges and coordinate choices have to be taken to stabilise the framework. Besides the gauge problems it is non-trivial to find the initial conditions for arbitrary spacetime configurations. Nevertheless NR has had multiple successes in describing the full theory of Einstein and describing the merger part of binary black hole mergers [114, 115]. The last part is what happens after the merger. The remnant black hole is formed in a highly excited state, that emits GWs to relax into its final state. This ringing down into the final stable state is called the ringdown and is described by quasinormal modes: the characteristic damped oscillations with which such a black hole deexcites [25, 114]. This will be discussed in much detail in Section 2.2.

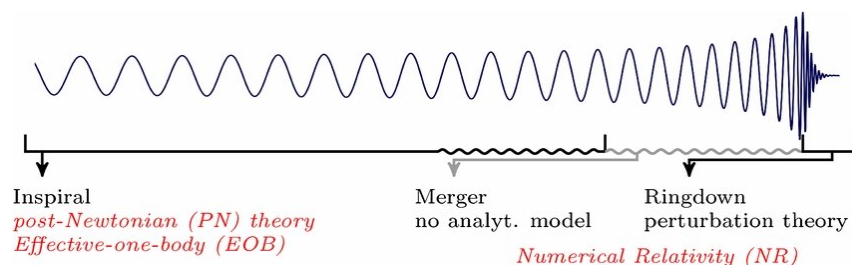


Figure 2.5: Gravitational waveform of binary black hole merger, split into its three parts. The needed frameworks are added. Figure from [116].

The compact objects that collide need not be black holes. Collisions of neutrons stars

have also been observed [117]. Neutron stars (NSs) are very dense stars built out of neutrons. Little is known about the inner workings of these stars [118]. These collisions are useful in multi-messenger astronomy [117]. By measuring the event through two different observables, one can learn a lot about NSs and the universe [119–121]. As an example, these observations give an estimate for the Hubble constant. The universe is expanding. The speed with which it expands is the Hubble constant (which is not a constant) [119]. A rigid non-spherically symmetric body which rotates is also a source of GWs, like a neutron star with ‘mountains’ on the surface [114].

The universe itself may also have been a source of GWs in the early universe. When the universe underwent a phase transition, the universe may have been very inhomogeneous for a small moment in time. This causes GWs, that still linger around in the universe to this day, albeit they have decayed substantially [114]. This background of GWs is unsurprisingly called the (stochastic) gravitational wave background. Stochastic means having a random pattern that may be analysed statistically but not predicted precisely. These stochastic signals may give information about the big bang, the inflationary epoch or QFT [114]. Another source of the stochastic gravitational wave background would be compact binary coalescences that are too feeble to be measured independently [114].

The opposite of these omnipresent backgrounds are burst signals. These are short-duration signals from unknown or unmodelled signals. An example is a supernova, the explosion of a star [114]. They produce a burst signal that is difficult to accurately model. Because they also emit EM radiation, they are prime candidates for multi-messenger astronomy [114]. The given sources may not be a complete list. Perhaps there are sources we do not know about.

GWs have been measured [122]. This is done through interferometry [123]. The idea traces back to the observation that GWs stretch and squeeze spacetime. The idea central to measuring GWs is measuring this stretching and squeezing [111]. One could say distances are becoming longer and then shorter again and one should measure this. But this is a coordinate dependent statement. A coordinate independent statement would be to say that the proper distance/time between two events shrinks and expands periodically [111]. GWs thus leave an imprint on the time it takes for a photon to make a round trip between two points, as the proper distance between the points changes over time. Therefore an interferometer can measure these imprints [111]. An interferometer uses interference of waves to extract information. In the typical Michelson interferometer light is split into two arms, reflected back and recombined at the beam splitter. The superposition of the two light waves is then measured. If the path lengths are equally long, the waves are in antiphase and cancel each other out. They are in antiphase because of a 180° phase shift at the beam splitter [124]. If one of the paths is half a wavelength of the light longer, the waves are in phase and constructively interfere. Therefore the interferometer can measure changes in the length of paths by looking at the intensity of the light at the detector [111]. More precisely it measures changes in the proper distance, the coordinate independent quantity. The longer the path of the interferometer, the smaller the relative change of length that can be measured. The typical GW detectors have armlengths of several kilometres long. The effective length of the path is often made longer through Fabry-Perot interferometers [111, 123].

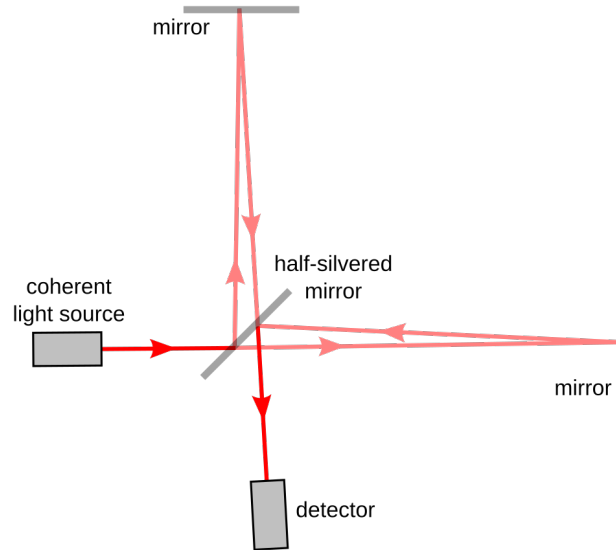


Figure 2.6: Schematic view of Michelson interferometer. Figure from [125].

Measuring such small length changes is non-trivial. Ideally, any change in the arm lengths of the interferometer would be due to GWs, but it is difficult to distinguish noise from GWs. This means that the detector must be stabilised as to minimise noise. Any length change gives a signal, whether it is from mini-earthquakes or from GWs [111]. A lot can be said about the sources of noise and how to mitigate them. Without giving details, the main sources of noise are [111]:

1. Seismic noise: The earth vibrates. This can range from mini-earthquakes to trucks driving nearby the detector.
2. Newtonian noise: Fluctuations in nearby mass distributions give rise to time-varying gravitational forces that move the mirrors of the interferometer.
3. Thermal noise: Objects vibrate at non-zero temperatures. This vibration changes the position of the mirror ever so slightly.
4. Shot noise: Light is constructed out of a discrete set of photons. This discrete nature sets a limit on the precision with which one can measure light intensity. It is a quantum noise source.
5. Radiation pressure: the laser light in the interferometer pushes against the mirror and moves it.
6. Other noise sources.

These noise sources are mitigated using many techniques, like active and passive damping, and squeezed light [111, 126]. We will not go into them any further. Even with all of this noise mitigation, it is non-trivial to filter out the signal from the noise. It turns out that with current detectors one is generally in a regime where $|n(t)| \gg |h(t)|$, the noise is larger than the signal [126]. However, provided one has information on $h(t)$ the waveform of the GW, one can still extract the signal from the data when the noise amplitude is larger than the amplitude of the signal. This is done through matched filtering [111, 114]. One has a

template $\tilde{h}(t)$ of the GW. If one integrates the GW-data with the template, the integral grows as a function of time. If one integrates the noise with the template, the integral will likely oscillate out to zero, as the noise is ‘random’. This ‘randomness’ assumption is made rigorously through claiming the noise is Gaussian and stationary. By integrating the data with the templates in a smart way, one can find if a GW is present in the signal, even if the noise is louder than the signal [111]. This requires one to know the templates. Matched filtering is thus most useful if one knows what to look for. Therefore techniques like PN, NR and QNMs are crucial in data processing, to model what one expects. It will not be further explained in this thesis, but there are many more diagnostics and tools to process the signal and data, like looking at excess power [111, 127]. For QNMs (Section 2.2) alone there is an entire zoo of methods for data analysis [25]. In Section 2.4 such data analysis methods will be explained in some detail.

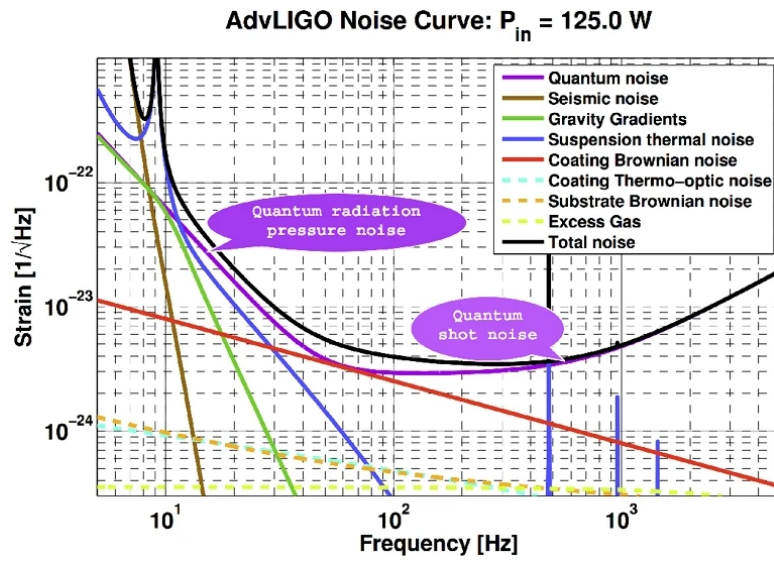


Figure 2.7: Noise curve of LIGO, decomposed into its different sources. The strain can be seen as the amplitude of the noise. It is given as a function of frequency. Figure from [128].

The most notable GW detectors present will now be mentioned. All of these are based on interferometry, except for the Pulsar Timing Array (PTA). Other methods that do not involve interferometry have been constructed [111], but thus far only interferometers have observed GWs [9], but strong evidence for gravitational waves exists through PTA measurements [129–131].

1. LIGO: Laser Interferometer Gravitational-Wave Observatory. This observatory consists of two interferometers with 4km arms. One is located in Louisiana, one in the state of Washington. They were the first detector to detect gravitational waves and three of their scientists got the Nobel Prize in 2017 for the direct detection of gravitational waves [9, 123].
2. Virgo: This observatory is a single interferometer with 3km arms. It is the main detector for gravitational waves after LIGO. It lies near Pisa in Italy [132].

3. KAGRA: Kamioka Gravitational Wave Detector. This observatory is a single underground interferometer with 3km arms. It is the third gravitational wave detector to detect gravitational waves. It uses cryogenics to reduce noise. It is located in Kamioka, Japan [133].
4. LISA: Laser Interferometer Space Antenna. This observatory will be an interferometer in space with arms of 2.5 million kilometres. It will be triangular in form with the corners of the triangle being satellites. It will measure gravitational waves with lower frequencies than LVK. It should be operational in 2035. Similar to this European project there are Chinese projects to build gravitational wave detectors in space, like Taiji. [134, 135]
5. ET: Einstein Telescope. This will be either two L-shaped interferometers with 15km of length, or one triangular interferometer with arms of 10km. It will either be built at the tripoint between Belgium, the Netherlands and Germany, or at Sardinia, or in Saxony. It will be built underground and use cryogenics to reduce noise. It should be operational in 2035. The possibility for it to come to Belgium/Netherlands is a big motivation for the work in this thesis [136].
6. CE: Cosmic Explorer. This observatory consists of two interferometers. One with 40km arms and one with 20km arms. It will be built in the United States and is meant to complement ET, with similar timelines [137].
7. PTA: Pulsar Timing Arrays. There are multiple array set-ups, like NANOGrav, EPTA or CPTA [129–131]. They measure pulses from multiple (millisecond-)pulsars across the sky. Gravitational waves can very mildly change the arrival times of the pulses. By measuring (quadrupolar) correlations between the arrival times of pulsar pulses, one can measure very long wavelength (so very low frequency) gravitational waves. More specifically it measures a stochastic gravitational wave background, probably mostly caused by the merger of supermassive black holes in the centre of merging galaxies. Strong evidence exists for the measurement of a stochastic GW background by PTAs [129–131].

Despite not explicitly using much of the information given in this section, it is useful to realise the observational aspects of the theory developed in this thesis. After all, the end goal is to measure the effects calculated in this thesis in one of these detectors, more specifically in interferometer set-ups. This will be studied in Section 2.4 and Chapter 4. ET and CE may be the best candidates to detect the effects central to this thesis.

2.2 Quasinormal Modes

2.2.1 What is a Quasinormal Mode?

Quasinormal modes are the characteristic damped oscillations of compact objects like black holes. When a (Kerr) black hole is perturbed, it will stabilise again towards the Kerr metric [22–25]. This stabilisation happens with characteristic complex frequencies. The frequencies are complex in the sense that the oscillations are damped: the black hole is stable and perturbations die out. This damping reflects the dissipative nature of black

holes, as the boundary conditions will make clear [22–25]. The real part of the frequency describes the ringing, the oscillatory frequency. The imaginary part describes the damping rate. One can write $\omega = \tilde{\omega} - \frac{i}{\tau}$ with τ the characteristic damping time $\tau > 0$. This can be seen by putting this frequency in the exponential: $e^{-i\omega t} = e^{-i\tilde{\omega}t} e^{-\frac{t}{\tau}}$. These frequencies are independent of the perturbation itself and depend only on the background black hole. QNMs are not only defined for black holes but for all (stable) compact objects, meaning they can also be calculated for NSs, boson stars or other exotic objects [22–25]. Because it is treated as a perturbation on a background, it is a linear phenomenon and the study of QNMs is often called black hole perturbation theory. (Strictly speaking black hole perturbation theory is a slightly broader subject [25, 138].) QNMs naturally appear in the context of GWs in a compact binary coalescence, e.g. a binary black hole coalescence. After the two compact objects have merged, a final state forms. The two objects have merged into one object, e.g. a black hole. This state is not formed in its ground state, it is still highly excited and rings down into its stable end state. This stabilisation is characterised by QNMs: the final compact object is perturbed (by its formation channel: the highly non-linear merger) and settles down with these damped oscillations called QNMs [25]. This last part of the GW signal is called the ringdown and is indeed an exponentially decaying GW: the QNMs. Because one is essentially measuring (complex) frequencies when studying this ringdown, one also calls the study of QNMs Black Hole Spectroscopy [22–25]. These are the complex frequencies which are of interest in this thesis. This thesis restricts its attention to black hole QNMs.

Black holes are ‘deceptively easy’ objects in GR. Black holes are completely described by their mass, spin, electric charge (which is negligible in most scenarios [101]) and magnetic charge (if this exists) [98–100]. This implies that their observables are also described by these four (but in practice 2) parameters. This means that one can constrain GR from the moment one has at least three observables from a black hole: they cannot be independent from each other. Such observables are quite naturally given in the form of the quasinormal frequency (QNF) spectrum: the excitation spectrum of a black hole [22–25]. When two black holes have merged into a final remnant and this remnant is ringing down, one can measure its GWs, and try to extract the QNFs. One frequency completely fixes the parameters of the black hole: the real and imaginary part of the frequency fix the mass and spin of the black hole. Any additional frequency one measures only tests the theory: all parameters have already been fixed. This explains why QNMs are so interesting to study: they are one of the best ways to test GR and the no-hair theorem [22–25]. This is exactly the goal of this thesis, to predict these frequencies in beyond GR scenarios, such that when one measures these frequencies in GW detectors, one can test the different beyond GR scenarios and put constraints on these theories. It is important to realise that this is not just a null test of GR but additionally we constrain alternative scenarios [5].

Apart from testing the frequencies of the QNMs, one can also test the amplitudes. Given a certain merger of two compact objects, GR predicts certain QNMs to be present and it also predicts the amplitudes. When testing beyond GR scenarios, one can therefore either test the frequencies themselves that shift or the amplitudes that shift. Both possibilities are being examined [11, 22–25, 139]. One should look at it like playing the piano. When two black holes merge and result in a ringing remnant, the universe is playing the piano. It plays a chord with several notes (that is, several frequencies), each of these notes is

played with different volume. The task of the scientist is to figure out which notes the universe plays and with what volume. GR predicts certain notes with certain volumes, but extensions of GR change these frequencies slightly. The goal of this thesis will be to find out how the black hole chords change in different modifications to GR.

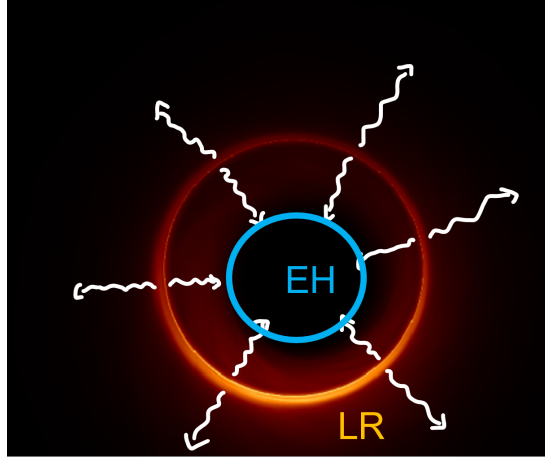


Figure 2.8: QNM boundary conditions, with the light ring as the transition region between ingoing and outgoing waves. Figure adapted from [109].

For Schwarzschild black holes, QNMs are described through the Regge-Wheeler or Zerilli equations. They directly describe perturbations on top of the Schwarzschild spacetime [25]. The solutions which obey the correct boundary conditions are QNMs. The two different equations describe the different ‘polarities’ of the modes. Regge-Wheeler describes the axial modes, Zerilli the polar modes [25]. The polar and axial modes have the same spectrum. This feature is called isospectrality [25], and will come back later. Once spin is added, QNMs are described through the Teukolsky equation, which itself is derived through the Newman-Penrose (NP) formalism [22–25, 140]. To understand the mathematics and physics of QNMs, it is therefore needed to first give an explanation as to what the NP formalism actually is, this will be done in Section 2.2.2. Furthermore to understand QNMs in theories beyond GR it will be necessary to modify the Teukolsky equation to introduce the beyond GR corrections. This modification is done in a tetrad formalism quite close to the NP formalism: the GHP formalism [48]. This Teukolsky equation is a differential equation. In essence it is a linearisation of gravity around a black hole background, making use of the NP formalism [112]. QNMs are then solutions to this Teukolsky equation with specific boundary conditions. The boundary conditions are given by the causal structure of the black hole background spacetime [22–25]: nothing can come out of an event horizon (EH). QNMs are thus purely ingoing at the EH. Nothing can come in from infinity. QNMs are purely outgoing at infinity. This means the phase velocity (which defines ingoing or outgoing) switches sign somewhere between the horizon and infinity. The region where it does is called the light ring, which was briefly mentioned in Section 2.1.3. From a geometric phase-velocity argument one can thus intuitively claim that QNMs are produced around the light ring, because they go out towards the horizon and infinity from here. This can be made more precise and formal using the eikonal limit, which will be explained later [110].

2.2.2 Newman-Penrose Formalism

General relativity as it has been introduced in Section 2.1 was characterised through a purely geometrical notion. Spacetime is a (differentiable) manifold and the metric of this object is given by solving the Einstein Equations with the necessary boundary conditions [81]. An equivalent way of looking at this would be to describe the spacetime at every point by an ‘observer’ at this point. When you characterise at each point in spacetime how the universe ‘looks’ locally for an observer, you characterise the spacetime completely. This rather hand-wavy way of looking at GR can be made more precise through the tetrad formalism [140]. One can construct at each point in spacetime a basis of vectors, which span the vector space (tangent space) at this point in spacetime. This locally defined set of (four) linearly independent vector fields is called a tetrad and can be chosen such that it reflects the physics of the spacetime [140]. This tetrad can be seen as a local choice of coordinate-basis, describing the manifold by some ‘local observer’. This link between observers and the tetrad formalism is one of its main strengths. The tetrad can be adapted to the null propagation of GWs and spacetime curvature [140]. In the language of differential geometry one describes the tangent bundle of the manifold (this is where the vector fields live) by a local choice of coordinates. All tensors are projected onto the tetrad [140]. The metric can be reconstructed by the tetrad using the inner products between the vectors. If one has vectors e^a with components e_μ^a in a chart with coordinates x^μ , one can find the metric using $g_{\mu\nu} = \langle e_a | e_b \rangle e_\mu^a e_\nu^b$. Here, $\langle . | . \rangle$ is the inner product induced by the metric, which makes the expression seemingly circular, but one should realise that the metric is a geometric object, and one is looking for a coordinate-dependent representation [140]. The idea that one can use the inner product acting on a vector basis to reconstruct a metric, is used more generally in mathematics and even in the study of quasinormal mode decomposition [141–149].

The Newman-Penrose (NP) formalism is a special case of the tetrad formalism, where the vector basis is a null tetrad: all four vectors have zero norm (they are all null). They consist of two real vectors and a complex conjugate pair [140, 150]. The formalism is well adapted to the treatment of the propagation of radiation in curved spacetime: the real vectors often point radially inward and radially outward asymptotically. This means it is a natural choice of tetrad for QNMs, where one has GWs being emitted by the black hole [140]. The details of the NP formalism will not be given, only the main ideas, as this suffices as background context.

The crucial quantities in the NP formalism are the spin coefficients, the Weyl scalars and the Ricci-NP scalars. The twelve spin coefficients are analogous to the Christoffel symbols, and the precise connection can be made via the vierbein formalism [140]. The Christoffel symbols describe how the metric changes across the manifold through derivatives, the spin coefficients describe how the tetrad changes across the manifold through derivatives. The Weyl scalars are the five scalars $(\psi_0, \dots, \psi_4)$ that encode information from the Weyl tensor, by projecting the tensor on different combinations of the tetrad vectors [140]. The Weyl tensor is a tensor that describes the curvature of spacetime. It differs from the Riemann tensor in that it does not encode local changes in the volume of spacetime but only of its shape. It can be seen as the traceless part of the Riemann tensor [81, 84, 140]. Unsurprisingly, the Ricci-NP scalars describe the Ricci tensor. The

Ricci tensor contains the information that is not captured by the Weyl tensor: how the local volume changes [81, 84]. Once again these scalars are obtained through projections on different combinations of the tetrad vectors. Because the Weyl scalar and Ricci scalar can be constructed out of the Riemann tensor, which itself can be constructed out of the Christoffel symbols, there is a link between their Newman-Penrose equivalents as well. These NP field equations relate spin coefficients, Weyl scalars and Ricci scalars with one another. Additionally, the EFE can be translated to the NP formalism [140].

The Weyl scalars are crucial in the NP description of the black hole spacetime. ψ_2 describes the Coulomb/background curvature, ψ_0 the ingoing (transverse) radiation and ψ_4 the outgoing transverse radiation at infinity. ψ_1 and ψ_3 describe longitudinal radiation and can be set to zero by a suitable transformation for any spacetime that is asymptotically flat (like the Kerr spacetime) [151]. They can thus in this thesis be seen as gauge quantities and of no physical interest. The other Weyl scalars are relevant for both describing the black hole and the perturbative radiation originating from it. Notice that what is canonically called ψ_4 can describe the outgoing radiation of GWs $\psi_4 = \ddot{h}_+ - i\ddot{h}_\times$ in the transverse traceless gauge [140, 152]. Therefore this scalar is crucial in understanding GWs and black hole ringdown.

A Kerr black hole is a Petrov type D spacetime [140]. This means a coordinate choice exists such that only the Weyl scalar $\psi_2^{(0)}$ is non-zero, with the superscript (0) indicating that this is a background quantity. Intuitively speaking, there exists a frame in which the radiation parts are zero and only the background curvature is present. It is in this context of Weyl scalars in the NP formalism that QNMs are typically studied, as one should study $\psi_4^{(1)}$ (and $\psi_0^{(1)}$) in the NP formalism, where (1) labels the perturbation contribution [140]. The NP field equations can be linearised around a black hole and what one is left with is the study of black hole perturbation theory. The linear perturbations generate $\psi_0^{(1)} \neq 0 \neq \psi_4^{(1)}$, which lead to decoupled equations which can be bundled into a single master equation: the Teukolsky equation [112, 153, 154]. This equation describes linear perturbations of matter or gravitational fields around the black hole in the NP formalism [140].

2.2.3 Teukolsky Equation

The Teukolsky equation, introduced in the previous Section 2.2.2, describes the presence of matter or gravitational fields perturbing the black hole [112, 153, 154]. This can be used to describe many phenomena. An example is an extreme mass ratio inspiral, where a stellar mass black hole orbits around a supermassive black hole. The small black hole can be treated as a perturbation around the larger one. Another example is the plunging of (a) particle(s) into a black hole [25, 138, 155].

The aspect central in this thesis is the study of quasinormal modes (QNMs). The Teukolsky equation can describe the damped oscillations of fields after a remnant black hole is perturbed: the ringdown. One should see the QNMs as the eigenstates of the Teukolsky operator with specific eigenvalues. The solution of this set of eigenvalues: quasinormal frequencies (QNFs), is then given by applying the correct boundary conditions to this

equation, as described in Section 2.2.1 [22–25, 140]. The Teukolsky equation is given by:

$$\begin{aligned}
& \left(\frac{(r^2 + a^2)^2}{r^2 - 2Mr + a^2} - a^2 \sin^2 \theta \right) \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{r^2 - 2Mr + a^2} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left(\frac{a^2}{r^2 - 2Mr + a^2} - \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \psi}{\partial \phi^2} \\
& - (r^2 - 2Mr + a^2)^{-s} \frac{\partial}{\partial r} \left((r^2 - 2Mr + a^2)^{s+1} \frac{\partial \psi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) \\
& - 2s \left(\frac{a(r - M)}{r^2 - 2Mr + a^2} + \frac{i \cos \theta}{\sin^2 \theta} \right) \frac{\partial \psi}{\partial \phi} - 2s \left(\frac{M(r^2 - a^2)}{r^2 - 2Mr + a^2} - r - ia \cos \theta \right) \frac{\partial \psi}{\partial t} \\
& + (s^2 \cot^2 \theta - s) \psi = 4\pi(r^2 + a^2 \cos^2 \theta) T.
\end{aligned} \tag{2.16}$$

ψ is the perturbation field of interest, s is the spin weight of the field (0 for scalar, ± 1 for vector, ± 2 for gravitational), (t, r, θ, ϕ) are the Boyer-Lindquist coordinates and T is the source term [112]. In studying the QNM spectrum, one is interested in the vacuum equation. In what follows we will thus set $T = 0$. The equation holds for all spin weights. Even more interesting, is that, in GR, $s = |s|$ and $s = -|s|$ result in the same QNM spectrum. This reflects the isospectrality of black holes in GR. The QNM spectrum of polar and axial modes is the same in GR. This isospectrality is broken in most beyond GR theories [156]. Notice how scalar QNMs ($s = 0$), that will be studied in this thesis, do not have a notion of isospectrality [22–25, 112, 140].

The Teukolsky equation is separable in its coordinates, meaning that the master equation can be separated in one angular equation and one radial equation [22–25]. The time and azimuthal angle can be separated out analytically with a prefactor of $\psi \rightarrow \tilde{\psi} e^{im\phi - i\omega t}$. This is possible because ∂_t and ∂_ϕ are Killing vectors: a Kerr BH is stationary and axially symmetric [81]. Killing vectors are vector fields that generate the continuous isometries (symmetries of the metric) of a spacetime, such as translations or rotations [81]. If the field is given by $\psi = e^{im\phi - i\omega t} S(\theta) R(r)$, the separated Teukolsky equations in vacuum are:

$$\begin{aligned}
& (r^2 - 2Mr + a^2)^{-s} \frac{d}{dr} \left((r^2 - 2Mr + a^2)^{s+1} \frac{dR}{dr} \right) \\
& + \left(\frac{((r^2 + a^2)\omega - am)^2 - 2is(r - M)((r^2 + a^2)\omega - am)}{r^2 - 2Mr + a^2} + 4is\omega r - A - a^2\omega^2 + 2am\omega \right) R = 0
\end{aligned} \tag{2.17}$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dS}{d\theta} \right) + \left(a^2\omega^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} - 2a\omega s \cos \theta - \frac{2ms \cos \theta}{\sin^2 \theta} - s^2 \cot^2 \theta + s + A \right) S = 0. \tag{2.18}$$

m is the azimuthal quantum number, A the separation constant connecting the two equations, and ω the QNF [22–25, 112]. Most important in this analysis are gravitational QNMs, with $s = \pm 2$. These are the modes one measures in ringdown analyses of black hole mergers. This thesis, however, will calculate scalar QNMs. Scalar QNMs are a natural starting point, being technically simpler than their gravitational counterparts due to the absence of backreaction. Although they are not the modes directly detected, they already reveal much of the relevant physics of the EFT of gravity [2–4, 47]. They solve the Klein-Gordon (KG) equation on a curved background $\square\psi = 0$. This KG equation

of a scalar field on top of a black hole is essentially the Teukolsky equation with $s = 0$ [22–25, 112].

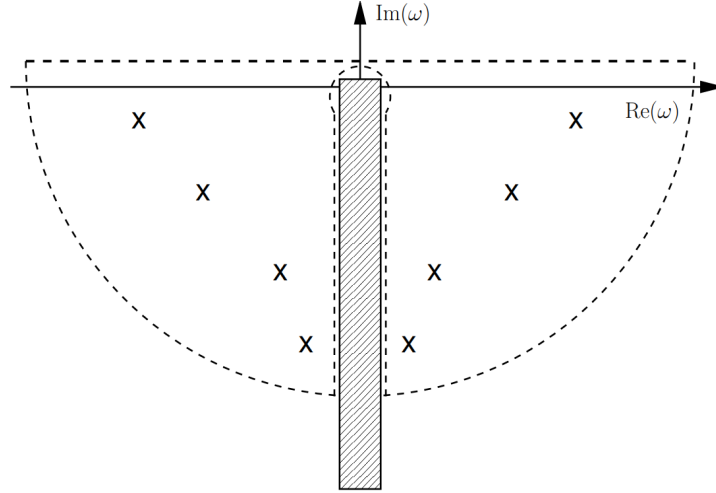


Figure 2.9: Green’s function contour in the complex contour domain. The dotted line shows the contour to integrate over to go back to the time domain. The crosses are the Green’s function poles: QNMs. The branch cut is visible in the form of a striped rectangle. Figure from [22].

The structure of black hole ringdown and the Teukolsky equation can best be understood from the Green’s function perspective. The Green’s function solves the equation with a point source with the correct boundary conditions (ingoing at the horizon and outgoing at infinity). One considers the Fourier transform of the field $\psi_\omega := \mathcal{F}\psi_t = \int_{t_0}^{\infty} \psi(t)e^{i\omega t}dt$. Solving such a wave equation in frequency space leads to the frequency domain Green’s function. Returning back to the time domain requires the inverse transformation. To calculate this integral, one closes the contour as described in the Figure 2.9 above [22–24]. The Green’s function has poles in the complex plane. These are called QNMs and make up the typical damped sinusoids in the ringdown signal. There is also a branch cut at the negative imaginary axis. This one is cut out of the integration and it contributes to the ringdown signal in what one calls a polynomial tail [22]. This signal is subdominant at early times but because it only decays polynomially in time, it dominates over the exponential decay of QNMs at late times. The tail corresponds to the backscattering off the background curvature. The last contribution comes from the arcs. This results in the prompt response, the wave that propagates directly from the source (e.g. directly after the merger) [22–25].

The structure of the Teukolsky equation thus predicts there are three phases in black hole ringdown. First one has the prompt response (which arguably arrives when the linear approximation may not be satisfied). Secondly one has the QNM content. This part is the most important part and is used to test GR. The last part is that of the polynomial tail [22–25].

2.2.4 The Mathematics and Physics behind QNMs

QNMs are the eigenstates of the Teukolsky operator, QNFs the eigenvalues [22–25]. Because the Teukolsky operator is a non-Hermitian operator, the eigenvalues are not real but complex. This non-Hermitian behaviour has its physical origins in the dissipative nature of QNMs: radiation leaks out towards the horizon and infinity. Because the operator is non-Hermitian, it is not guaranteed that its eigenstates form a complete basis for the solution space [22–25]. It is still an open question at the moment of writing, whether or not QNMs form a complete set [141–144]. It is typical to call the frequency spectrum the QNMs. Technically speaking this is not true, the solutions to the Teukolsky equation are the QNMs. They have their accompanying eigenvalues: the QNFs.

An interesting property of gravitational QNMs is that they are isospectral, as mentioned in Section 2.2.3. Remember that QNMs are perturbations of the black hole and thus they can be even or odd in parity (or a combination of the two). Now in GR the polar and axial QNMs have the same frequencies [22–25]. This no longer holds in most beyond GR theories: isospectrality is lost. There are however cases where isospectrality is kept in the lowest-order EFT beyond GR: a specific quartic gravity case [156]. What these words mean will be explained in Section 2.3.

QNMs are characterised by three mode labels (one can call them quantum numbers, but keep in mind there is nothing quantum about QNMs). The angular sector gives rise to two labels: the angular momentum indices l and m . m is the azimuthal angular momentum index and characterises the azimuthal behaviour of the QNMs completely as $e^{im\phi}$, as described in Section 2.2.3. l is the orbital angular momentum index and characterises the polar angle behaviour of the QNMs. Intuitively it is seen as a multipole number ($l = 2$ is quadrupole,...) but this only holds exactly for Schwarzschild black holes: for Schwarzschild black holes in GR the angular behaviour is given by spin-weighted spherical harmonics [22–25]. For Kerr black holes the angular Teukolsky equation has to be solved. This gives rise to spin-weighted spheroidal harmonics $S_{lm}^{a\omega}(\theta)$. They generalise spherical harmonics and form a complete basis in the angular space [22–25, 157]. The third label comes from the radial sector. It is the overtone number n . It characterises the different frequencies with the same l, m . For Schwarzschild, the n increases as the damping time of the QNFs decreases. For Kerr, the labelling is adopted from Schwarzschild, but this ordering is not completely preserved [25]. Higher overtones damp out faster, which explains why the fundamental mode ($n = 0$ the longest lived mode) is the easiest to measure experimentally [22–25]. Strictly speaking these three labels are insufficient. Every labelling still leads to two modes/frequencies: a prograde and a retrograde mode. As can be seen in Figure 2.10, there are QNMs with $Re(\omega) > 0$ and modes with $Re(\omega) < 0$, which have the same labels. In GR, there is a symmetry between the two, in that if $\omega_{l,m,n}$ is a QNF, then $-\omega_{l,-m,n}^*$ is as well. For Schwarzschild, all m have the same spectrum, which explains the $-\omega^*$ symmetry visible in Figure 2.10 as a mirror symmetry around the branch cut [25]. One possible definition of prograde modes is that $\text{sign}[Re(\omega)] = \text{sign}[m]$, where retrograde modes have $\text{sign}[Re(\omega)] = -\text{sign}[m]$. The notion of prograde and retrograde modes therefore is not defined for $m = 0$. So if we call $\omega_{l,m,n}^+$ the modes with $Re(\omega_{lmn}^+) > 0$ and ω_{lmn}^- the modes with $Re(\omega_{lmn}^-) < 0$, we have fully labelled the QNMs with $(l, m, n, +/ -)$ [25].

The radial and angular Teukolsky equations are both instances of confluent Heun Functions. These are special functions defined in mathematics and discovered by Heun [25, 158, 159]. It is a second-order linear ordinary differential equation (ODE) of the form [158]:

$$\frac{d^2 H}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon \right) \frac{dH}{dz} + \frac{\alpha z - q}{z(z-1)} H = 0. \quad (2.19)$$

It has regular singularities at 0 and 1 and an irregular singularity at ∞ . The equation is known in many different coordinate choices. This is one of them. Because the Confluent Heun equation and its solutions have been studied by mathematicians, one can use this knowledge when characterising and solving for QNMs [158, 159].

Many ways exist to compute QNMs. The most widely used are (pseudo)spectral methods, Leaver's method, Confluent Heun methods and WKB-methods [25, 26, 158–162]. Each of these methods have their own strengths. Now these methods will be described very briefly. For an example of an exact implementation of a pseudospectral method, we refer to Section 3.2, as this is the method we will use [1]. By far the most used method to calculate QNMs in GR is Leaver's method [160]. Leaver's method is based on the Frobenius method of solving an ODE. At the horizon and at infinity the QNM functions diverge in Boyer-Lindquist coordinates. Like in many methods, first one separates through a known ansatz the divergent behaviour at the boundaries. This is done in the form of $\psi = \alpha f$, with the divergent prefactor α as in Eq. (3.27) [160]. Implicitly this is equivalent to a change of coordinate slicing. One can choose different coordinate charts to map the manifold and this amounts to a different differential equation describing the same Teukolsky equation. In essence, to solve the differential equation one changes coordinates to a chart where the QNMs are regular across the physical domain. After separating the coordinates, both the angular and radial equation become completely regular [160, 163]. In this case this is done through hyperboloidal coordinates. Without going into the details, this amounts to compactifying the radial coordinate, having the radial coordinate reach null infinity instead of spatial infinity at the infinite boundary and have the behaviour at the horizon regular as well [163]. Once the radial coordinate is compactified such that the horizon lies at $\xi = 0$ and null infinity at $\xi = 1$, one can use the Frobenius method to solve for the QNMs [143, 163]. ξ is defined as $\xi = \frac{r-r_+}{r-r_-}$, with r the radial coordinate, $r_+ = M + \sqrt{M^2 - a^2}$ the EH, and $r_- = M - \sqrt{M^2 - a^2}$ the inner horizon. This radial coordinate will also be used in Chapter 3³. After compactifying the domain and removing divergences, Leaver's method solves the equations using the Frobenius method, where one expands the radial and angular solutions as a power series. The frequencies for which a minimal solution exists, are the QNFs [160].

Another often used approach is a (pseudo)spectral approach [1, 161]. Once again one typically separates the divergent boundary behaviour. Instead of doing a Taylor expansion $f = \sum_{n=0}^{\infty} a_n \xi^n$ as in Leaver's method, one decomposes in another basis, often Chebyshev polynomials or Legendre polynomials. One then has to solve for the coefficients of the basis functions. This only has converging solutions for certain frequencies: the QNFs.

³Technically speaking it will be a different coordinate, because that coordinate is chosen such that the horizons remain at the same position beyond GR, see Section 2.3.3.

One of the big strengths of such a method is that it does not require separability of the Teukolsky equation. This makes it more broadly applicable to beyond GR scenarios where the Teukolsky equation is no longer separable [1]. In a spectral method the amplitudes are calculated directly and explicitly. In a pseudospectral method one does this rather implicitly [1, 68–71]. One solves the differential equation at a finite number of points, a certain grid. Implicitly this also means decomposing in a certain basis, often Chebyshev polynomials. This idea will be used in this thesis to compute QNM shifts and the method will be further elaborated upon in Chapter 3.

Often one uses confluent Heun functions to evaluate QNMs, as they are the natural framework in which they arise [158, 159]. By using the mathematically known asymptotic behaviours of QNMs, one can construct algorithms to find the QNFs at which the asymptotic behaviours of the confluent Heun solutions match that of the physical boundary conditions of QNMs [159].

Finally, from a more theoretical point of view it is interesting to solve QNMs using WKB analysis [110, 162]. A WKB analysis is based on the idea that the potential is slowly varying and that the wavefunction has a phase which changes a lot faster than the coefficients/amplitudes. This approximation becomes ‘exact’ in the large l limit [64, 110, 162]. This method gives analytical expressions for the QNFs. It gives expressions for asymptotic behaviour of QNMs, like a large l limit. The large l limit (large multipole limit) is often called the eikonal limit. Many studies of beyond GR modifications make use of WKB analysis because it simplifies calculations (in the large l limit) and leads to analytical predictions [64, 110, 162, 164].

It was already mentioned that QNMs originate from the light ring of a black hole. In a WKB analysis this is justified by a link between properties of the light ring and properties of the QNMs in the eikonal limit [110]. The light ring is canonically defined as being the circular unstable geodesic of lightlike particles. So the light ring is the circular path around a black hole that light can follow, but it is unstable [81]. If Ω is the angular velocity of massless particles on the light ring (unstable circular geodesic) and λ_{Lyap} is the Lyapunov exponent of the light ring, then one sees that $\omega = \Omega l - i(n + 1/2)|\lambda_{Lyap}|$, with ω the QNF in the eikonal limit. This is for a Schwarzschild black hole, but can be generalised to rotating black holes [110, 165]. The Lyapunov exponent determines how unstable the geodesic is. It describes how fast neighbouring paths diverge from the light ring. This gives a geometrical interpretation to QNMs. Every time the particles complete an orbit around the black hole, the QNM oscillates l times (for large l). The imaginary part of the QNF is given by $-(n + 1/2)|\lambda_{Lyap}|$. This means that the larger the instability of the null geodesic and thus the larger λ_{Lyap} , the larger the damping of the QNM. So if the particles ‘leak’ out of the unstable photon ring faster, the QNM will damp out faster. Gravitational QNMs are ‘gravitons’ leaking from the null geodesic [110, 165]. Keep in mind that strictly speaking this geometrical interpretation only holds in the eikonal limit.

Even though the first calculation of the QNM spectrum occurred half a century ago, there are still uncertainties about the QNM spectrum in GR itself [158, 159, 166]. The troubling modes are called non-traditional modes and consist of two kinds of modes. The first kind consists of the unconventional modes and they are the QNMs that lie beyond the branch

cut. As mentioned in Section 2.2.3, there is a branch cut on the negative imaginary axis. Beyond this branch cut lies the second Riemann sheet. The confluent Heun functions are ‘multivalued’, akin to a logarithm in the complex plane, leading to multiple Riemann sheets [166]. A logarithm is the inverse of an exponential, but this inverse is not unique. If one traverses a full circle in the complex plane, this corresponds to a phase shift of $2\pi i$, because $e^{2\pi i} = 1$. As a result, one can always add $2n\pi i$ to a logarithm without changing the exponential $z = e^{\ln z} = e^{\ln z + 2n\pi i}$. Each such choice defines a different sheet and crossing a branch cut is the passage from one sheet to the next. Together, these sheets form a Riemann surface [167]. The situation is similar for the confluent Heun functions that describe QNMs. Here too different choices give rise to different sheets, and the unconventional modes lie on the second sheet. At the moment of writing, there is no conclusive evidence as to whether or not these unconventional modes are physical. They occur for black holes of all spin [159]. These modes are visualised for a Schwarzschild black hole in Figure 2.10, with the blue 1 label.

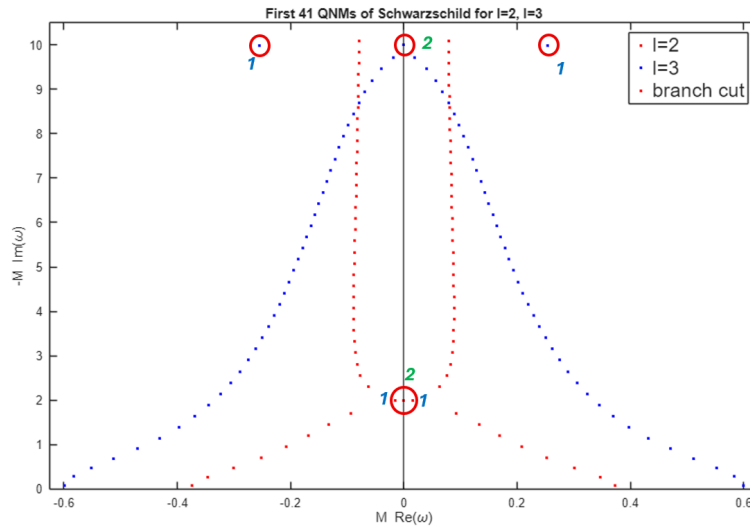


Figure 2.10: Quasinormal mode spectrum for Schwarzschild black hole for the $l = 2, 3$ multipoles and the lowest 41 overtones. Blue denotes the $l = 3$ modes, red the $l = 2$ modes. The circles denote the non-traditional modes whose physical interpretation remains unclear. 1 denotes the unconventional modes, 2 the purely damped modes, which lie exactly on the branch cut (black line). The overtone number increases upwards along the vertical axis.

The second kind of modes consists of the purely damped modes. These are modes that reside on the negative imaginary axis (where the branch cut lies). They have no real, oscillatory, part to their frequency, only the imaginary damping part. Some modes cross the branch cut as spin increases, at which point they become purely damped. Some modes originate from the negative imaginary axis in the Schwarzschild limit (algebraically special modes). Research has been done to study whether such purely damped modes exist [158, 159, 166]. Multiple classifications have been made, but ultimately no physical proof of their (non-)existence has been given [158, 159, 166]. An important caveat here is that the boundary conditions do not exist for purely damped modes, because the phase velocity is ill-defined/zero. Therefore one can only study analytic continuations of boundary conditions. Some purely damped modes have been proven to be unphysical, like the alge-

braically special modes [158]. These modes are visualised for a Schwarzschild black hole in Figure 2.10, with the green 2 label. The modes that are most observationally relevant for current and (near-)future observations however are well understood and one can study their beyond GR behaviour [25].

Although the immediately relevant frequencies are well understood, the amplitudes and excitation of QNMs are not. This leads to the observational aspects of QNMs. As GWs have been measured, QNMs have as well [10, 11], but extracting QNMs is a non-trivial issue. The reason for this is that damped sinusoids are prone to overfitting [25]. The goal is to take the strain data from the detector and find which QNMs are present and to what degree. This means decomposing the signal into damped sinusoids with characteristic frequencies and amplitudes. Many techniques have been applied to infer the amplitudes and frequencies present in the data. However, it turns out to be challenging to extract many QNMs. It is difficult to determine how many quasinormal modes are present in the data, and what the associated amplitudes are [11, 25]. In real data, at most two QNMs have been extracted in a single signal. In only one event (GW250114) the overtone has been detected with certainty. In other events only indications of an overtone have been detected [11, 16]. The frequencies are consistent with GR [11, 16].

Many approaches use the data from the inspiral and merger GW signal to constrain the parameters of the remnant black hole. This is not done in stand-alone tests. Here one aims to use simply the ringdown signal. This may reduce constraining power but does not introduce biases from earlier parts of the signal [11, 16, 25]. Ringdown-only tests only use the simplicity of the QNM spectrum to null-test GR and constrain beyond GR. This is what will be done in this thesis. Both Bayesian methods and least-square methods have been used in the past for such tests [11, 16, 25, 168]. The exact algorithms to be used to decompose the ringdown into its constituents in the next generation of detectors are an open problem [25, 104, 134, 136]. Apart from using the frequencies of the QNMs as probes for deviations of GR, one can also use the amplitudes as probes for beyond GR effects [25, 139]. Using NR, one can construct models to predict the amplitudes of the different QNMs [169]. It is then possible to test GR by comparing the measured amplitudes with the ones GR predicts.

An extra complication is that QNMs are not the end of the story. QNMs are a linear effect. GR is not a linear theory. As long as the perturbations are small, the linear perturbation is valid and QNMs are a good approximation. This means that close after the merger the QNM approximation may not be valid [25, 148, 168]. But even in the linear regime, sufficiently after merger, QNMs are not the complete picture. Apart from the poles in the contour integral of Section 2.2.3, there is also the contribution of the branch point and branch cut. Extra contributions are a prompt response (direct wave, arguably quite close to the merger) and a polynomial tail. This latter starts out small and remains negligible for a relatively long period after the merger. Because it decays more slowly, however, it becomes the dominant effect at late times. This has not been observed in GW data yet, but is important to keep in mind because this signal should be present, although small [25, 147, 170].

In short, many ways exist to solve for quasinormal modes, which have given us a good un-

derstanding of QNFs in GR and their physical meaning, except for non-traditional modes. Beyond GR knowledge is more limited, and this knowledge will be expanded in this thesis. However, amplitudes are not well understood, making detection analysis complicated.

2.3 Beyond General Relativity from an EFT point of view

2.3.1 Why we need Quantum Gravity

Einstein's theory of gravity is not reconcilable with quantum mechanics [105, 171]. Reconciling the two is one of the most fundamental questions of theoretical physics. How can one construct a theory that describes both the smallest scales of quantum mechanics and the largest scales of gravity? General relativity (GR) is a classical theory in the sense that every quantity can be measured unambiguously, with no fundamental limit on precision. This is not necessarily true in quantum mechanics, where e.g. the momentum and position of a particle cannot be determined simultaneously with arbitrary precision [105, 171]. To show that GR and quantum mechanics cannot trivially be reconciled, R. Feynman came up with a thought experiment [172]. Imagine a mass in a superposition of two positions A and B. GR then describes how this mass curves spacetime and how spacetime determines the path the mass takes. The mass roughly follows a geodesic and influences the spacetime by its own mass. Because the mass is in a superposition of two positions and/or paths, this implies that the gravitational field/spacetime has to be in a superposition of two configurations as well. This shows the tension between the two theories: gravity should be 'quantum' if it interacts with a 'quantum object' [172]. This intuitive explanation shows why it is unsatisfactory to end the story with GR. Furthermore classical GR is not to be trusted in regions where the curvature of spacetime approaches the Planck scale and ultimately becomes singular. It is often said (in a way that does not do justice to the subtleties that arise) that GR predicts its own demise, realised by e.g. singularities [105, 171]. One should therefore look for a quantum description of GR, and this is done using the framework of QFT. The Einstein-Hilbert action describes GR and one could start from this action to create a quantised theory of gravity. However, if one tries to quantise GR, one is left with a non-renormalisable field theory. This means that in the theory divergences appear in Feynman diagram expansions that cannot be absorbed by a finite number of counterterms, rendering the theory ill-defined [105, 171, 173].

The road to a consistent description of quantum gravity has turned out to be a long, difficult one. Currently there is no working theory that describes quantum gravity in our universe [105, 171]. Multiple proposals are being worked on, string theory being the most well-known [105]. String theory unifies GR with quantum mechanics and it keeps an explicit connection with QFT and a general relativistic description of spacetime. At low energies it describes GR, gauge theories and matter [171]. Its hypothesis is that, at the fundamental level, matter does not consist of point-particles but rather of tiny strings [105]. A problem inherent to quantum gravity theories, like string theory, is the bewildering array of possibilities of low energy theories: these quantum gravity theories are defined at the UV of the Planck Scale ($\sim 10^{19} \text{GeV}$) and need to describe our universe

all the way down at low energies [105].



Figure 2.11: Visual interpretation of an EFT, where the lowest-order corrections are the first step beyond GR, ending in some high energy UV completion. One does not need to know the exact form at the top to study the first layer. Figure adapted from [174].

Because of the lack of a complete theory and the large number of possibilities, there is little use in starting from quantum gravity at the Planck scale and working our way down, when confronting theory with observations. Instead of trying to find a theory of everything, this work adopts the reverse philosophy: starting from the well-known low energy scales of the Standard Model (SM) and GR, one builds up effective theories that describe and incorporate corrections at higher energies [59, 85–88]. Within this effective field theory (EFT) framework the low energy limit is the SM and GR, while deviations from these theories appear as higher-order operators suppressed by an appropriate energy/length scale. EFT provides a framework that is agnostic to the UV completion to incorporate beyond GR effects in the description of gravity [59, 85–88]. The EFT framework, despite going beyond GR, remains in itself a classical theory. The ‘goal’ should be that this originates from a unified theory of quantum gravity in the UV limit. The corrections to the action should encode the influence of this UV physics, without requiring the explicit knowledge of the underlying theory [59, 85–88]. This is the strength of an EFT. It is directly confrontable with observables and EFT allows a large family of theories to be mapped to a small set of constraints [33].

EFTs only have a limited range of applicability [59, 85–88]. An EFT is only valid within a certain range of energies. For example the SM is only valid up to energies of (at least) the energy of the Large Hadron Collider. This means the lowest-order correction to GR, so GR plus a lowest-order change to the action, has a limited range of energy and curvature where it is valid [59]. When one gets closer to the singularity of the black hole, at some point the curvature will be too large for the lowest-order modification to be enough, and

higher-order terms have to be taken into account.

Since GR is the low energy limit and corrections are higher-order in energy and thus curvature, beyond GR effects are most pronounced where the curvature is largest [33, 59, 85–88]. This implies that the best places to look for quantum gravity effects are at black holes or at the very early universe (near the big bang). Therefore one will typically use the strong field regime to test GR and quantum gravity effects. Thus far no deviations have been observed. GR has passed every test that was ever laid on it thus far [5, 6, 11, 13, 15, 73–77]. These observations put constraints on the amplitude of corrections of the Einstein-Hilbert action. Therefore it is expected that any deviation from GR that is seen in observables (like GWs) will be small. This assumes that observables behave smoothly in the GR limit [59]. For many quantities, like the QNMs that have been examined [2, 3], this holds. In a more general setting this may not always hold. This justifies that typically only the lowest-order corrections in the EFT are retained. In this thesis, too, only the first correction beyond GR will be taken into account. The question that one tries to answer with this is: what is the lowest-order deviation from GR in observables [28, 29, 32–34, 37, 38, 40, 41, 45–51]? In this thesis the question will be: what are the lowest-order shifts to the QNFs when one goes from GR to the first-order EFT? It is crucial to keep track of the order in EFTs. Results will only be valid in the lowest-order. It is also important to remain within the regime of validity of the EFT. Because it is unknown what the following non-vanishing contribution will be, it is hard to make exact statements. One can make general assumptions and make power counting arguments to study this regime of validity [4]. In the rest of the thesis, one will stay within the lowest-order correction systematically, to stay consistent with the EFT approach. This is non-trivial.

2.3.2 Higher-Curvature Gravity

The exact form of the EFT action depends on the symmetries one assumes to have. It is important to keep general diffeomorphisms (and local Lorentz invariance) as a symmetry in the system, even when going beyond GR, as our current understanding of causality uses these premises [81]. After making this assumption, one is left with only a handful of possibilities for the EFT. The correction to the action is then given by an extra term built out of the Riemann Tensor, the dual Riemann tensor, contractions thereof and (co-variant) derivatives with respect to the coordinates [59]. These possibilities are classified by the number of derivatives present in the action. EFTs built on these assumptions are called higher-derivative gravities or higher-curvature gravities [59, 85, 86, 175, 176]. In this thesis we call them higher-curvature gravities. All the theories discussed in the remainder of the thesis are examples of higher-curvature gravities, except when mentioned otherwise. The Einstein-Hilbert Lagrangian is simply given by the Ricci scalar, second-order in derivatives [81]. The possible first-order corrections can have 4 derivatives: quadratic gravity, 6 derivatives: cubic gravity, 8 derivatives: quartic gravity,... One can continue increasing the number of derivatives [86]. Loosely speaking, one can say that when increasing the number of derivatives even higher, the chances of this theory being the lowest-order correction and not some higher-order correction, if present, shrink. Therefore in the remainder of this thesis only the quadratic and cubic gravity cases will be considered. In the introduction, the quartic case will also be discussed briefly, because

together with quadratic and cubic gravity it is one of the main EFT theories [4, 86, 177].

An EFT starts from the Einstein-Hilbert action and adds an extra part. The lowest number of derivatives one can have in the modification is 4: quadratic gravity. In an EFT perspective, quadratic gravity is trivial or topological if you only allow for higher order operators from the metric tensor. Only when non-minimally coupling a scalar field, does the theory become non-trivial and/or non-topological [59]. There are two theories within the quadratic gravity case: scalar Gauss-Bonnet (sGB) and dynamical Chern-Simons (dCS). If one combines the Lagrangians of the theories, one gets what is called ‘Axi-Dilaton’ [59]. The Lagrangian of scalar Gauss-Bonnet (sGB) (excluding a term with higher derivatives on the scalar, as this is typically neglected [59, 178]) is given by:

$$\mathcal{L} = R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \alpha f(\phi)(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2). \quad (2.20)$$

Here α is the sGB parameter/amplitude, ϕ is a scalar field that is added to the theory with potential $V(\phi)$ and non-trivial coupling with the metric through $f(\phi)$, $f(\phi)$ is a coupling function of the scalar field [59, 61, 62]. The coupling has no constant part, because the term $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is topological, it does not alter the equations of motion of the theory, only when coupled to a scalar field [59, 176]. The theory has even parity symmetry. The Lagrangian of dynamical Chern-Simons (dCS) is given by:

$$\mathcal{L} = R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \alpha f(\phi)R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}. \quad (2.21)$$

Once again α is the dCS parameter/amplitude [59]. Here $\tilde{R}^{\mu\nu\rho\sigma}$ is the dual Riemann tensor, constructed out of the Riemann tensor and the Levi-Civita symbol $\epsilon^{\rho\sigma\alpha\beta}$ as follows [61, 62]:

$$\tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\rho\sigma\alpha\beta}R^{\mu\nu}_{\alpha\beta}. \quad (2.22)$$

The dCS-contribution has odd parity. After the quadratic theories one climbs up in order of curvature and derivatives and one finds cubic gravity, the first non-trivial purely gravitational theory [59, 176]. Here one has terms with six derivatives. Once again there are two possibilities, one with even parity and one with odd parity [59, 63, 176]. Cubic gravity is the lowest-order contribution to the beyond GR action which does not require extra scalar fields. Combining the cubic even and cubic odd terms, the Lagrangian becomes [59, 63, 176]:

$$\mathcal{L} = R + \alpha_{ev}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu} + \alpha_{odd}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}\tilde{R}_{\delta\gamma}{}^{\mu\nu}. \quad (2.23)$$

Here α_{ev} is the parameter/amplitude of the cubic even term and α_{odd} of the odd term [59]. If both the quadratic and cubic terms do not appear in the EFT, the lowest-order to appear is one with 8 derivatives: quartic gravity [59, 86]. Here one has multiple possible terms. The Lagrangian is given by:

$$\mathcal{L} = R - \alpha\mathcal{C}^2 - \tilde{\alpha}\tilde{\mathcal{C}}^2 - \alpha_-\tilde{\mathcal{C}}\mathcal{C}. \quad (2.24)$$

With $\mathcal{C} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $\tilde{\mathcal{C}} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$. The first two terms in the beyond GR part of the Lagrangian are even. The last term is odd. The α ’s are once again the parameters/amplitudes [86, 177]. In more exotic situations one can make other perturbations to

the Einstein-Hilbert action, but this thesis will not deal with this topic.

Each of these theories, quadratic, cubic and quartic, have reasons to be studied. Each of them can naturally appear as a lowest-order correction to GR from a UV quantum gravity theory [85, 86, 175, 176, 179–184]. From a UV point of view, quartic gravity is expected to be the lowest non-vanishing contribution beyond GR in most scenarios [86, 184, 185]. However, in some string theories or other quantum gravity theories both cubic gravity or quadratic gravity can be the lowest non-vanishing order [59, 85, 87, 88, 175]. Quadratic gravity is often studied because it allows for extra dynamics in the form of a scalar field and because it is the lowest possible order correction to the Einstein-Hilbert action. Cubic gravity is often studied because it does not have any additional scalar fields and therefore is the lowest-order correction that does not give rise to new degrees of freedom [59, 85, 87, 88, 175]. Quartic gravity is often studied as some subtheories within quartic gravity have interesting properties like isospectrality of QNMs and because it naturally follows from string theory as being the lowest non-vanishing contribution [86, 156, 186].

2.3.3 Black Holes in Higher-Curvature Gravity

A natural requirement when going beyond GR is that solutions of GR are not lost entirely. Black holes solutions should still exist and look like their GR counterpart. They should reduce ‘somewhat’ smoothly to GR as the beyond GR coupling λ is taken to zero [59, 61–63]. In the higher-curvature gravity cases this continuity is quite strictly enforced, as one will see below. The reasoning for this is that black holes have been observed and have within observational limits agreed with GR predictions. This holds strictly speaking only for Kerr black holes, because Schwarzschild black holes do not exist in astrophysical scenarios, but one can observe slowly spinning black holes [10, 11, 25, 122]. Extremal black holes are a more subtle case [4, 64]. The metric of a modified Kerr BH can be split up as:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \lambda g_{\mu\nu}^{(1)} + \mathcal{O}(\lambda^2), \quad (2.25)$$

where λ is a parameter that controls the deviations beyond GR. At zeroth order in λ , the metric is the usual Kerr metric. For quadratic gravity, one has $\lambda = \frac{\alpha^2}{M^4}$ with α given in the previous Section 2.3.2 and M the mass of the black hole. For cubic and quartic gravity, one has $\lambda = \frac{\alpha}{M^4}$ [59, 61–63, 177]. This implicitly incorporates a continuous limit from beyond GR into GR. In higher-curvature gravities discussed in this thesis, one can write the first-order metric as follows in modified Boyer-Lindquist coordinates:

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu = (g_{\mu\nu}^{(0)} + \lambda g_{\mu\nu}^{(1)}) dx^\mu dx^\nu = & - \left(1 - \frac{2M\rho}{\Sigma} - H_1 \right) dt^2 - (1 + H_2) \frac{4Ma\rho(1-x^2)}{\Sigma} dt d\phi \\ & + (1 + H_3) \Sigma \left(\frac{d\rho^2}{\Delta} + \frac{dx^2}{1-x^2} \right) + (1 + H_4) \left(\rho^2 + a^2 + \frac{2a^2 M \rho (1-x^2)}{\Sigma} \right) (1-x^2) d\phi^2. \end{aligned} \quad (2.26)$$

Here ρ is the radial coordinate [59]. It is very important to note that we are using a different coordinate ρ in the radial direction. This radial coordinate ρ behaves differently from r and has different asymptotic behaviour (see Eq. (3.8) in [59]). The coordinate is chosen in such a way that the horizons are still at the same position ($\rho_\pm = r_\pm$) for

the modified Black holes as for the GR one with the same mass and spin [59]. One has $x = \cos \theta$ the cosine of the polar angle, $\Sigma = \rho^2 + a^2 x^2$, $\Delta = \rho^2 - 2M\rho + a^2 = (\rho - \rho_+)(\rho - \rho_-)$ with $\rho_{\pm}$ the radii of the horizons of the Kerr black hole [59]. H_i are the functions that describe the first-order modifications to the metric of the Kerr BH. To make sure a and M still carry the physical meaning of spin and mass of the black hole, the boundary conditions of the H_i are the following [59]:

$$h_1^{(0)} = 0, h_3^{(0)} = h_4^{(0)} = -\frac{h_3^{(1)}}{M}, h_2^{(0)} = -\frac{h_3^{(0)}}{2}, h_1^{(1)} = -Mh_3^{(0)}, \quad (2.27)$$

where the H_i have been written asymptotically at infinity as: $H_i = h_i^{(0)} + \frac{h_i^{(1)}}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)$. In this case the superscripts do not denote the expansion in λ but the asymptotic expansion. An important but subtle remark to make is that the extremality of black holes changes in higher-curvature theories [59, 62]. In GR the extremality condition is that $a = M$ [81]. This is no longer true for beyond GR. The proper way to determine the extremality condition, is to look at the vanishing of the surface gravity of the black hole [81]. Intuitively this means that an extremal black hole is one where the two horizons merge to one. In the coordinates chosen above, the horizons seem to coincide at $a = M$ like in GR, because it was mentioned that in these coordinates the horizon is at the same position as in GR [59, 62]. This is however purely a gauge artefact. Changing coordinates, it can become clear that the horizons do not actually merge and the surface gravity does not vanish at $a = M$. The metric perturbation as it is written above, is thus only valid below extremality and not at extremality [59]. This is different from GR, where one can take some (this is subtle) limit $a \rightarrow M$ to describe the extremal black hole [81]. This is a first indication that extremal black holes are more difficult to capture in beyond GR theories. This will reappear in Chapter 3. The modified Boyer-Lindquist coordinates used above, are thus only valid in the region between $a = 0$ and strictly less than the smaller of the beyond GR extremality bound and the black hole mass M (GR extremality). If beyond GR the extremality condition lies beyond $a = M$, then the coordinates are valid up to (but not including) $a = M$ [59].

Finding black hole solutions in higher-curvature gravity is then equivalent to solving for the H_i functions. These are found by solving the equations of motion of the modified GR action. In GR this results in the EFE [81]. For beyond GR theories this gives rise to a modification of the EFE. One can keep everything first order in the beyond GR parameter λ by solving the modified EFE in lowest-order in λ . This results in $H_i \rightarrow \lambda \tilde{H}_i$ with $\tilde{H}_i$ independent of λ [59, 61–63]. Up until 2025, the only solutions to the modified black hole were given by a spin expansion [50, 55–60]. This means to express the H_i as a Taylor expansion of the dimensionless spin parameter $\chi = a/M$, and solving the modified EFE for each order in spin. Because of the slow convergence towards higher spins, solutions were reliable up to $0.7 - 0.9$ [59]. Only in 2025 came the first solutions for the entire astrophysically relevant spin range [61–63]. This was done by solving these equations with spectral techniques. This means one decomposes the signal into a basis and then searches for the amplitudes corresponding to the different basis functions (in this case Legendre and Chebyshev polynomials have been used) [1, 68–71]. The black hole spacetime is then described through H_i and the H_i are given as a superposition of different Chebyshev and Legendre polynomials with corresponding amplitudes. Notice

that one then approximates the H_i as a polynomial function. For the radial coordinate this necessitates compactifying. Chebyshev polynomials are typically defined on $[-1, 1]$. This means infinity needs to be mapped to a finite (radial) coordinate. The choice of compactification is still free [61–63].

Since up until recently [61–63] there were no higher spin black hole solutions, there were no QNM solutions for highly spinning black holes in beyond GR theories [50, 59, 60]. This was unfortunate, since astrophysical black holes have rather large spins, and beyond GR features are more pronounced for larger spins [4, 10, 25]. This thesis is focused on solving this gap in theoretical knowledge. Studying QNMs in beyond GR theories is highly important because, as said before, the no-hair theorem tells us that QNMs are one of the best ways to test GR and look for beyond GR theories [25]. Therefore it is important to study these QNMs for astrophysically relevant spins. The thesis succeeds in developing a method to calculate and observationally test these QNMs [2, 3]. This will be described in Chapters 3 and 4.

2.4 Detection and Data Analysis

If one wants to measure beyond GR effects, one needs GW detection techniques. A lot of research is being done in this context [17, 25, 126]. This section sketches the relevant background for using ringdown observations to test GR. In current GW detectors the noise amplitude is larger than the signal amplitude. This makes it difficult to reliably extract the relevant observables from the data. To solve this problem, sophisticated detection statistics are used [126]. The first important step is to find the signal buried under the noise. For this, it is crucial to understand not only the waveform one expects, but also the properties of the noise. Most suited in this context is the matched filtering method [111, 126].

Matched filtering is the idea that if one knows what one is looking for, the signal can be found even when the noise overwhelms the signal, provided we understand the noise as well. It integrates the data with a template waveform. This integrates out the noise, because this averages out. The optimal filter to integrate with is not simply the template waveform, but the waveform weighted by the power spectral density (PSD), called a Wiener filter [111]. The PSD describes the noise: it can be understood as being the amount of power the noise has for a given frequency. There is an entire template bank, containing template waveforms for black hole coalescences. This multi-dimensional space is probed efficiently and in practice this is how GWs are detected [9, 111, 114].

Despite this being the first pipeline in the detection statistical framework, it is not the end of the story. Matched filtering is fit for searching for events, but not suited for a detailed statistical analysis for source parameters. For this, we use Bayesian statistics [126].

2.4.1 Bayesian statistics

Bayesian statistics is the approach to statistics where probabilities express a measure of belief, based on prior knowledge and likelihood of an event [187]. This differs from the

frequentist approach in how probability is interpreted. In the Bayesian framework (used in this thesis), the data is what it is, but the uncertainty describes that the parameters behind the event are unknown. Posterior probability distributions describe how strongly one believes a parameter to have a certain value. In the frequentist approach, the parameters behind the event are fixed, but the data fluctuates due to ‘randomness’. Probabilities describe this randomness and are in a sense ‘inverted’ to describe the probability that the intervals capture the true fixed (but unobserved) parameters [187, 188].

Bayesian statistics are based on Bayes’ theorem for conditional probabilities: $p(A|B, I) = \frac{p(B|A, I)p(A|I)}{p(B|I)}$. Here A, B are two propositions, and I some background information relevant to the propositions. Consider now a model M which describes GWs or the observation of interest, the data d and the parameter(s) ϑ . The Bayes’ theorem is then used to calculate the probability distribution of the parameters one wants to know [187]. One thus uses:

$$p(\vartheta|d, M) = \frac{p(d|\vartheta, M)p(\vartheta|M)}{p(d|M)}. \quad (2.28)$$

In the context of observations, one explains this as follows: The posterior $p(\vartheta|d, M)$ is given by the product of the likelihood $p(d|\vartheta, M)$ and the prior $p(\vartheta|M)$ divided by the evidence $p(d|M)$. The posterior describes the probability of the parameters ϑ given the observed data d and assuming the model M . The likelihood describes how likely the data is for a given set of parameters and model. The prior describes our knowledge on the parameters prior to the observation [187]. It describes what we initially believe the distribution of the parameters to be in this model. In many cases this is unknown. In that case one often uses a flat (‘uninformative’) prior, where all physically possible values are assigned equal weight [111, 187, 189, 190]. Finally, the evidence describes the probability of the data being generated by this model. Indirectly it can be understood as a measure of the probability of the model itself. It plays an important role in comparing different models, which is done with the Bayes factor. When comparing two models, the Bayes factor is the ratio between the evidences of the models. The most probable model will have the largest evidence. If one is simply interested in the posterior, the evidence is of no importance as it is a scaling of the probability, equivalent to normalising the probability [187].

The focus will lie on constraining beyond GR corrections from observational data. This means finding the posterior distributions of the beyond GR parameters, with the data d being the ringdown part of GW detections. The prior on the EFT coupling will be taken flat, as no prior knowledge on the beyond GR coupling can be assumed.

2.4.2 Ringdown analysis

Apart from the prior, one also needs the likelihood to evaluate the posterior. Calculating the likelihood is the key part of any Bayesian statistical pipeline [187]. To calculate the likelihood, we need to make assumptions on how the noise looks. The noise is assumed to be stationary coloured Gaussian noise [25, 111, 126]. Stationary means that the properties of the noise do not change over time, which implies that frequency bins are also assumed to be uncorrelated. For current detectors and lengths of the signal, this is a good

approximation. Coloured Gaussian means that the noise is distributed as a Gaussian, but where the variance is a function of frequency. Not every frequency has the same power of noise. This is described through the PSD $S_n(f)$ [25, 111, 126].

Under these assumptions, the probability density function describing the noise $\mathbf{n}$ is given by a multivariate normal distribution [25]:

$$p(\mathbf{n}|M) = \frac{1}{\mathcal{N}} e^{-\frac{1}{2} \sum_{k=0}^{Q-1} \mathbf{n}_k^T \mathbf{C}_k^{-1} \mathbf{n}_k}. \quad (2.29)$$

Here $\mathbf{C}_k$ is the covariance matrix of the noise in detector k , where there are Q detectors. $\mathcal{N}$ is the normalisation. $\mathbf{n}$ is the vector of noise realisations sampled every Δt seconds [25]. We assume the data from our detectors are given by $\mathbf{d}$, that the noise $\mathbf{n}$ is described by the distribution above and that the data is simply the sum between noise $\mathbf{n}$ and signal $\mathbf{h}$. The residual $\mathbf{d} - \mathbf{h}$ is therefore expected to look like noise [25, 111, 126]. We can obtain a likelihood by realising that the distribution for a certain dataset for a given set of parameters ϑ is simply given by the distribution that the corresponding noise is given by $\mathbf{n} = \mathbf{d} - \mathbf{h}(\vartheta)$. We can thus say that the likelihood is given by [25]:

$$p(\mathbf{d}|\vartheta, M) = \frac{1}{\mathcal{N}} e^{-\frac{1}{2} \sum_{k=0}^{Q-1} (\mathbf{d}_k - \mathbf{h}_k(\vartheta))^T \mathbf{C}_k^{-1} (\mathbf{d}_k - \mathbf{h}_k(\vartheta))}. \quad (2.30)$$

If we assume the noise in the different detectors has been characterised and that the signals $\mathbf{h}$ are known, the main remaining difficulty is to probe this likelihood accurately and efficiently [5, 18, 25, 191–193]. A first difficulty lies in calculating the covariance matrix and inverting it. To do this as efficiently as possible, one uses the structure of the matrix. First of all the assumed stationarity makes it such that the covariance matrix is only dependent on time differences and not on absolute time, rendering the matrix in a form called a Toeplitz matrix [25]. Even more efficient inversion techniques exist if one assumes the noise has periodic boundary conditions. This renders the matrix circulant, which allows a very efficient inversion of the covariance matrix. As long as the periodic boundary condition approximation holds, this allows for a more efficient building of the sampler [25]. The sampler is the algorithm that explores the parameter space and builds up the posterior distribution described earlier. The sampler used in this thesis will be discussed in a few paragraphs [194].

In ringdown data it is however not justified to approximate periodic boundary conditions, as imposing these periodic boundary conditions leads to biased or lost information at the boundaries of the data segment [25]. If one can window the data such that unimportant data is present at the edges, this is not a problem. However, this is not possible because the ringdown model starts at a certain time, at which the signal is immediately relevant [25]. One cuts out data before the merger and from the moment the ringdown data starts, there is crucial information present. Many techniques or ideas to bypass this have been thought of, but many turn out to be insufficiently good. Two solutions are possible for this problem. The first solution is to add pre-merger data in the model, the second solution is to give up on the faster inversion of the covariance matrix [25]. In this thesis we only use information from the ringdown and thus focus on the second solution.

As the noise characterisation is done in the frequency domain, the construction of the

covariance matrix typically involves an inverse Fourier transform, which implicitly imposes circularity. To break this, the transform must be truncated. This is valid because typically the time segment used to characterise the noise PSD is a lot longer than the time segment of ringdown data [25]. In doing so one constructs the covariance matrix and one can evaluate the likelihood. It is important to realise that the duration of the dataset of interest is necessarily longer than the duration of the ringdown signal. This follows from the observation that the noise is coloured Gaussian noise and thus correlated in time beyond the signal time [25].

Apart from the noise model, the model to describe the QNMs themselves is also crucial. This is the M in the notation used earlier and it determines the parameters of interest. Different models to describe ringdown data exist, each with their own level of flexibility [5, 11, 12, 16, 18, 25, 74, 191–193]. The most abstract and flexible model is called **damped sinusoids**. It fits damped sinusoids on the signal and does not assume any underlying black hole structure. This means it provides an agnostic test of our ringdown knowledge and the Kerr hypothesis [25, 195]. The strain is given by:

$$h(t) = \sum_{n=0}^N A_n e^{-(t-t_0)/\tau_n} e^{-2\pi f_n(t-t_0)+i\phi_n}. \quad (2.31)$$

n indexes the modes. A is the amplitude with ϕ the phase. τ is the damping time and f the frequency. $N+1$ denotes the number of modes considered and t_0 is the starting time of ringdown [25, 195]. A more involved, physically motivated, model is the **Kerr** model. In this case the modes are no longer agnostic but informed by the Kerr black hole model. The modes are labelled by the indices l, m, n and correspond to the QNMs associated with them. It assumes the frequency and damping time are given by the Kerr parameters [13, 25]. The strain is given by:

$$h(t) = \sum_{l,m,n} A_{l,m,n} e^{-(t-t_0)/\tau_{lmn}(M,a)} e^{-2\pi f_{lmn}(M,a)(t-t_0)+i\phi_{lmn}}. \quad (2.32)$$

The parameters one extracts here are the amplitudes and phases of the QNMs and the mass and spin of the remnant black hole. It allows for the immediate connection of data with the astrophysical properties of the remnant [13, 25]. It is this model which is relevant for the thesis. The only difference is that we add an extra degree of freedom: the beyond GR coupling parameter. The QNMs are then given by the beyond GR spectra [5]. This will be covered in more detail in Chapter 4. It allows for explicitly constraining beyond GR parameters directly. Another way to test the Kerr hypothesis is by doing an agnostic test, based on measuring deviations to the QNMs without relying on an underlying theory. These techniques, however, are less sensitive [25]. More elaborate models exist, where not only the frequencies but also the amplitudes are modelled. These ringdown models are based on amplitudes as they come out of NR. The intrinsic parameters one infers are then the initial masses and spins of the two merging black holes [25]. In this thesis we will not assume any amplitude model.

There are some conceptual problems with ringdown observations. First and foremost the description of ringdown by QNMs is limited in time. When one gets too close to the merger, the signal is no longer linear and non-linear modes have to be taken into account

[25, 148]. In the late time regime the QNMs are no longer sufficient and a polynomial tail becomes dominant. The latter is for current detectors negligible. The former, however, is of crucial importance. On the one hand one wants to start the ringdown analysis as early as possible after the merger to maximise signal-to-noise ratio (SNR). On the other hand one wants to start the ringdown analysis sufficiently late after the merger such that the linear regime is a good approximation. At the moment of writing, no well-motivated prescription is known for the starting time of the QNM analysis [25]. Associated with this is the fact that damped sinusoids are prone to overfitting. This means that the data may be well described by QNMs, even if this does not make sense from a physical point of view. If the starting time is before the merger, you may well be able to fit QNMs very well to the data, despite the linear regime not yet being valid at such early times (one may not even have a remnant black hole yet) [25]. Another issue regarding the starting time, is that this may be different for different detectors. To match these times, one is required to fix the sky-location of the signal before the analysis. Methods that avoid these issues rely at least to some degree on pre-merger data. Although useful, these methods lack the simplicity of black hole spectroscopy [25]. These issues surrounding the starting time can be somewhat mitigated by varying it and examining the resulting differences. More systematic issues may exist, but will not be covered in this thesis introduction.

We now know how to evaluate the likelihood and with it the posterior. The posterior is sampled through nested sampling. This is an algorithm used to probe complicated and/or unknown posterior distributions and calculate the evidence of such a distribution [194, 196, 197]. It generates randomly a starting set of live points, sampled from the prior. At each point one calculates the likelihood. Then one iteratively removes points and replaces them by new points. At each iteration the point with the lowest likelihood is removed. This is called a dead point [196, 197]. The point is replaced by a new live point with a higher likelihood. As one iteratively increases the minimum likelihood of the sampled live points, the probed size of the parameter space shrinks associated with this. If there are N points, each point constitutes on average a fraction $1/N$ of the volume of the parameter space. Removing the point with the lowest likelihood with a point of higher likelihood takes away approximately such a volume. One immediately sees the probed volume shrinks (on average) exponentially as one iterates through the live points [196, 197].

Nested sampling probes the prior from the worst to the best likelihood, until only the very highest likelihoods are left as live points. At this point one has an idea of the posterior distribution, by having probed the prior parameter space with the associated likelihoods (what is needed to calculate the posterior) [196, 197]. A big advantage of the nested sampling algorithm, is that it immediately also calculates the evidence. It does so by noting that each dead point p_i shrank the probed parameter space by some approximate volume δV_i . The contribution to the evidence is then given by $\delta V_i \cdot L_i$ with L_i the likelihood of that dead point. Remember that the evidence was the integration of the likelihood over the parameter space. The evidence is needed to compare models in a Bayesian statistics framework [187, 196, 197].

The exact point at which the nested sampling should be stopped can be determined in many ways, but intuitively it becomes clear that at some point all the live points will

be near the very top of the likelihood functional. At this point little progress can be made and it becomes computationally best to terminate. The accuracy of the sampling is determined by the number of live points used. Because of the exponential decrease in the probed volume, the method is useful for highly dimensional inference problems [196, 197]. The curse of dimensionality says that typically decomposing the parameter space grows exponentially in computational cost when the dimensionality increases. A vast part of the parameter space will be relatively unimportant for the posterior and evidence as it will have very low likelihood. However, the computational cost of nested sampling does not grow exponentially as a function of dimensionality, because of the exponential damping of volume. There are thus three main benefits for using nested sampling. First, it is useful for multi-modal problems (when the likelihood has multiple local extrema). Secondly, it not only calculates the posterior but also the evidence. And third it adapts well to highly dimensional problems [196, 197].

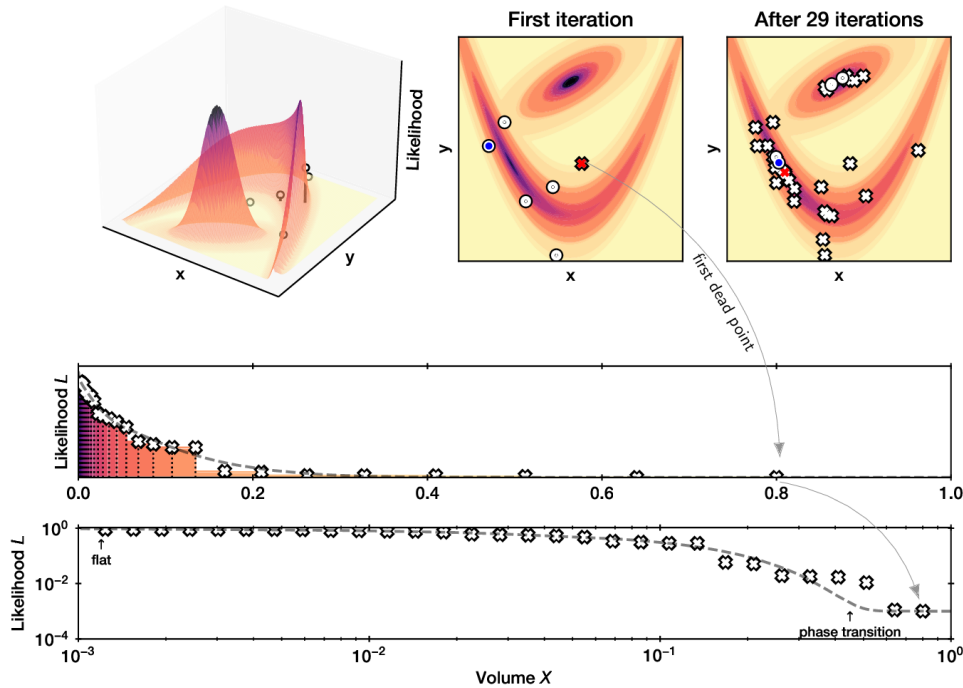


Figure 2.12: Schematic example of nested sampling. Firstly, the prior gets sampled through live points. Live points die, become dead points and are replaced by higher-likelihood live points. The volume shrinks exponentially and the method works well with multi-modal distributions like the one visualised here. Figure from [197].

What is most difficult in nested sampling, is the choice of a new live point once the live point with the lowest likelihood has been killed. This point must have a likelihood higher than the dead point, but this leaves one with many techniques to sample the new live point [196, 197]. In CPNest, the nested sampler used by `pyRing`, Markov chain Monte Carlo (MCMC) methods are used. `pyRing` is the codebase developed by LIGO to analyse ringdown GW signals [7]. It is also the codebase used in this thesis. MCMC is a framework used to probe difficult and/or unknown probability densities [194, 198]. It generates sequences (Markov Chains) of random samples from a probability distribution. The chain consists of a sequence of samples that explore (at least a little bit) the parameter space. The sequence is a Markov Chain, meaning that the proposal of the next step is only

dependent on the current step and not any previous information. The original points of the chains are either chosen randomly, or are recently deceased points. After some while the random walk of the MCMC finishes and the final step will be proposed as the new live point for the nested sampler [196–198]. First, we need to describe the iteration scheme within the MCMC itself.

At each new iteration, a new sample value for the sequence is proposed. This proposal is either accepted and used for the next iteration, or rejected, and the old value is reused in the following iteration. The acceptance or rejection is often based on the probability distribution one is trying to sample. The acceptance chance is *for example* calculated by comparing the value of the probability function at the old value and the new value [198]. The acceptance ratio is given by the ratio of the probabilities. Important is that one only needs to know a function proportional to the probability, not the probability itself. This removes the need to a priori know the normalisation of the probability. One then generates a number uniformly between 0 and 1. If this number is smaller than the acceptance ratio, the new value is accepted. It is rejected otherwise [196–198]. One can design more elaborate sampling schemes than the very simplistic example shown here. The probability that is sampled is called the constrained prior. This is given by:

$$\pi^*(\vartheta) \sim \begin{cases} \pi(\vartheta) & \text{if } L(\vartheta) > L_{thresh} \\ 0 & \text{otherwise,} \end{cases} \quad (2.33)$$

where L_{thresh} is the threshold minimum likelihood [196].

Proposing new samples is done through a certain probability function $g(x|y)$ where x is the new sample and y the old one. One can *for example* choose a Gaussian distribution around y to not make the jumps too large. Multiple choices can be made for this function. In practice some prior knowledge using e.g. the known likelihoods and live points must be used. Realise that the step proposal for the MCMC changes as the live points change, as the needed likelihood increases and the probed parameter space shrinks [194, 196–198]. At the end of the MCMC one returns to the nested sampler with a nearly independent sample, with higher likelihood. The number of steps needed in the MCMC depends on the implementation and the balance between minimising the number of steps of the MCMC and making sure the sampling space is probed well. Since one is dealing with a Markov Chain, the elements in the sequence are correlated. This means additional elements add less new information than independent samplings, reducing the efficiency of the algorithm [198]. If one makes the stepping size (e.g.) smaller in the sequence or one takes less iterations in the sequence, the autocorrelation becomes larger, the efficiency decreases and the computational cost might increase. At the end of the sequence, the last point (which will be a new live point) must be roughly independent of the starting point for the MCMC [196, 197].

The construction of the posterior distribution has been explained, involving a prior, a likelihood, a sampling algorithm and a model [7, 194]. This ringdown data analysis has been developed into a codebase called **pyRing** [7]. **pyRing** contains the pipeline to calculate the relevant detection statistics and posteriors associated to GW-ringdown signals. The work done in this thesis starts from this codebase, developed by LVK [7, 194].

Chapter 3

Scalar Quasinormal Modes

This chapter describes the theoretical results obtained in this thesis. We calculate scalar QNMs in multiple beyond GR theories across a large set of spins and modes, as mentioned in Chapters 1 and 2. These results have been submitted for publication and were released as preprints: arXiv:2604.02214 [2] and arXiv:2604.11755 [3]. The framework is in part based on [1]. In the remainder of this chapter, we will use units in which $c = G = M = 1$. The spin a then goes from 0 to 1 in GR.

3.1 Klein-Gordon Equation

In this thesis, we are interested in how scalar fields damp out and stabilise around a black hole when it is perturbed. More specifically we will decompose the perturbations in quasinormal modes: the damped oscillations of the scalar field [2, 3, 25]. The goal will be to find how the QNFs (so the frequencies and damping times of these characteristic damped oscillations of the scalar field) change in different beyond GR scenarios. For this one requires the Klein-Gordon (KG) equation, which is the relativistic wave equation for a scalar field. It is often written down in position space. One then writes: $(\square + m^2)\phi = 0$ with m a mass term. This is for a scalar field ϕ with a mass term as the potential. One can add extra potential terms to increase the complexity of the KG equation. Intuitively one can see the KG equation as the differential equation form of the energy-momentum relation for a scalar field: $-E^2 + (|p|c)^2 + (mc^2)^2 = 0$ [81, 199]. Here we reinstated the factors of c . E is the energy, p the three-momentum and m the mass. In solving the KG equation for a scalar field on the curved background of the black hole, one solves its relativistic wave equation [81, 199]. Finding such solutions (the QNMs) will be the core of this chapter. We will restrict ourselves to massless scalar fields that are minimally coupled to the metric. The latter means there is no direct coupling between the scalar and curvature invariants. As mentioned in Section 2.3, this thesis only considers leading-order corrections to GR.

To calculate the frequencies for scalar QNMs in higher-curvature gravity theories, we need to solve the KG equation

$$\square\psi = 0 \tag{3.1}$$

on a curved background. The d'Alembertian (box operator $\square$) for a scalar field on a

curved background is given by [200]:

$$\square\psi = \frac{1}{\sqrt{-g}}\partial_\mu (g^{\mu\nu}\sqrt{-g}\partial_\nu\psi). \quad (3.2)$$

For scalar QNMs in higher-curvature gravity, the background in question is a modified Kerr BH [59], which can be split up as described in Section 2.3.3. At zeroth order in λ , the metric is the usual Kerr metric, as described in Section 2.1.3. Now we switch to higher-curvature gravity, where Kerr is no longer a solution to the equations of motion of the theory. However, since we are working perturbatively in λ , it is possible to recast the solution for a stationary, axially symmetric BH in the form of Eq. (2.26). Here, the four functions $H_i(\rho, x)$ are linear in λ and encode the corrections to the Kerr geometry [59, 61–63]. Recall also that the coordinates have been chosen, such that the horizons of this modified BH are still at the roots $\rho_\pm = M \pm \sqrt{M^2 - a^2}$ of Δ . The boundary conditions of H_i were given in Eq. (2.27) [59].

The modified Kerr background is still stationary and axially symmetric, hence this metric admits the Killing vectors ∂_t and ∂_ϕ . As exhibited in Section 5 of [47], we can separate these variables out readily, and make the ansatz

$$\psi = \int_{-\infty}^{\infty} d\omega \sum_{m=-\infty}^{\infty} e^{i(m\phi - \omega t)} \psi_{m,\omega}(\rho, x), \quad (3.3)$$

such that the KG equation simplifies to

$$\square\psi = \int_{-\infty}^{\infty} d\omega \sum_{m=-\infty}^{\infty} e^{i(m\phi - \omega t)} \mathcal{D}_{m,\omega}^2 \psi_{m,\omega}. \quad (3.4)$$

Every component $\psi_{m,\omega}$ now satisfies a Laplace equation of the form

$$\mathcal{D}_{m,\omega}^2 \psi_{m,\omega} = 0. \quad (3.5)$$

The square on the $\mathcal{D}$ denotes that the differential operator is quadratic in derivatives. In practice, it is this equation one needs to solve to find the QNMs [2, 3]. The QNMs are the solutions to this equation that solve the physical boundary conditions. These boundary conditions are given by the causal structure of spacetime. One is interested in the spectrum of complex frequencies ω for which there is a solution to the equation above, which is purely ingoing at the horizon and purely outgoing at infinity. Realise that this integral in Eq. (3.4) is over the real line of frequencies. As in Section 2.2.3, one can change this into a contour integral of a frequency space Green's function and one can see the QNMs as the poles inside the contour [22–25]. Effectively this means one should find the complex frequencies which solve this Eq. (3.5) with boundary conditions. We continue now with finding these solutions. Since we're working perturbatively in λ , again we can decompose

$$\mathcal{D}_{m,\omega}^2 = \mathcal{D}_{(0)m,\omega}^2 + \lambda \mathcal{D}_{(1)m,\omega}^2 + \mathcal{O}(\lambda^2). \quad (3.6)$$

The expressions for this Laplacian are given below [47]. At zeroth order, we have

$$\begin{aligned} \mathcal{D}_{(0)}^2 \psi = & \frac{1}{\Sigma} \partial_\rho (\Delta \partial_\rho \psi) + \frac{1}{\Sigma} \partial_x ((1 - x^2) \partial_x \psi) \\ & - \frac{\psi}{\Delta \Sigma} \left[\frac{(\Sigma - 2M\rho)m^2}{1 - x^2} + 4Ma\rho m\omega - \left(2M\rho a^2 + \frac{\Sigma(\rho^2 + a^2)}{1 - x^2} \right) \omega^2 \right], \end{aligned} \quad (3.7)$$

which is the usual Teukolsky equation in the $s = 0$ scalar case. The first-order operator reads

$$\mathcal{D}_{(1)}^2 \psi = -\frac{H_2 \Delta}{\Sigma} \partial_\rho^2 \psi - \partial_x^2 \psi + \frac{P}{\Sigma} \partial_\rho \psi + \frac{Q}{\Sigma} \partial_x \psi + \frac{S}{\Sigma} \psi, \quad (3.8)$$

where the expressions of P , Q and S as a function of H_i with $i \in \{1, 2, 3, 4\}$ are given by: [3, 47].

$$\begin{aligned} P = & -\frac{H_2^{(1,0)}}{\Sigma^2} 4a^2 M^2 \rho^2 (x^2 - 1) + 2H_3 (M - \rho) \\ & + \frac{(\Sigma - 2M\rho)H_4^{(1,0)} - \Sigma H_1^{(1,0)}}{2\Sigma^2} (a^4 x^2 + a^2 \rho (-2Mx^2 + 2M + \rho + \rho x^2) + \rho^4) \\ & + \frac{4a^2 (H_4 - 2H_2)}{\Delta \Sigma^3} M^2 \rho (x^2 - 1) (a^4 x^2 - a^2 \rho (Mx^2 + \rho) + \rho^3 (3M - 2\rho)) \\ & + \frac{H_1 M}{\Delta \Sigma^2} (-a^6 x^2 + a^4 \rho^2 (1 - 2x^2) + a^2 \rho^3 (4M(x^2 - 1) - \rho(x^2 - 2)) + \rho^6) \end{aligned} \quad (3.9)$$

$$\begin{aligned} Q = & \frac{H_2^{(0,1)}}{\Delta \Sigma^2} 4a^2 M^2 \rho^2 (x^2 - 1)^2 - \frac{2a^2 H_1}{\Delta \Sigma^2} M \rho x (x^2 - 1) (a^2 + \rho^2) + 2H_3 x \\ & - \frac{(x^2 - 1)H_4^{(0,1)}}{2\Delta \Sigma^2} (a^6 x^4 + a^4 \rho x^2 (\rho(x^2 + 2) - 2Mx^2) + a^2 \rho^2 (4M^2 (x^2 - 1) - 4M\rho x^2 + \rho^2 (2x^2 + 1)) \\ & + \rho^5 (\rho - 2M)) + \frac{4a^2 (H_4 - 2H_2)}{\Delta \Sigma^3} M^2 \rho^2 x (x^2 - 1) (a^2 (x^2 - 2) - \rho^2) \\ & + \frac{(x^2 - 1)H_1^{(0,1)}}{2\Delta \Sigma} (a^4 x^2 + a^2 \rho (-2Mx^2 + 2M + \rho + \rho x^2) + \rho^4) \end{aligned} \quad (3.10)$$

$$\begin{aligned} S = & \frac{4aH_2}{\Delta^2 \Sigma^2} M \rho [2am^2 M \rho (a^2 x^2 + \rho(\rho - 2M)) - 8a^2 m M^2 \rho^2 (x^2 - 1) \omega - m\omega \Delta \Sigma^2 \\ & + 2aM \rho (x^2 - 1) \omega^2 (a^4 x^2 + a^2 \rho (-2Mx^2 + 2M + \rho + \rho x^2) + \rho^4)] \\ & + \frac{H_1}{\Delta^2 \Sigma} (a^4 x^2 \omega + a^2 \rho \omega (-2Mx^2 + 2M + \rho + \rho x^2) - 2amM \rho + \rho^4 \omega)^2 \\ & - \frac{H_4}{(x^2 - 1) \Delta^2 \Sigma^2} (a^4 x^2 + a^2 \rho (-2Mx^2 + 2M + \rho + \rho x^2) + \rho^4) \\ & \cdot (a^2 m x^2 - 2aM \rho (x^2 - 1) \omega + m \rho (\rho - 2M))^2. \end{aligned} \quad (3.11)$$

Here the superscript $^{(i,j)}$ denotes that we take the i -th derivative in the radial coordinate and j -th derivative in the angular coordinate. Here we use $x = \cos \theta$. It is important to realise that the H_i are present here, because it is the change in the background which causes the scalar QNMs to be modified beyond GR [2, 3]. Remember that the Teukolsky equation is separable in GR [25]. As a consequence the KG equation is separable in zeroth order in λ . In this case, the operator can be decomposed as (notation from [1])

$$\mathcal{D}_{(0)}^2 \equiv \frac{1}{\Sigma} (L_\rho + L_x), \quad (3.12)$$

where [47, 112]

$$L_x \Theta(x) \equiv (1 - x^2) \frac{d^2}{dx^2} \Theta(x) - 2x \frac{d}{dx} \Theta(x) + \left(A_{lm}(a\omega) + a^2 \omega^2 x^2 - \frac{m^2}{1 - x^2} \right) \Theta(x), \quad (3.13)$$

$$L_\rho R(\rho) \equiv \frac{d}{d\rho} \Delta \frac{d}{d\rho} R(\rho) + \frac{1}{\Delta} (a^2 m^2 - 4amM\rho\omega + \omega^2(\rho^4 + a^2 \rho^2 + 2a^2 \rho M)) R(\rho) - A_{lm} R(\rho). \quad (3.14)$$

We can solve this system Eq. (3.6) by making the perturbative ansatz (see Eq. (115)-(116) of [1])

$$\psi = \psi_{(0)} + \lambda \psi_{(1)} + \mathcal{O}(\lambda^2), \quad \omega = \omega_{(0)} + \lambda \omega_{(1)} + \mathcal{O}(\lambda^2), \quad (3.15)$$

separating the different orders in the wavefunction and frequency. Grouping together terms of the same order in λ and neglecting higher-order terms $\mathcal{O}(\lambda^2)$, we obtain the following [2, 3]:

$$\begin{aligned} 0 &= \mathcal{D}_{m,\omega}^2 \psi \\ &= (\mathcal{D}_{(0),m,\omega}^2 + \lambda \mathcal{D}_{(1),m,\omega}^2) (\psi_{(0)} + \lambda \psi_{(1)}) + \mathcal{O}(\lambda^2) \\ &= (\mathcal{D}_{(0),m,\omega_{(0)}}^2 + \lambda \mathcal{D}_{(1),m,\omega_{(0)}}^2 + \lambda \omega_{(1)} \partial_{\omega_{(0)}} \mathcal{D}_{(1),m,\omega_{(0)}}^2) (\psi_{(0)} + \lambda \psi_{(1)}) + \mathcal{O}(\lambda^2) \\ &= \mathcal{D}_{(0),m,\omega_{(0)}}^2 \psi_{(0)} + \lambda \left(\mathcal{D}_{(1),m,\omega_{(0)}}^2 \psi_{(0)} + \mathcal{D}_{(0),m,\omega_{(0)}}^2 \psi_{(1)} + \omega_{(1)} \partial_{\omega_{(0)}} \mathcal{D}_{(0),m,\omega_{(0)}}^2 \psi_{(0)} \right) + \mathcal{O}(\lambda^2). \end{aligned} \quad (3.16)$$

This results in the system of equations [2, 3]:

$$\begin{cases} \mathcal{D}_{(0),m,\omega_{(0)}}^2 \psi_{(0)} &= 0 \\ \mathcal{D}_{(0),m,\omega_{(0)}}^2 \psi_{(1)} &= -\omega_{(1)} \delta \mathcal{D}_{(0),m,\omega_{(0)}}^2 \psi_{(0)} - \mathcal{D}_{(1),m,\omega_{(0)}}^2 \psi_{(0)} \end{cases} \quad (3.17)$$

where we have introduced a new operator $\delta \mathcal{D}_{(0)}^2$ defined by

$$\delta \mathcal{D}_{(0)}^2 [\phi] \equiv \left(\frac{\partial}{\partial \omega} \mathcal{D}_{(0)}^2 \right) \Big|_{\omega=\omega_{(0)}} [\phi]. \quad (3.18)$$

This modified KG Eq. (3.17) has to be solved order by order: every order in λ has to be solved separately. In what follows, the subscripts m and $\omega_{(0)}$ will be omitted. At zeroth order the differential equation is simply the GR $s = 0$ Teukolsky equation. This equation will be solved with a Newton-Raphson pseudospectral method, as explained in Section 3.2.1 [1–3]. At first order, given by the second line of Eq. (3.17), things are somewhat more complicated due to the source term $\mathcal{D}_{(1)}^2 [\psi_{(0)}]$. We wish to solve this system of equations using pseudospectral methods. In order to do this, we need to compactify the radial coordinate r and regularise the equations by removing the divergent behaviour at the boundaries. This also has to be done for the Teukolsky equation. In GR, they are related in that both the compactification and the regularisation are coordinate artefacts, and both problems are solved simultaneously [143]. The big simplification for the zeroth-order case is that the divergent behaviour at the boundaries is well known and many methods already exist to solve this Teukolsky equation [1–3, 158, 160, 201]. First of all, we adopt the radial compactification

$$z = \frac{\rho - \rho_+}{\rho - \rho_-}. \quad (3.19)$$

In this coordinate the outer horizon ρ_+ is located at $z = 0$ and spatial infinity is located at $z = 1$. To remove the divergent boundary behaviour, we have to find a factorisation,

$$\psi(z, x) \equiv \alpha_\omega(z, x) \tilde{\psi}(z, x) \quad (3.20)$$

such that the function $\tilde{\psi}$ is regular on the whole domain [2, 3]. To implement this in Eq. (3.17), we also expand the regularising function α in terms of λ and introduce two regular functions $\tilde{\psi}_{(0)}$ and $\tilde{\psi}_{(1)}$, such that

$$\psi = \alpha_{(0)} \tilde{\psi}_{(0)} + \lambda \left[\alpha_{(0)} \tilde{\psi}_{(1)} + \alpha_{(1)} \tilde{\psi}_{(0)} \right] + \mathcal{O}(\lambda^2), \quad (3.21)$$

where we have dropped the subscript ω that indicated the frequency dependence. In other words, we have chosen [2, 3]

$$\psi_{(0)} \equiv \alpha_{(0)} \tilde{\psi}_{(0)}, \quad \psi_{(1)} \equiv \alpha_{(0)} \tilde{\psi}_{(1)} + \alpha_{(1)} \tilde{\psi}_{(0)}. \quad (3.22)$$

This gives the regularised equations

$$\tilde{\mathcal{D}}_{(0)}^2(\omega_{(0)})[\tilde{\psi}_{(0)}] = 0 \quad (3.23)$$

$$\tilde{\mathcal{D}}_{(0)}^2(\omega_{(0)})[\tilde{\psi}_{(1)}] + \tilde{\mathcal{D}}_{(0)}^2(\omega_{(0)})[\beta \tilde{\psi}_{(0)}] + \omega_{(1)} \delta \tilde{\mathcal{D}}_{(0)}^2[\tilde{\psi}_{(0)}] = -\tilde{\mathcal{D}}_{(1)}^2(\omega_{(0)})[\tilde{\psi}_{(0)}], \quad (3.24)$$

where we have the modified operators

$$\tilde{\mathcal{D}}_{(0)}^2 \equiv \frac{1}{\alpha_{(0)}} \mathcal{D}_{(0)}^2 \alpha_{(0)}, \quad \delta \tilde{\mathcal{D}}_{(0)}^2 \equiv \frac{1}{\alpha_{(0)}} \left(\frac{\partial}{\partial \omega} \mathcal{D}_{(0)}^2 \right) \Big|_{\omega=\omega_{(0)}} \alpha_{(0)}, \quad \tilde{\mathcal{D}}_{(1)}^2 \equiv \frac{1}{\alpha_{(0)}} \mathcal{D}_{(1)}^2 \alpha_{(0)}, \quad (3.25)$$

and the ratio [2, 3]

$$\beta(z, x) \equiv \frac{\alpha_{(1)}(z, x)}{\alpha_{(0)}(z, x)}. \quad (3.26)$$

The regularising functions are now to be chosen such that they regularise the equations of motion [2, 3]. $\alpha_{(0)}$ regularises the zeroth-order equation. The term $\tilde{\mathcal{D}}_{(0)}^2[\tilde{\psi}_{(1)}]$ is therefore manifestly regular, as $\alpha_{(0)}$ is chosen to eliminate the divergences of the unperturbed Teukolsky operator [2, 3]. In contrast, $\delta \tilde{\mathcal{D}}_{(0)}^2[\tilde{\psi}_{(0)}]$ and $\tilde{\mathcal{D}}_{(1)}^2[\tilde{\psi}_{(0)}]$ generally develop singular behaviour at the horizon and at spatial infinity, reflecting the $\mathcal{O}(\lambda)$ modification of the boundary conditions. The auxiliary function β (which also defines $\alpha_{(1)}$) must therefore be constructed to cancel these divergences, ensuring that Eq. (3.24) remains regular throughout the domain [2, 3]. To look at how this function β behaves, one should first look at $\alpha_{(0)}$, the divergent boundary behaviour of the Teukolsky solutions in Boyer-Lindquist coordinates. $\alpha_{(0)}$ is found by studying the asymptotic behaviour of the Teukolsky equation and the associated singular structure, which encodes the boundary conditions (ingoing at horizon and outgoing at infinity, which lead to coordinate dependent divergences at $\rho = \rho_\pm$, $\rho = \infty$, $x = \pm 1$) [158]. The regulation function for the Teukolsky operator in terms of the radial coordinate ρ is given by [158, 160]

$$\alpha_{(0)}(\rho, x) \equiv (\rho - \rho_+)^{\xi} (\rho - \rho_-)^{(\eta - \mu)} e^{\zeta \rho} (1 - x)^{\frac{|m+s|}{2}} (1 + x)^{\frac{|m-s|}{2}} \quad (3.27)$$

where

$$\zeta = i\omega \quad (3.28)$$

$$\xi = \frac{1}{2}(-s - (s + 2i\sigma_+)) \quad (3.29)$$

$$\eta = \frac{1}{2}(-s - (s - 2i\sigma_-)) \quad (3.30)$$

$$\sigma_+ = (2\omega Mr_+ - ma)/(r_+ - r_-) \quad (3.31)$$

$$\sigma_- = (2\omega Mr_- - ma)/(r_+ - r_-) \quad (3.32)$$

$$\mu = 1 + s + \xi + \eta - 2M\zeta + si\omega/\zeta. \quad (3.33)$$

For the compactified radial coordinate z , the terms $\delta\tilde{\mathcal{D}}_{(0)}^2[\tilde{\psi}_{(0)}]$ and $\tilde{\mathcal{D}}_{(1)}^2[\tilde{\psi}_{(0)}]$ have singularities of the form z^{-1} and z^{-2} for the horizon, and $(z-1)^{-1}$ and $(z-1)^{-2}$ for spatial infinity [2, 3]. These singularities will have to be cancelled by our choice of β . We make the ansatz that $\alpha_{(1)}(\rho, x)$ takes the same form as Eq. (3.27), but with slightly different exponents. This is justified based on earlier work [2, 3, 45, 50, 51, 59], where the change in boundary conditions is linked to the change of these exponents through a change in surface gravity and angular velocity at the horizon. Expanding this in leading order of λ , this results in the following ansatz:

$$\alpha(\rho, x) = (\rho - \rho_+)^{\tilde{\xi}}(\rho - \rho_-)^{(\tilde{\eta}-\tilde{\mu})}e^{\tilde{\zeta}\rho}(1-x)^{\frac{|m+s|}{2}}(1+x)^{\frac{|m-s|}{2}} \quad (3.34)$$

$$= (\rho - \rho_+)^{\xi}(\rho - \rho_-)^{(\eta-\mu)}e^{\zeta\rho}(1-x)^{\frac{|m+s|}{2}}(1+x)^{\frac{|m-s|}{2}}(\rho - \rho_+)^{\lambda\delta\xi}(\rho - \rho_-)^{\lambda(\delta\eta-\delta\mu)}e^{\lambda\delta\zeta\rho} \quad (3.35)$$

$$\beta(\rho, x) = \frac{1}{\lambda}((\rho - \rho_+)^{\lambda\delta\xi}(\rho - \rho_-)^{\lambda(\delta\eta-\delta\mu)}e^{\lambda\delta\zeta\rho} - 1)_{\mathcal{O}(\lambda)} \quad (3.36)$$

$$= \delta\xi \ln(\rho - \rho_+) + (\delta\eta - \delta\mu) \ln(\rho - \rho_-) + \delta\zeta\rho + \mathcal{O}(\lambda) \quad (3.37)$$

$$\beta(z, x) = A \ln(z) + B \ln(z-1) + C \frac{\rho_- z - \rho_+}{z-1}. \quad (3.38)$$

The variables with a tilde were the exponents in full non-linearity. They encode the boundary conditions as a function of λ in different order. We expanded them up to first order such that they change as $\tilde{\xi} = \xi + \lambda\delta\xi$. The β is then found by expanding the boundary conditions up to first order, and realising that they are such that the equation is regular across the physical domain [2, 3]. Note that this expression will not introduce new logarithmic divergences, since at zeroth order Eq. (3.23) guarantees that $\beta\mathcal{D}_{(0)}^2\psi_{(0)} = 0$, so all logarithms will cancel. The only persisting terms are derivatives of β :

$$\frac{\partial\beta}{\partial z} = \frac{A}{z} + \frac{B}{z-1} + C \left(\frac{\rho_-}{z-1} - \frac{\rho_- z - \rho_+}{(z-1)^2} \right) \quad (3.39)$$

$$\frac{\partial^2\beta}{\partial z^2} = -\frac{A}{z^2} - \frac{B}{(z-1)^2} + C \left(\frac{2(\rho_- z - \rho_+)}{(z-1)^3} - \frac{2\rho_-}{(z-1)^2} \right) \quad (3.40)$$

These terms are exactly of the form we need to cancel the irregular parts of the other operators. Realise that there will be no $(z-1)^{-3}$ term in the equation. This follows from the fact that the second derivative has a prefactor of the form $(z-1)^2$ [2, 3]. Therefore, we can determine the form of β by matching the coefficients of z^{-1} , z^{-2} , $(z-1)^{-1}$ and $(z-1)^{-2}$ in $\mathcal{D}_{(0)}^2[\beta\psi_{(0)}]$ to those in $\delta\tilde{\mathcal{D}}_{(0)}^2[\tilde{\psi}_{(0)}]$ and $\tilde{\mathcal{D}}_{(1)}^2[\tilde{\psi}_{(0)}]$ by solving the resulting system

of equations for A , B and C [2, 3]. Thus the unknown parameters A, B, C , which encode the change in exponents, are fixed by demanding that the differential equation (3.24) is regular across the domain. Enforcing regularity is equivalent to demanding the boundary conditions to be satisfied. Any smooth solution to the equations Eq. (3.23) and Eq. (3.24) that is non-zero at the boundaries is guaranteed to be a scalar QNM of the modified black hole [2, 3]. The prefactors A, B and C are found analytically. It must be noted that certain terms in the equation vanish exactly due to the symmetries of the H_i s. However, since the H_i s are given numerically, this cancellation may be non-exact in practice [61–63]. When such residual terms appear in the numerator of a fraction with a zero in the denominator, the result can become numerically unstable. To solve this problem, these symmetry-induced cancellations have been enforced explicitly in the implementation. The next section will explain the numerical techniques used to solve these partial differential equations. Note how the divergence is a coordinate effect. In the hyperboloidal setting implicitly used, the Teukolsky equation is regular in GR and one modifies the similarity transformation prefactor α up to first order to obey the boundary conditions beyond GR. Now one has the differential equation and the boundary conditions of the KG equation, all what remains is to solve Eq. (3.23) and Eq. (3.24) to find the QNM shifts in higher-curvature gravity. It is important to note that this method is general and does not require one to assume a certain beyond GR theory, as long as one can write the metric changes as H_i [2, 3].

3.2 Pseudospectral Method

The equations defined in the previous section define a regularised set of equations defined on a compactified domain. The problem is reduced to finding the solutions $(\psi_{(0)}, \psi_{(1)})$ and its associated eigenvalues $(\omega_{(0)}, \omega_{(1)})$. This will be solved using pseudospectral methods. A pseudospectral method, much like a spectral method, solves a differential equation by decomposing in a basis [68]. The difference lies in the evaluation of the associated amplitudes of the different basis elements. In spectral methods these are explicitly calculated, often using integration (scalar product). In pseudospectral methods the differential equation is evaluated on a grid with weights associated to the grid points, such that the evaluation on that grid implicitly corresponds to the scalar product evaluation. The solution is then found by remarking that the error is orthogonal to the subspace of used basis functions [68–71]. One ends up with a discretised version of the differential equation, resulting in a matrix equation. Pseudospectral methods yield exponential convergence as one increases the number of collocation points, as long as both the basis functions and the function that is approximated are analytic functions [2, 3, 68–71].

We will discretise the problem using Chebyshev polynomials in both radial and angular directions. Chebyshev polynomials are chosen because its associated grid points and weights are analytically known [68]. We will use Chebyshev-Gauss-Lobatto collocation points because this scheme incorporates the boundaries, and the grid is denser near the boundaries [2, 3, 68]. In the radial direction, the nodes are given by:

$$z_i = \frac{1}{2} \left(1 - \cos \left(i \frac{\pi}{N_r} \right) \right), i = 0, \dots, N_r. \quad (3.41)$$

In the angular coordinate we use:

$$x_j = \cos\left(j \frac{\pi}{N_x}\right), j = 0, \dots, N_x. \quad (3.42)$$

The weights are $w_0 = w_{N_{r/x}} = \frac{\pi}{2N_{r/x}}$ and $w_i = \frac{\pi}{N_{r/x}}$ [68]. The derivatives are transformed into a differentiation matrix, using [68]:

$$a_n \rightarrow b_n = \frac{2}{1 + \delta_{0n}} \sum_{p=n+1, p+n=\text{odd}}^N p a_p. \quad (3.43)$$

Here a_n are the coefficients of the (Chebyshev) basis polynomials that decompose the function and b_n the transformed coefficients after acting with a derivative. This transformation produces the differentiation matrix, mapping the differential equation into a linear algebra problem (matrix equation) [68]. The matrix elements of such a differentiation matrix are given by (with ξ being the collocation points):

$$D_{ij} = (1 - \delta_{ij}) \frac{(1 + \delta_{i0} + \delta_{iN})(-1)^{i-j}}{(1 + \delta_{j0} + \delta_{jN})(\xi_i - \xi_j)} - \delta_{ij} \sum_{k \neq i}^N \frac{(1 + \delta_{i0} + \delta_{iN})(-1)^{i-k}}{(1 + \delta_{k0} + \delta_{kN})(\xi_i - \xi_k)}. \quad (3.44)$$

The end goal is to find the lowest-order shift in the frequency $\omega_{(1)}$ corresponding to a certain $\omega_{(0)}$. This will be done in two stages. First one solves the scalar Teukolsky equation in GR, resulting in a $\tilde{\psi}_{(0)}$ and $\omega_{(0)}$. This will be explained in the following section. After this, the corresponding beyond GR correction will be calculated. Both steps use the pseudospectral techniques introduced above [2, 3].

3.2.1 Zeroth-order equation

In GR, the equation of interest is separable, which means we can reduce the equations to a radial equation with radial wavefunction $R(z)$, and an angular equation with angular wavefunction $\Theta(x)$. $\mathbf{R}$ is the vector of length $N_r + 1$ that describes the discretised evaluations of the radial function on the z -grid. Likewise $\mathbf{\Theta}$ does so for the angular sector [2, 3]. The derivatives in expression Eq. (3.13) are replaced by the corresponding differentiation matrices as defined through Eq. (3.43). The equation is fully discretised by evaluating prefactors on the Gauss-Lobatto pseudospectral grid [68]. We obtain matrices $\mathbf{L}_r$ and $\mathbf{L}_x$ with sizes of respectively $(N_r + 1) \times (N_r + 1)$ and $(N_x + 1) \times (N_x + 1)$. They are parametrised by m, a and have two eigenvalues: the QNF $\omega_{(0)}$ and the separation constant A_{lm} . The separation constant links the angular and radial equation and prohibits the system from being solved trivially [2, 3].

The zeroth-order equation has thus been transformed into:

$$\begin{cases} \mathbf{L}_r(\omega_{(0)}, A_{l,m}) \mathbf{R} &= 0 \\ \mathbf{L}_x(\omega_{(0)}, A_{l,m}) \mathbf{\Theta} &= 0. \end{cases} \quad (3.45)$$

To solve this set of coupled equations, we use a Newton-Raphson approach. This is an algorithm designed to find roots, and iteratively gets closer to the root, when the initial guess is sufficiently close to the correct solution [69]. In the Schwarzschild limit $a \rightarrow 0$, the

angular equation is solved analytically through spin-weighted spherical harmonics, with $A_{lm} = l(l+1)$. Then one has to solve solely for the radial equation [25].

One approximates the root by solving iteratively $\mathbf{J}_F(\mathbf{x}_n)\delta\mathbf{x} = -\mathbf{F}(\mathbf{x}_n)$, where n denotes the iteration. J denotes the Jacobian and F the function(al) one needs to find the root of. Simply put, one calculates the tangent of the functional at a point and searches the point where this tangent crosses zero [2, 3, 69]. This will be the next starting point. In this set-up, this results in a matrix equation [2, 3]:

$$\begin{pmatrix} \mathbf{L}_x & \mathbf{0} & \partial_\omega \mathbf{L}_x \cdot \boldsymbol{\Theta} & \partial_A \mathbf{L}_x \cdot \boldsymbol{\Theta} \\ \mathbf{0} & \mathbf{L}_r & \partial_\omega \mathbf{L}_r \cdot \mathbf{R} & \partial_A \mathbf{L}_r \cdot \mathbf{R} \\ 2\boldsymbol{\Theta}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{R}^T & \mathbf{0} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\mathbf{R} \\ \delta\omega^{(0)} \\ \delta A_{lm} \end{pmatrix} = - \begin{pmatrix} \mathbf{L}_x \cdot \boldsymbol{\Theta} \\ \mathbf{L}_r \cdot \mathbf{R} \\ \boldsymbol{\Theta}^T \cdot \boldsymbol{\Theta} - 1 \\ \mathbf{R}^T \cdot \mathbf{R} - 1 \end{pmatrix}. \quad (3.46)$$

Solving this equation iteratively modifies the wavefunction, the frequency and the separation constant. The iteration is repeated until the residual $\delta\omega_{(0)}$ between two iterations is smaller than the working precision [2, 3]. The first line describes the angular sector, the second line the radial sector. The upper right block describes the parametric dependence on $\omega_{(0)}$, $A_{l,m}$, through a derivative [2, 3]. The derivatives are taken analytically. The lower two lines normalise the corresponding eigenstates with $\boldsymbol{\Theta} \cdot \boldsymbol{\Theta} = 1$ and $\mathbf{R} \cdot \mathbf{R} = 1$. Eigenvectors can be normalised freely, and enforcing a normalisation relieves this redundancy in the equations and renders the matrix invertible in a neighbourhood of the solution [2, 3].

The initial vector and eigenvalues should be chosen sufficiently close to the actual solution for the Newton-Raphson method to converge. For fundamental modes this demand is not very stringent and we initialise with a spherical harmonic in the angular sector (which is exactly correct for Schwarzschild) and a constant array in the radial sector [2, 3]. The separation constant is initialised to the Schwarzschild one and the frequency is initialised from a list of scalar Kerr QNMs, in order to be close to the correct one [22]. For the overtones one must watch out to make sure it does not jump to another mode. To avoid this, the radial vector is initialised with the solution constructed using a Leaver solver [2, 3].

Now we have the initial ‘guesses’ and the framework to calculate the zeroth-order equation, one can use these results as input for the first-order equation.

3.2.2 First-order equation

At first order, the zeroth-order solution enters as a source term. The equation is no longer separable, which means the radial and angular part need to be solved simultaneously in a 2D pseudospectral method [1–3]. As a consequence, the vector of grid evaluations will be of size $(N_r + 1) \cdot (N_x + 1)$ and the discretised operators (matrices) will now have a size of $((N_r + 1) \cdot (N_x + 1)) \times ((N_r + 1) \cdot (N_x + 1))$. The zeroth-order solution is simply the (flattened tensor-)product of the radial and angular solutions acquired in the previous section: $\tilde{\Psi}_{(0)} = \mathbf{R} \otimes \boldsymbol{\Theta}$ [2, 3]. Once again the derivatives are transformed into differentiation matrices and the prefactors are evaluated on the Gauss-Lobatto collocation grid. We denote the radial and angular derivatives as $\partial_z \rightarrow \mathbf{D}_z \otimes \mathcal{I}_x$ and $\partial_x \rightarrow \mathcal{I}_z \otimes \mathbf{D}_x$,

with $\mathcal{I}$ the identity matrices [2, 3]. Together, this reduces the sourced first-order equation into a linear matrix problem:

$$\begin{pmatrix} \tilde{\mathbf{D}}_{(0)} & (\delta\tilde{\mathbf{D}}_{(0)} \cdot \tilde{\Psi}_{(0)})^T \\ (\tilde{\Psi}_{(0)})^T & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\Psi}_{(1)} \\ \omega_{(1)} \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{D}}_{(1)} \cdot \tilde{\Psi}_{(0)} \\ 0 \end{pmatrix}, \quad (3.47)$$

with the tilde denoting the regularisation of the equation. The linear equation solves for the modifications $\psi_{(1)}$ and $\omega_{(1)}$ [2, 3]. The upper line in the equation above denotes the first-order modification to the KG equation, Eq. (3.24). The lower lines once again removes the normalisation redundancy [2, 3]. It enforces the modification to be orthogonal with respect to the GR eigenvector: $\tilde{\Psi}_{(0)} \cdot \tilde{\Psi}_{(1)} = 0$. This removes the freedom $\tilde{\Psi}_{(1)} \rightarrow \tilde{\Psi}_{(1)} + \epsilon \tilde{\Psi}_{(0)}$. Removing this freedom makes sure that the matrix is non-singular and a unique solution can be found [2, 3]. The exact choice of normalisation should not matter for the eigenvalue perturbation $\omega_{(1)}$. Notice also how one needs the zeroth-order solution $\tilde{\Psi}_{(0)}$ to calculate the matrices for the first-order equation. The total technique can thus be summarised as follows [2, 3]:

1. Separate radial and angular equations at zeroth order and discretise into a matrix problem (Eq. (3.45)).
2. Find $\omega_{(0)}$ and $\tilde{\Psi}_{(0)}$ with the Newton-Raphson method applied to the previous matrix equation (Eq. (3.46)).
3. Use the previously found solution as input for the first-order equation. Discretise it into a matrix problem (Eq. (3.47)).
4. Find $\omega_{(1)}$ by solving the linear problem above.

3.3 Arbitrary Spin Quasinormal Frequencies

Using the method outlined in the previous two sections, we have calculated scalar QNM frequency shifts in different beyond GR theories [2, 3]. As explained in Section 2.3, we look at lowest-order corrections to GR, described through the formalism of EFT. We have tested sGB, dCS, axi-dilaton, cubic even and cubic odd theory. We have calculated the fundamental modes for $l \leq 5$ for all associated m up to a spin of $a = 0.99$. The spin a is incremented in steps of $\Delta a = 0.05$ up to $a = 0.9$, after which steps of $\Delta a = 0.01$ have been used. For cubic even gravity, the first overtone has been calculated for the $2 \leq l \leq 5$ modes, for all m and spins up to $a = 0.99$ [3]. For sGB and dCS, the first overtone has been calculated for the $l = 2$ modes, for all m and spins up to $a = 0.99$ [2]. In calculating all these modes for all the different theories, we show the viability of using pseudospectral methods for calculating QNM deviations in higher-curvature gravity [2, 3].

In axi-dilaton, one introduces a parity odd term to the sGB and dCS action. Cubic odd gravity also adds a parity odd term to the action. Because of the parity of these theories, scalar QNMs do not receive any correction [47]. We have numerically verified that the lowest-order modifications $\omega_{(1)}$ vanishes, using the method outlined in the previous section. The remainder of this work will restrict attention to the sGB, dCS and cubic even

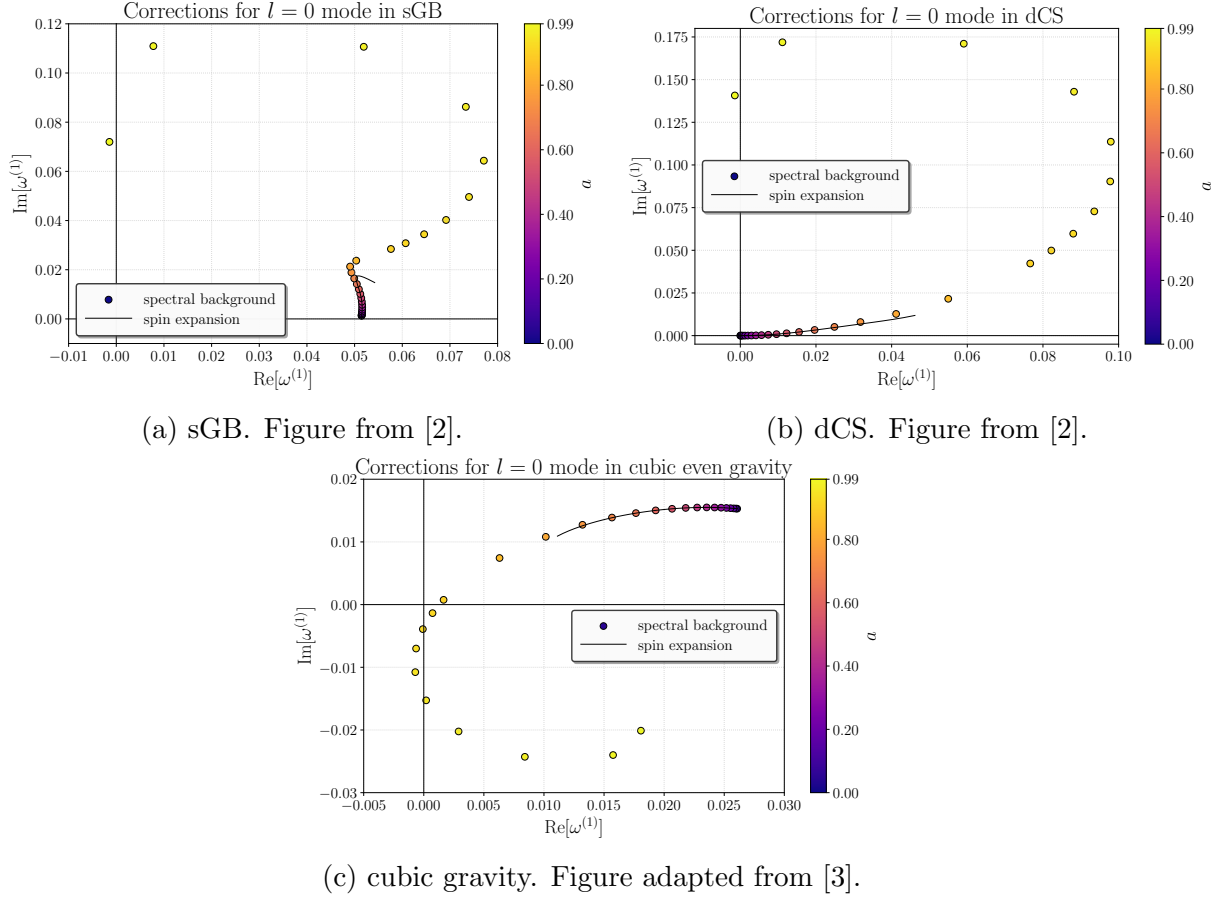


Figure 3.1: Lowest-order corrections $\omega_{(1)}$ to the $l = m = 0$ fundamental scalar QNM in different EFT modifications to GR, visualised in the complex plane. The circles represent the results obtained with the numerical background. The colour-scale denotes the spin, going up to $a = 0.99$. The line describes the results of the spin expansion, up to a spin of $a = 0.8$ [2, 3].

gravity cases, as they provide the non-trivial modifications [47].

We have used the numerically calculated black hole backgrounds, as described in Section 2.3.3 [61–63], to calculate the corrections for spins up to $a = 0.99$. To verify the results in the low spin regime, one can use a perturbative spin expansion for the background. This solves the black hole background up to some order in dimensionless spin and this perturbative approximation breaks down at high spin [59]. The pseudospectral method from Section 3.2 is also verified by comparing to an eigenvalue perturbation method applied on the spin expansion background [49, 202, 203]. At low spins, the results are in excellent agreement with each other. At higher spins, the spin expansion results are no longer reliable and deviate from the fully numerical results. A spin expansion up to $\mathcal{O}(a^{14})$ has been used, which remains accurate up to $a \approx 0.7$ [2, 3]. This can be seen in Figures 3.1, 3.2a, 3.2b and 3.2c.

Computing $\omega_{(1)}$ with the numerical background takes close to two orders of magnitude longer in time than using the spin expansion background for the pseudospectral method

with the same grid size. To speed up the pseudospectral method, we use memoization. This means that functions that need to be evaluated multiple times are saved after the first evaluation such that the results can quickly be returned when the same expression is called again [204].

Figure 3.1 shows the $l = 0$ results for sGB, dCS, and cubic even gravity [2, 3]. It visualises how the lowest-order modifications evolve in the complex plane as the spin increases from Schwarzschild towards the near-extremal regime. The spin expansion results follow the numerical results closely up to $a \approx 0.7$, where they suddenly deviate strongly, especially in the sGB case, where the correction turns around for the spin expansion result [2, 3].

Figures 3.2a, 3.2b and 3.2c display the $l = 2$ fundamental modes for the different m for sGB, dCS and cubic even gravity. These modes are a good representation of the general behaviour of $l \neq 0$ modes. The corrections show remarkable behaviour at large spins. In the large spin limit, certain modes grow large [2, 3]. An extensive physical reasoning will be given in Section 3.4. This seemingly divergent behaviour for certain $\omega_{(1)}$ signals that near-extremal black holes amplify beyond GR effects. Most notable for $l = 2$ is the $l = m = 2$ mode. It becomes orders of magnitude larger between $a = 0.9$ and $a = 0.99$. This seemingly divergent behaviour also occurs for the $l = m = 1$, for $l = 3$ at $m = 2, 3$, for $l = 4$ at $m = 3, 4$, and for $l = 5$ at $m = 3, 4, 5$ for both sGB and dCS. For cubic gravity it is the same, except that the $l = 3, m = 2$ mode does not seem to diverge [2, 3]. Note, however, that ‘seemingly diverging’ is a subjective measure. The observation that this amplification is at least qualitatively theory independent will be established firmly in the following Section 3.4.

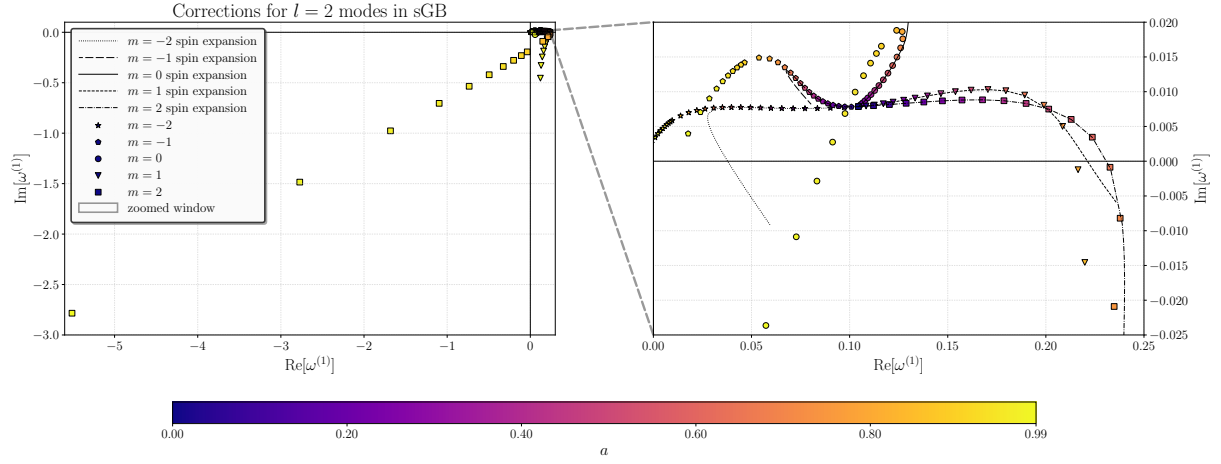
We have computed the scalar QNMs with different resolutions of the collocation grid, to estimate the error. The error is estimated as the difference in frequency between the solutions at different resolutions, which turns out to be a rather conservative estimate [2, 3]. We have confirmed the results converge exponentially as a function of the pseudospectral grid size, as seen in Figure 3.3. Similar results are found for cubic even gravity [3]. To validate that the near-extremal behaviour is not a numerical artefact, the convergence is additionally tested by changing the resolution of the numerical background. For a background with lower resolution, the mode corrections shift less than 10^{-16} for all the tested modes [2]. This proves that the numerical background resolution is less of a limiting factor than the collocation grid resolution. As the spin increases, the convergence becomes slower [2, 3]. In general, convergence becomes better when increasing l : higher multipoles converge faster. All modes have been calculated with relative errors smaller than 10^{-3} and for $l \geq 2$ the relative errors are smaller than 10^{-6} [2, 3].

The scalar QNM corrections $\omega_{(1)}$ can be found in the GitHub repository¹.

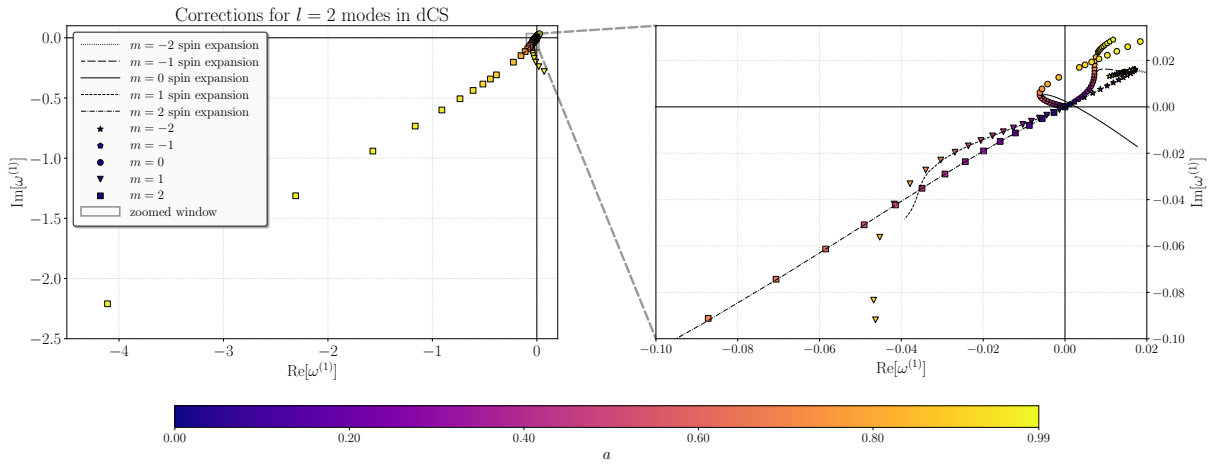
3.4 Physical Interpretation

In the previous section the results have been presented. For black holes of low and medium spin, the modifications remain rather small. However, for certain modes the corrections

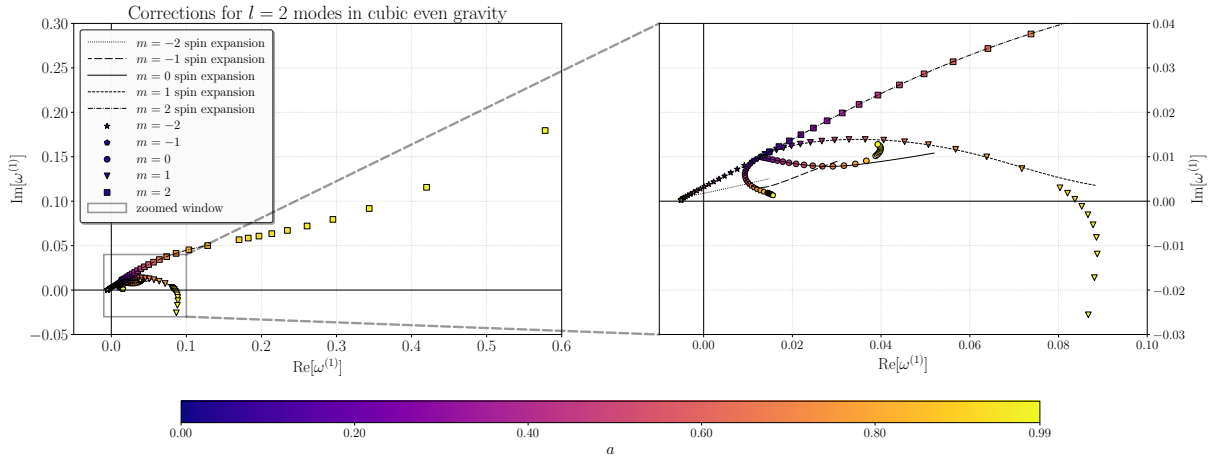
¹ scalar QNMs higher-derivative gravity



(a) sGB. Figure from [2].



(b) dCS. Figure from [2].



(c) cubic gravity. Figure adapted from [3].

Figure 3.2: Lowest-order corrections $\omega_{(1)}$ to the $l = 2$ fundamental scalar QNM in different EFT modifications to GR for different m , visualised in the complex plane. The markers represent the results obtained with the numerical background, with stars corresponding to $m = -2$, pentagons to $m = -1$, circles to $m = 0$, triangles to $m = 1$ and squares to $m = 2$. The colour-scale denotes the spin, going up to $a = 0.99$. The line describes the results of the spin expansion, up to a spin of $a = 0.8$, with the dotted line representing $m = -2$, the long-dashed line $m = -1$, the solid line $m = 0$, the short-dashed line $m = 1$ and the dot-dash line $m = 2$ [2, 3].

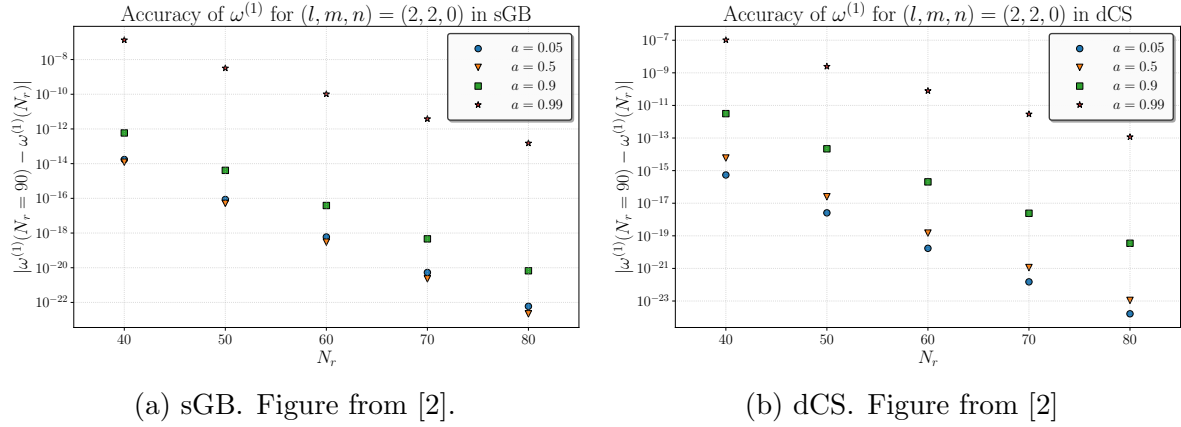


Figure 3.3: Convergence of leading-order corrections to scalar $l = m = 2, n = 0$ QNMs in different EFT modifications to GR, for different spins. The convergence is as a function of radial grid size with the corresponding angular grid size given by $N_x = N_r - 20$. Blue circles correspond to a spin of $a = 0.05$, orange triangles to $a = 0.5$, green squares to $a = 0.9$, and red stars to $a = 0.99$. The plot is semi-logarithmic and shows exponential convergence.

grow large in the large spin limit. This section will discuss the cause of this ‘blow up’ and its physical interpretation [2, 3].

To understand the divergent behaviour, a study of the GR spectrum of near-extremal black holes is important. In the extremal limit certain modes become infinitely long lived: zero damped mode(s) (ZDM). The imaginary part of these modes vanishes in the extremal limit. In a sense, extremal black holes are metastable. Perturbations do not die out but become infinitely long lived [4, 25, 95, 96]. Not all QNMs become zero damped in the extremal limit, some modes retain a finite damping: damped mode(s) (DM). The phase boundary separates the region where only DMs exist from the region where ZDMs and DMs coexist [25, 95, 96]. In the ZDM region most modes become zero damped, except for a few, which undergo something called a ‘resonance mode avoidance’ [25]. An example of such a mode is the $l = m = 2, n = 5$ mode, which is an $l = m = 2$ mode which remains damped in the extremal limit. This phase boundary can be calculated analytically in the eikonal limit, i.e. the large l limit. It is given by $m_{cr}(l) = \bar{\mu}_{cr}(l + \frac{1}{2})$ with $\bar{\mu}_{cr} = \frac{1}{1 + \frac{S}{\pi}} \approx 0.744$ with $S = \int_{\arcsin(\sqrt{3}-1)}^{\pi - \arcsin(\sqrt{3}-1)} d\theta \sqrt{\frac{7}{4} - \frac{1}{\sin^2 \theta} + \frac{\cos^2 \theta}{4}}$. Modes with $m \geq m_{cr}$ lie in the region where most modes become zero damped. Modes with $m \leq m_{cr}$ retain a finite damping in the extremal limit [95, 96].

As shown in Figure 3.4, it is the modes close to the phase boundary that appear to grow large in the large spin limit. This suggests that the observed theory-independent amplification of beyond GR stems from this phase boundary [2–4]. If one assumes the lowest-order approximation remains valid in this regime and that a spin of $a = 0.99$ is sufficiently close to extremality, modes near the phase boundary *typically* acquire stronger damping for sGB and dCS and reduced damping for cubic gravity. This would effectively push the ZDMs and the associated phase boundary to larger m for sGB and dCS and smaller m for cubic, increasing $\bar{\mu}_{cr}$ for sGB and dCS and reducing it for cubic [2, 3].

However, there are serious concerns about the two assumptions made above. The lowest-order approximation is expected to break down in the near-extremal regime, exactly because of this movement of the phase boundary, as will soon be discussed [2, 4]. Furthermore $a = 0.99$ is not sufficiently close to extremality to reliably make strong assumptions about quantitative changes [2–4].

If there are corrections for which $\text{Im}[\omega_{(1)}] > 0$, the QNMs may become unstable in the linear regime, if $\text{Im}[\omega_{(0)} + \lambda\omega_{(1)}] > 0$. Within this leading-order approximation, one finds that instabilities occur for sGB when $\lambda \gtrsim 0.81$, for dCS when $\lambda \gtrsim 0.52$, and for cubic even gravity when $-2.34 \gtrsim \lambda$ or $\lambda \gtrsim 0.15$, for the modes and spins examined. These values are above the current observational constraints for solar mass black holes, for sGB and dCS. For sGB the bound is $\ell = M(|\lambda|)^{1/4} = 0.38 \text{ km}$, found for a black hole mass of $3.6M_\odot$. ℓ is the length scale of the beyond GR theory and is observation independent. For this event, this is equivalent to $\lambda = 0.019$. This bound is found using post-Newtonian predictions of a BH-NS merger [2, 205]. For dCS the bound $\ell \leq 8.5 \text{ km}$ is obtained from measurements of NS properties and is for this observation equivalent to $\lambda = 0.23$ [2, 206]. For cubic even gravity the bound $\ell \leq 34 \text{ km}$ is obtained using ringdown tests, leading to $\lambda = \pm 0.09$ for an event like GW250114 [5]. λ is the dimensionless constant and is thus dependent on the length scale of the relevant object, as it goes like $(\text{length scale})^{-4}$. As a consequence, if the black hole has a sufficiently small mass, the λ can still be large enough without being inconsistent with observations. This however would necessitate the existence of sub-solar mass (primordial) black holes (for sGB and dCS), which have not been observed and of which no confirmed formation channel exists [207].

Therefore our results do not imply linear instabilities in the observationally relevant part of parameter space. It does however signal that the leading-order expansion breaks down in the near-extremal limit. The corrections become arbitrarily large and thus the approximation that the solution is perturbatively close to the GR result breaks down [2–4]. Earlier research suggested that the corrections saturate or diminish as one gets even closer to extremality, but the divergence with respect to the GR values still holds (as the imaginary part of certain GR modes themselves go to zero) [4]. This does not imply a breakdown in the EFT itself. The EFT itself only breaks down when higher-order operators are no longer negligible in the (effective) Lagrangian [2, 3]. One can never truly predict whether the EFT will have broken down, as one does not know when higher-order operators become relevant. Therefore this framework cannot be evidence of a breakdown of the EFT [2–4]. The study is limited to the breakdown of the leading-order expansion. However, one can do an educated guess on the breakdown of the EFT, based on assumptions on the length scale at which new physics arises. If one follows a power counting argument, it can be proven that it is not guaranteed the EFT breaks in the extremal limit [4]. Apart from the EFT breakdown, the black hole background solution itself might break in the near-extremal regime.

As discussed earlier, the observation that EFT corrections of certain modes become large in the large spin limit is linked to the presence of a phase boundary between ZDMs and DMs. Beyond GR corrections modify this phase boundary and break the leading-order approximation [2–4]. This originates from the fact that a shifting phase boundary not

only leads to a quantitative change to the spectrum, but even to a qualitative change where the nature of modes can change. A mode can be zero damped in GR but become damped in beyond GR or vice versa. This order one change in the qualitative behaviour of modes leads to a divergence in the leading-order corrections to the spectrum when compared to the GR values, as this could lead to a non-perturbative change [2–4]. The leading-order approximation breaks down when the distance from the GR phase boundary becomes comparable to the beyond GR frequency shift [2]. Once this happens, one has to expand the solution simultaneously in both beyond GR coupling and distance from the phase boundary [2–4].

The origin of the amplification of beyond GR effects in the near-extremal regime is the observation that the phase boundary shifts when modifying GR. Therefore one expects this feature to be theory independent, as long as the theory modifies the phase boundary. Even more, the effect is spin-weight independent. Scalar, vector and tensor (gravitational) QNMs all have ZDMs and DMs in the extremal regime, and therefore a phase boundary [2–4, 95, 96]. Despite the results being obtained for scalar QNMs, the underlying mechanism is expected to hold also for gravitational QNMs. As a consequence, rapidly rotating black holes are expected to be one of the most promising candidates for observing beyond GR modifications to the ringdown spectrum [2, 3].

The regime of validity of the leading-order approximation changes as a function of spin. For a spin a finite distance from extremality, the corrections remain finite with respect to the GR values [2, 3]. However, as the extremal limit is approached, the corrections grow for modes sufficiently close to the phase boundary, shrinking the domain of validity for the leading-order perturbative expansion in λ . The regime of validity vanishes in the extremal limit, as an additional expansion in the distance to the phase boundary becomes necessary [2, 3].

Strictly speaking, the extremal limit is subtle to describe beyond GR, as the spin-extremality condition changes. An extremal black hole is defined as having a vanishing surface gravity $\kappa = 0$. In GR this is equivalent with demanding $a = 1$ [81]. This extremality condition acquires corrections at lowest-order in λ , such that extremality no longer corresponds to $a = 1$. The leading-order correction to the surface gravity shifts the extremality condition to $a > 1$ for quadratic gravity (dCS and sGB), for $\lambda > 0$. In cubic gravity λ can have both signs such that one cannot say whether the condition becomes $a < 1$ or $a > 1$. If $a_{ex} > 1$, true extremality lies beyond the validity of the background expansion and the true extremal solution remains inaccessible with current methods [2, 61, 62].

Zero damped modes arise from the near-horizon region of the black hole. In the extremal limit, this region can be seen as an infinitely long throat on which quasinormal modes can survive without decay. The modes with $m \approx l$ are the ones that become zero damped. These are the modes that move along the direction of rotation of the black hole [25, 95, 96, 208]. As the black hole is rotating at extremality, these modes are co-rotating with the black holes and pulled into the circulation, preventing them from decaying. The modes take infinitely long to escape the throat because of this co-rotation [25, 95, 96]. As one modifies GR, the phase boundary shifts and the modes which become zero damped

change. In other words, the near-horizon geometry changes such that modes that were locked into co-rotating no longer are or modes that were not locked into co-rotating now are. From an intuitive point of view, the structure near the horizon changes such that the rotational effects are modified.

This can be understood even more fundamentally by comparing this to ideas from string theory like fuzzballs, as shown in Figure 3.5 [209–211]. In this case the black hole does not truly possess a singular structure, but is rather a smooth object made out of strings. From afar this structure looks like a black hole, but it is a quantum gravity description of a black hole. Now because of this smooth structure, the throat for a given microstate of an extremal black hole is no longer infinite [209–211]. The macrostate describing the fuzzball has some boundary describing the shape and size of the fuzzball where the throat no longer has infinite length [209–211]. This fundamentally changes the near-horizon structure and may alter the behaviour or existence of ZDMs, even if perhaps slightly. Even though this is only an example, one can thus see that quantum gravity effects alter the structure of the throat region, which alters the QNM structure. This links naturally the idea that quantum gravity effects alter the high-curvature structure of black holes, described in lowest EFT order by e.g. sGB, dCS or cubic gravity, with the observable amplification of beyond GR effects in the extremal regime. Figure 3.5 shows an explicit example of how the throat region can get modified beyond GR, probably leading to a change in the phase boundary structure of QNMs [210].

In conclusion, theory-independent amplifications of beyond GR effects are found for all tested theories, where the leading-order perturbation breaks down in the large spin limit. This complements earlier theoretical research that studied quartic gravity, mostly in the eikonal limit [2–4]. The suggestion that modifications are most pronounced for rapidly rotating black holes is crucial for the following Chapter 4, where the observational aspects are discussed. The data analysis should focus on black holes with large spins and low mass. The low mass comes from the scaling $\lambda \sim M^{-4}$ [61–63] and can be understood from the fact that the characteristic curvature increases as the black hole mass decreases, and one wants to probe as high a curvature as possible.

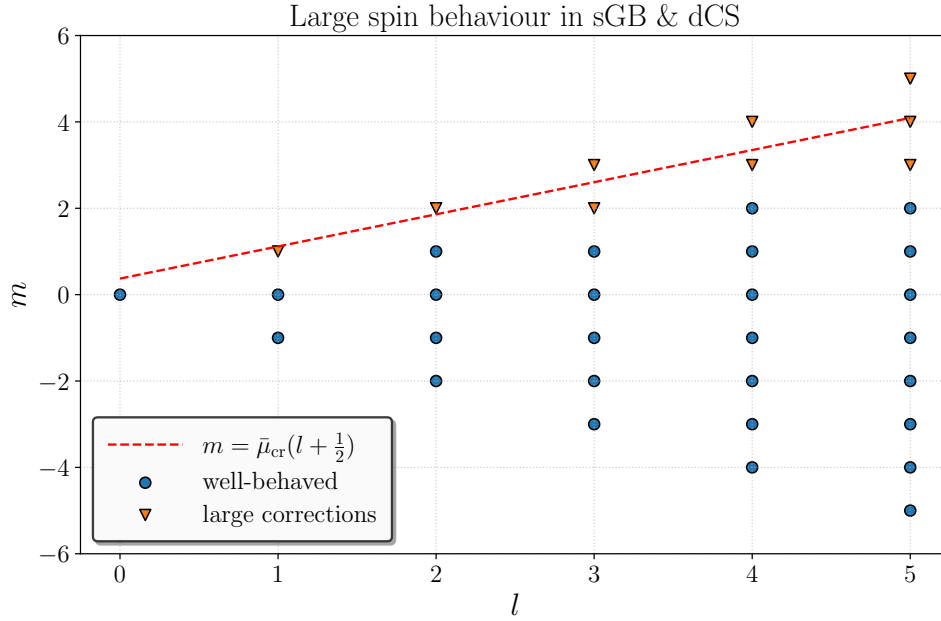


Figure 3.4: Phase space diagram of QNMs of (near-)extremal black holes, illustrating which modes seem to diverge in the large spin limit, in the l, m plane. Orange triangles denote such divergent modes. Blue circles denote modes that remain well behaved in the spin regime probed in this thesis. The red dashed line represents the ZDM-DM phase boundary in the eikonal limit. Above this line one has ZDMs, below only DMs. Notice that the modes which grow large in the large spin limit are identical for sGB and dCS, and except for $l = 3, m = 2$ also for cubic even gravity [2, 3]. Figure from [2].

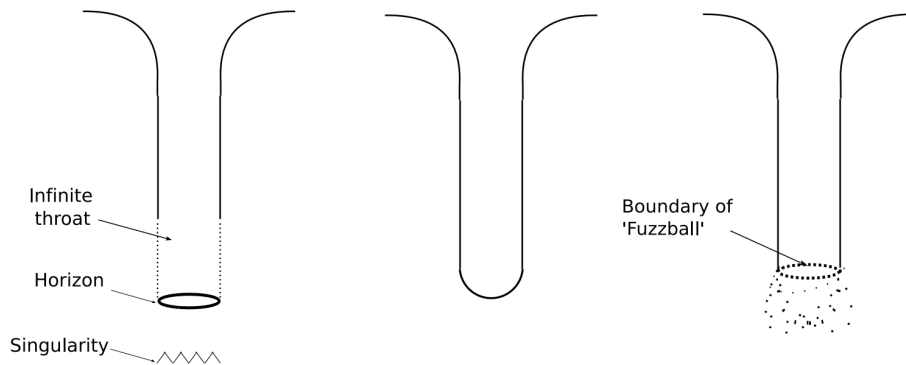


Figure 3.5: Schematic visualisation of infinite throat of extremal black hole in GR and how it gets modified in beyond GR scenarios. The left figure shows the throat in GR, the middle figure a specific microstate and the right figure the general fuzzball geometry. Figure from [210].

Chapter 4

Observational Aspects

In this chapter, the theoretical predictions from the previous Chapter 3 will be tested on observational data. In Chapter 3, we computed the corrections that scalar quasinormal modes obtain in various beyond GR scenarios. This chapter will look for these deviations in gravitational wave data obtained by LIGO. More specifically the focus will be restricted to the loudest signals and the black holes with the highest observed spin, as Chapter 3 has shown that the corrections are expected to be largest in this regime.

The ringdown signal of multiple events will be studied, using the codebase `pyRing` [7]. First, we implement the previously obtained results in `pyRing`. After this, we can use the adapted nested sampling pipeline of `pyRing` to obtain posteriors for our signal parameters [7, 194]. This includes QNM amplitudes and phases, distance, polarisation, inclination angle, remnant black hole mass and spin, and beyond GR coupling. We will study sGB, dCS and cubic even gravity, as we have obtained non-trivial results for these theories [2, 3].

We calculated scalar QNM modifications. However, we measure gravitational QNMs in gravitational wave detectors [25]. We overcome this challenge by using the scalar corrections for the gravitational wave data analysis. This will be explained in more detail in Section 4.1, where we explain the implementation details. In Section 4.2 we present the results of the data analysis. In Section 4.3 we analyse the findings and the approximation of using scalar modifications for gravitational wave observations.

4.1 Scalar QNMs as a Proxy

The ringdown signal after two black holes have merged is described by QNMs, which have frequencies characteristic of the underlying theory of gravity. In an ideal case, one would know the frequencies of different theories and be able to measure precisely the frequencies of the observed remnant. Neither of these are currently realised [25]. We do not know the gravitational frequencies for arbitrary high spin in beyond GR scenarios [2, 3] and we cannot perfectly measure these frequencies [11]. During the writing of the thesis, $l = m = 2, n = 0$ cubic and quartic gravity gravitational QNMs of high spins were computed [177]. However, we require multiple QNMs to test the Kerr hypothesis, if one restricts attention to the ringdown [25]. To circumvent this limitation, one needs a proxy to approximate gravitational QNMs in beyond GR scenarios. In this thesis, scalar QNMs

will act as a proxy for the gravitational ones: we will approximate the gravitational corrections $\omega_{(1)}^{s=2}$ with the scalar corrections $\omega_{(1)}^{s=0}$ [8].

Recent work studied the validity of this proxy. Within current detector sensitivities, using scalar QNM corrections are said to be sufficiently accurate [8]. Because scalar modes are governed roughly by the same physics as gravitational modes, they have similar qualitative behaviour. As argued in Section 3.4, the amplification of beyond GR signatures in the large spin limit happens for both scalar and gravitational modes. This justifies using our scalar modes in the Bayesian statistics pipelines [2–4, 8].

As mentioned in Section 2.2.3, scalar QNMs do not have a notion of isospectrality, whilst gravitational QNMs do. This isospectrality is broken in the theories studied in this thesis, meaning that polar and axial gravitational modes obtain different frequencies in sGB, dCS, and cubic even gravity [41, 45, 46, 51, 177]. When using a scalar mode as a proxy for a gravitational mode, one loses this bifurcation, as no such separation between axial and polar modes exists for scalar perturbations [8, 25]. As a consequence, this proxy assumes either that isospectrality breaking is subdominant or negligible, or that only the polar or axial component of the mode is present in the data. This approximation has been made before in similar contexts. It can be justified by claiming that either only the least damped mode is observable, or that the excitation mechanism only or primarily excites either polar or axial contributions [38, 75, 212]. For non-precessing systems (when the spins of the inspiralling black holes are aligned), the $l = m = 2$ mode only contains polar contributions within GR, justifying that isospectral breaking is sub-dominant. In this thesis, this symmetry will be assumed, see Section 4.2 [25].

Because for cubic even gravity results exist for the $l = m = 2, n = 0$ gravitational modes, we can compare them to the $l = m = 2, n = 0$ scalar modes we calculated for cubic even gravity, visualised in Figure 4.1 [177]. Notice how the oscillatory behaviour throughout the spin range is only present in the gravitational corrections. However, the amplification of beyond GR effects in the large spin limit is visible and similar in all three modes, strengthening the idea that scalar QNMs capture this general feature of EFTs. The scalar and gravitational corrections are generally of roughly the same order of magnitude for most spins, but the proxy does not seem suited for the high-precision studies expected with next-generation detectors. This is elaborated upon in Sections 4.2 and 4.3.

Figure 4.1a studies how similar the absolute corrections are. The recent paper that examines scalar modes as a proxy uses relative corrections $\frac{\omega_{(1)}}{\omega_{(0)}}$ [8]. As Figure 4.1b shows, the picture is structurally the same for relative corrections and both relative and absolute corrections are equally good as a proxy. This is observed also in Section 4.2, where the data analysis with the relative or absolute corrections as proxy give nearly identical results. Therefore the data analysis that is presented in Sections 4.2 and 4.3 will show exclusively the results with absolute corrections $\omega_{(1)}$.

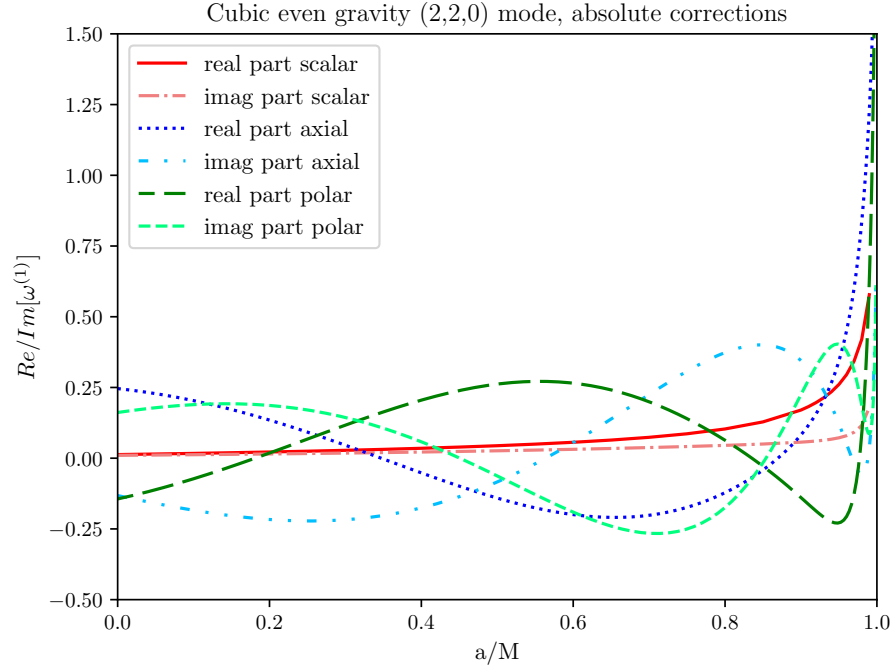
In the introduction Section 2.4.2 the ringdown waveform model that is used in this thesis was already introduced as being [7, 13, 25]:

$$\begin{aligned}
h(t) &= \frac{M}{D} \sum_{l,m,n} (h_{lmn}^+ + h_{lmn}^-) \\
h_{lmn}^+ &= A_{l,m,n}^+ S_{l,m,n}(\iota, \phi) e^{-(t-t_0)(\frac{1}{\tau_{lmn}^{(0)}(M,a)} + \frac{\lambda}{\tau_{lmn}^{(1)}(M,a)})} e^{-2\pi i(f_{lmn}^{(0)}(M,a) + \lambda f_{lmn}^{(1)}(M,a))(t-t_0) + i\phi_{lmn}^+} \\
h_{lmn}^- &= A_{l,m,n}^- S_{l,-m,n}(\iota, \phi) e^{-(t-t_0)(\frac{1}{\tau_{lmn}^{(0)}(M,a)} + \frac{\lambda}{\tau_{lmn}^{(1)}(M,a)})} e^{+2\pi i(f_{lmn}^{(0)}(M,a) + \lambda f_{lmn}^{(1)}(M,a))(t-t_0) + i\phi_{lmn}^-}
\end{aligned} \tag{4.1}$$

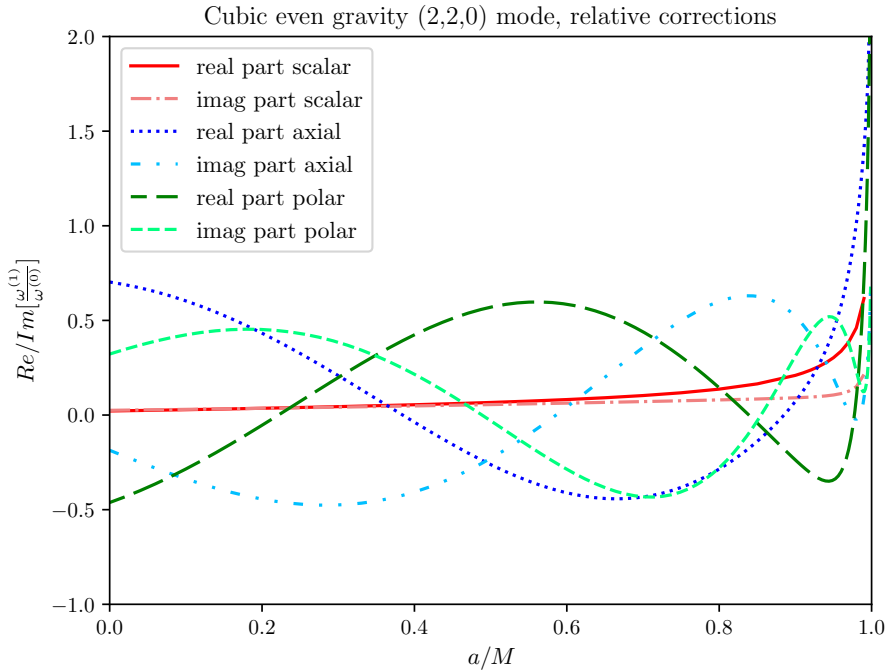
Here M is the remnant mass, which determines the overall amplitude, together with the luminosity distance D . Both are free parameters. A_{lmn} are the amplitudes of the QNMs of interest, which will be free parameters in this Bayesian posterior estimation pipeline. ϕ_{lmn} are the phases of the QNMs and will also be free parameters. $S_{l,m,n}(\iota, \phi)$ is the spheroidal harmonic describing the orientation of the event. ϕ is the azimuthal orientation and not a free parameter. ι describes the inclination angle, and is a free parameter, provided different amplitudes are related to one another, which is the case if one assumes the non-precessing symmetry [7, 25]. In this case there is a reflection symmetry which connects the amplitudes of two different modes [7, 25]. As we mentioned in Section 2.2.4, there exist modes with $\text{Re}(\omega_{l,m,n}) > 0$ and with $\text{Re}(\omega_{l,m,n}) < 0$, which are connected through the symmetry $\omega_{l,m,n} \rightarrow -\omega_{l,-m,n}^*$ [25]. This symmetry is used in the model, as denoted by the $\pm$ superscripts in Eq. (4.1) [7]. If the reflection symmetry is obeyed, we have that the amplitudes of these modes are connected through $A_{lmn}^- = (-1)^l (A_{lmn}^+)^*$, when the amplitudes and phases have been grouped into complex amplitudes [7]. This reduces the number of free parameters and is often a good approximation [25]. t_0 is the starting point of the ringdown waveform. We will use different starting times, because there is no clear method to determine the ‘ideal’ starting point [13, 16, 25]. One has to balance maximising SNR versus being in the linear perturbation regime. $\tau_{lmn}^{(0)}$ and $\tau_{lmn}^{(1)}$ are the damping times of the mode in GR and the correction beyond GR respectively. $f_{lmn}^{(0)}$ and $f_{lmn}^{(1)}$ are the oscillation frequencies of the mode in GR and the correction beyond GR respectively. They are functions of mass and spin of the black hole, which will be the free parameters to be determined [5, 6, 25]. Additionally, $\text{sign}(\lambda) \cdot \ell$, the EFT coupling length scale, will also be a free parameter. This length scale is constructed as $\lambda = \text{sign}(\lambda) \cdot \frac{\ell^4}{M^4}$ (λ is positive for sGB and dCS, and can have both signs for cubic even gravity) [5, 6]. ℓ is the free parameter because this is an independent quantity of the theory, whereas λ depends on the curvature length scale of the individual system, which is the remnant mass M in this case [5].

Note additionally that for certain (large) λ and a it can happen that $\frac{1}{\tau_{lmn}^{(0)}(M,a)} + \frac{\lambda}{\tau_{lmn}^{(1)}(M,a)} < 0$. In this case the mode is unstable and exponentially growing. This is not observed and these regions are excluded from parameter space [5, 7].

This ringdown model was for a large part already implemented into `pyRing` [7], apart from the fact that isospectrality was explicitly broken in the model and that the corrections $f_{(1),lmn}$ and $\tau_{(1),lmn}$ were undefined for large spins, rendering the beyond GR study of rapidly rotating black holes impossible within the EFT framework [5, 6]. We at first implemented the isospectral structure of scalar QNMs as a proxy. This is simply done by



(a) Absolute corrections [8, 177].



(b) Relative corrections [8, 177].

Figure 4.1: Scalar corrections (absolute $\omega_{(1)}^{s=0}$ or relative $\frac{\omega_{(1)}^{s=0}}{\omega_{(0)}^{s=0}}$) and gravitational corrections (absolute $\omega_{(1)}^{s=2}$ or relative $\frac{\omega_{(1)}^{s=2}}{\omega_{(0)}^{s=2}}$) of the $l = m = 2, n = 0$ mode in cubic even gravity. The red line shows the real part of the scalar corrections as a function of spin and the red dot-dash line the imaginary part, the blue dotted line and double-dotted-dash line the real and imaginary part of the axial gravitational corrections and the green long-dashed and short-dashed line the real and imaginary part of the polar gravitational corrections.

‘not’ splitting the modes in two different corrections. After this, we need to implement the scalar QNMs into the model. It is important to note that the mass and spin of a black hole are continuous quantities, but we have obtained results for a discrete set of spin values, with an analytical dependence on the mass [2, 3, 25]. The discrete set must be promoted into a continuous function, which we achieve by using spline interpolation. A spline is a piecewise polynomial function, interpolating between the different points. We use a cubic spline, which means that the spline consists of cubic polynomials and is $\mathcal{C}^2$ –continuous [213]. This allows the nested sampler to probe the spin space and calculate the frequency corrections continuously.

Since we have now implemented the theoretical results into the `pyRing` codebase, we can test our findings in observational data.

4.2 GW250114 & GW231123 Analysis

At the time of writing, LVK has measured a few hundred binary black hole merger events [10, 72, 122]. These observations are separated into five catalogues¹. These catalogues correspond to the different observing runs. The detectors are not permanently up and running. They have periods in which they observe (observing runs) intermediated by periods of upgrades and commissioning [10, 72, 122, 123, 132].

These events can be tested with the model described in the previous Section 4.1. The first three observing runs have been used to constrain EFT parameters using ringdown. They used previously obtained spin expansion results for low and intermediate spins [5, 6]. The fourth (and most recent) observing run has not been analysed to obtain a new constraint. In this section, we will use the two most relevant events from observing run O4: GW250114 and GW231123 [11, 12, 214]. These events have been observed exclusively by LIGO [11, 214]. We have also analysed GW150914, the very first gravitational wave observation, as a test case, but since this event has already been studied very extensively, this will not be discussed, but results can be found in appendix B [9, 11, 12, 25, 195, 214].

Since earlier research has shown that few sources dominate the constraints, it is justified to focus on the events that are most likely to alter results [5, 6]. GW250114 is the loudest event to date, and is the only event which has the confirmed presence (with 5σ certainty) of an overtone. This allows one to accurately test the Kerr hypothesis and find tight constraints [11, 12].

GW231123 is arguably one of the most controversial detections of the last observing run (e.g. [214–218]). It was very short $\approx 0.1 - 0.2s$, with glitches (non-stationary noise artefacts) present in both LIGO Hanford and Livingston detectors 1 second and 2–3 seconds before the event respectively [214]. The black holes that collided were rapidly rotating and the colliding black holes are the heaviest detected so far. Due to the short duration of the signal, alternative sources (like cosmic strings) have been proposed to explain the event, rather than the standard interpretation of a binary black hole merger [214, 218, 219].

¹Two weeks before the thesis submission deadline, the fifth catalogue was released [122].

Despite the heavy mass of the remnant suppressing EFT corrections, the high spin and the speculative nature of the signal justify analysing this event. If one finds signatures or hints of EFT corrections, this would not necessarily point towards beyond GR effects, but would simply indicate that something is not fully understood about the signal.

For the statistical analysis, we make use of the `amp-non-prec-sym` option [7, 25]. This means we will use the assumption that the binary black hole merger is non-precessing. The implications of this have been discussed in Section 4.1.

The fundamental $l = m = 2, n = 0$ and the first overtone $l = m = 2, n = 1$ have been observationally confirmed [11]. These will be the modes that will be taken into account in the data analysis for both events, alongside their symmetric counterparts (the $+$ and $-$ modes). We use an equatorial sky-frame. This coordinate system is widely used in astronomy [7, 111]. We will test the three theories separately. As the two events in question were only observed in LIGO, we can only use data from these two detectors [11, 214]. We use the following sampler settings [7, 194]:

```
nlive=4096 (number of live points in nested sampler)
poolsize=100 (number of points in the proposal pool)
maxmcmc=4096 (maximum number of steps MCMC may use when replacing a dead point)
nthreads=1 (number of parallel threads)
seed=1234 (to make the results reproducible).
```

These settings are based on the typical analyses done on LVK data [7, 220].

4.2.1 GW250114

We have analysed the ringdown part of GW250114 in GR, sGB, dCS, and cubic even gravity. As starting times we take $4M, 6M, 8M, 10M$ after the peak of the strain. The production-level analysis used $t_0 = 6M$ (and $3M$ and $9M$) after merger [11]. We fixed the following parameters for the priors [11]:

```
mf-time=68.40897469707073 $M_\odot$  (the mass used in the time after merger)
ra=2.3332345218980706 (right ascension sky coordinate)
dec=0.19024356075903648 (declination sky coordinate)
phi=0 (azimuthal orientation GW)
 $M \in [40, 80]M_\odot$  (mass of remnant)
 $a/M \in [0.2, 0.99]$  (spin of remnant)
```

The first four parameters are fixed to the values of the LVK analysis [11]. The mass and spin range have been limited to be around the values expected from inspiral-merger-ringdown data. By doing so, unphysical values are avoided. The ranges are however taken sufficiently broad such that the posteriors fall off within the prior boundaries [11].

Figures 4.3, 4.4, 4.5 show the corner plots of the data analysis beyond GR for $t_0 = 6M$. These results can be compared to Figure 4.2, which visualises the posterior of the event within GR for $t_0 = 6M$. A corner plot visualises the posterior in the highly dimensional parameter space. The diagonal plots denote the one-dimensional marginalisation of the posterior on a single parameter. It is the statistical distribution of possible parameter

values. The off-diagonal plots describe the two-dimensional marginalisation of the posterior on two parameters: the cross section of the posterior for two parameters. This allows one to look for correlations between multiple parameters [187, 221].

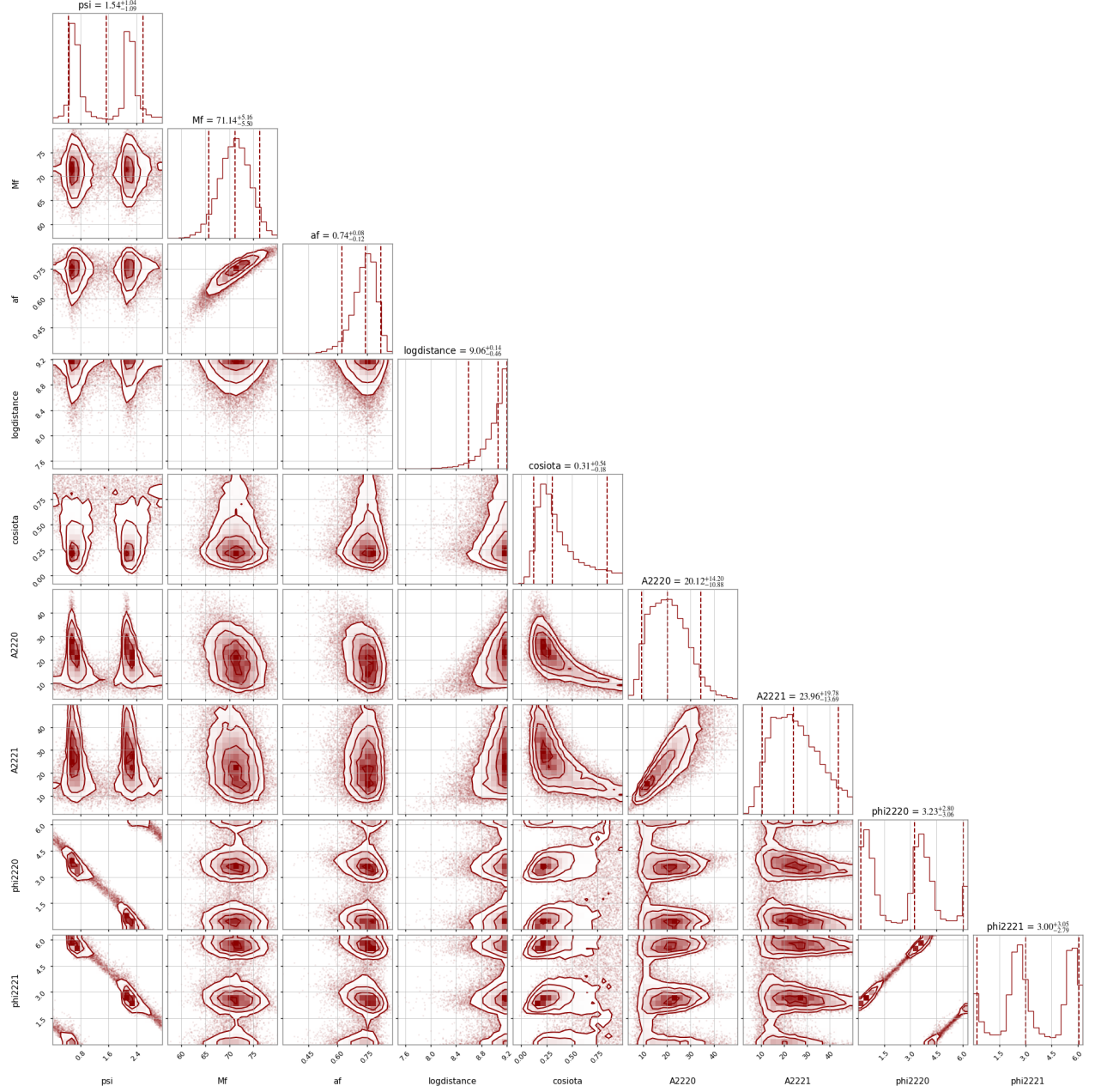


Figure 4.2: Corner plot of GW250114, within GR, constraining astrophysical quantities.

First of all, all corner plots show no deviation from GR. The EFT coupling parameter ℓ (ell) is given in kilometres and the support of the posterior is completely constrained around $\ell = 0$. The 90% confidence region for sGB and dCS are approximately $\ell \lesssim 40.0$ km and $\ell \lesssim 22.8$ km respectively. For cubic gravity, where λ can be both positive and negative, we have $\ell \cdot \text{sign}(\lambda) \in [-30.9, 35.9]$ km approximately. When converting to λ and using the median value for the black hole mass, one finds respectively: $\lambda \lesssim 0.020$, $\lambda \lesssim 0.002$,

and $\lambda \in [-0.008, 0.014]$. For cubic even gravity and sGB these constraints are comparable with ringdown results obtained from the entire earlier catalogue, or slightly better. For dCS, the constraints are notably tighter [5, 6].

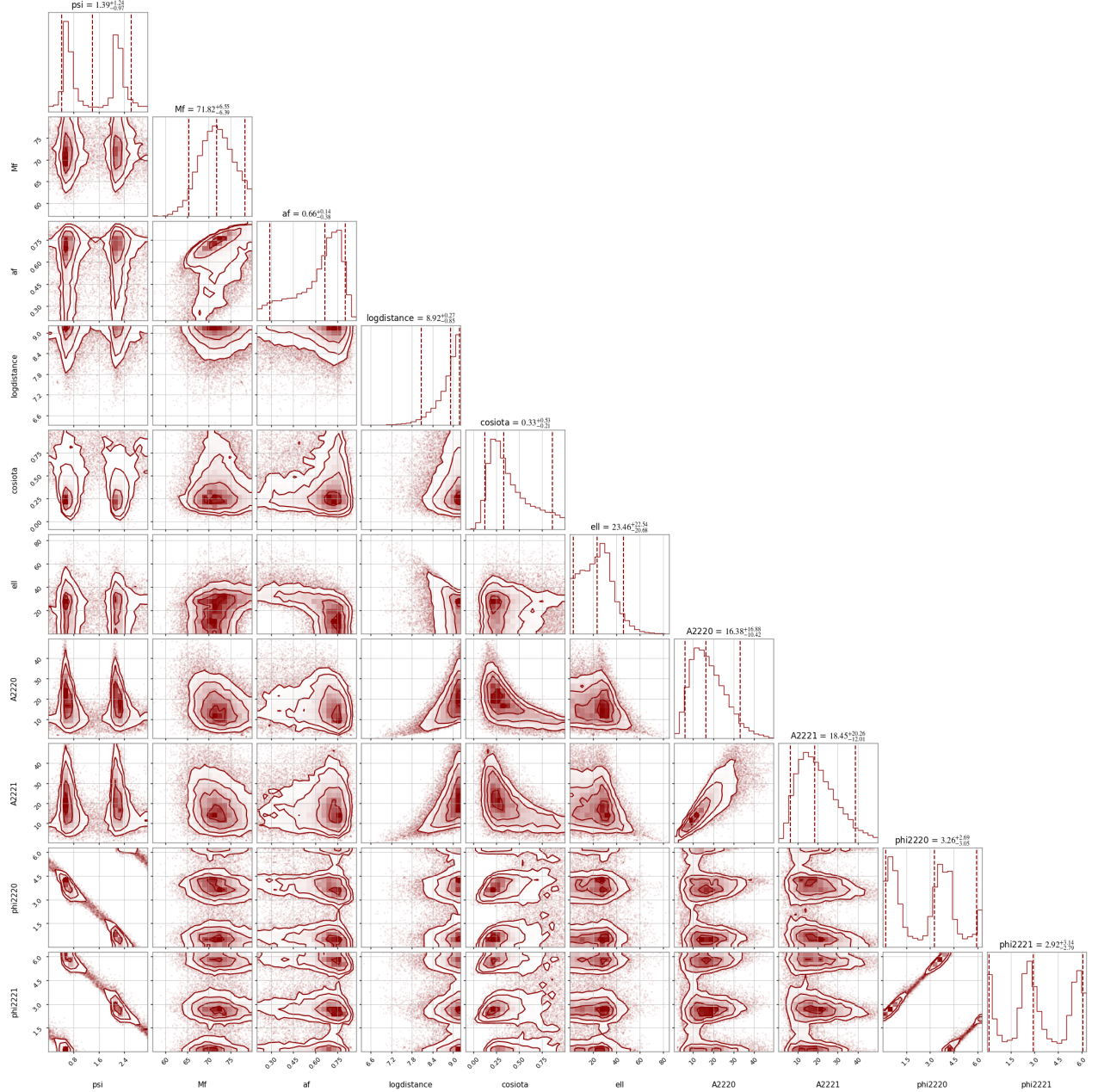


Figure 4.3: Corner plot of constraining scalar Gauss-Bonnet parameter(s) using GW250114. ℓ (ell) denotes the length scale of the coupling.

The fact that no deviation from GR is measured, can be confirmed by looking at the Bayes factors. The Bayes factor $\mathcal{B}$ is the ratio of the evidences of two models, and thus quantifies which model best describes the data [187]. In this context, we compare the beyond GR theory with GR. The fact that the logarithms are negative indicates that GR is preferred, confirming that no deviations from GR are observed. One finds for sGB, dCS and cubic even gravity respectively: $\ln(\mathcal{B}_{sGB}) = -1.2$, $\ln(\mathcal{B}_{dCS}) = -1.5$, and $\ln(\mathcal{B}_{cbc}) = -1.4$.

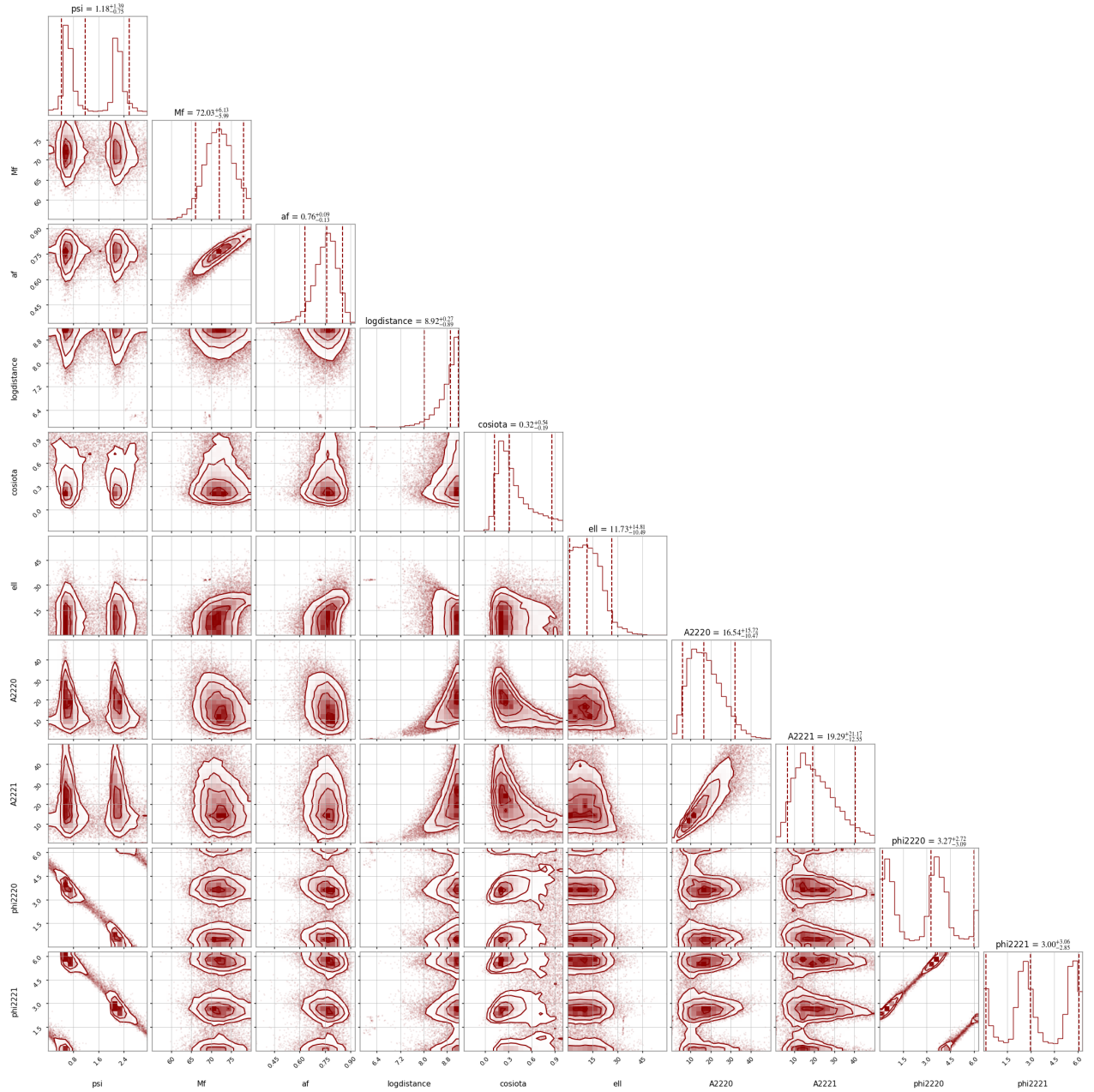


Figure 4.4: Corner plot of constraining dynamical Chern-Simons parameter(s) using GW250114. ℓ (ℓ) denotes the length scale of the coupling.

Also notice the joint posterior distribution between mass and spin. This follows a banana-like shape, which comes from the fact that there is some degeneracy between mass and spin, where the mass is simply a scaling of the frequencies [25, 190]. Moreover note the clear evidence of the presence of an overtone. This can be seen by the posterior of A_{221} , which has no support near zero.

Varying the starting time of the model does not change the results significantly. The constraints on M , a become less precise as the starting time is delayed. The amplitudes of the QNMs do vary, because the later the starting time, the more the modes have decayed, and the smaller the amplitudes. By $t_0 = 10M$ a lot of the evidence for an overtone has

died out, because the overtone damps out faster than the fundamental one [25]. In a rough estimate we have $\tau_{221} \approx 4M$ and $\tau_{220} \approx 10M$ [22]. In going from $t_0 = 6M$ to $t_0 = 10M$ the fundamental mode has diminished by 33%, whereas the first overtone has diminished by 63%. The constraints on ℓ change relatively little when varying starting times.

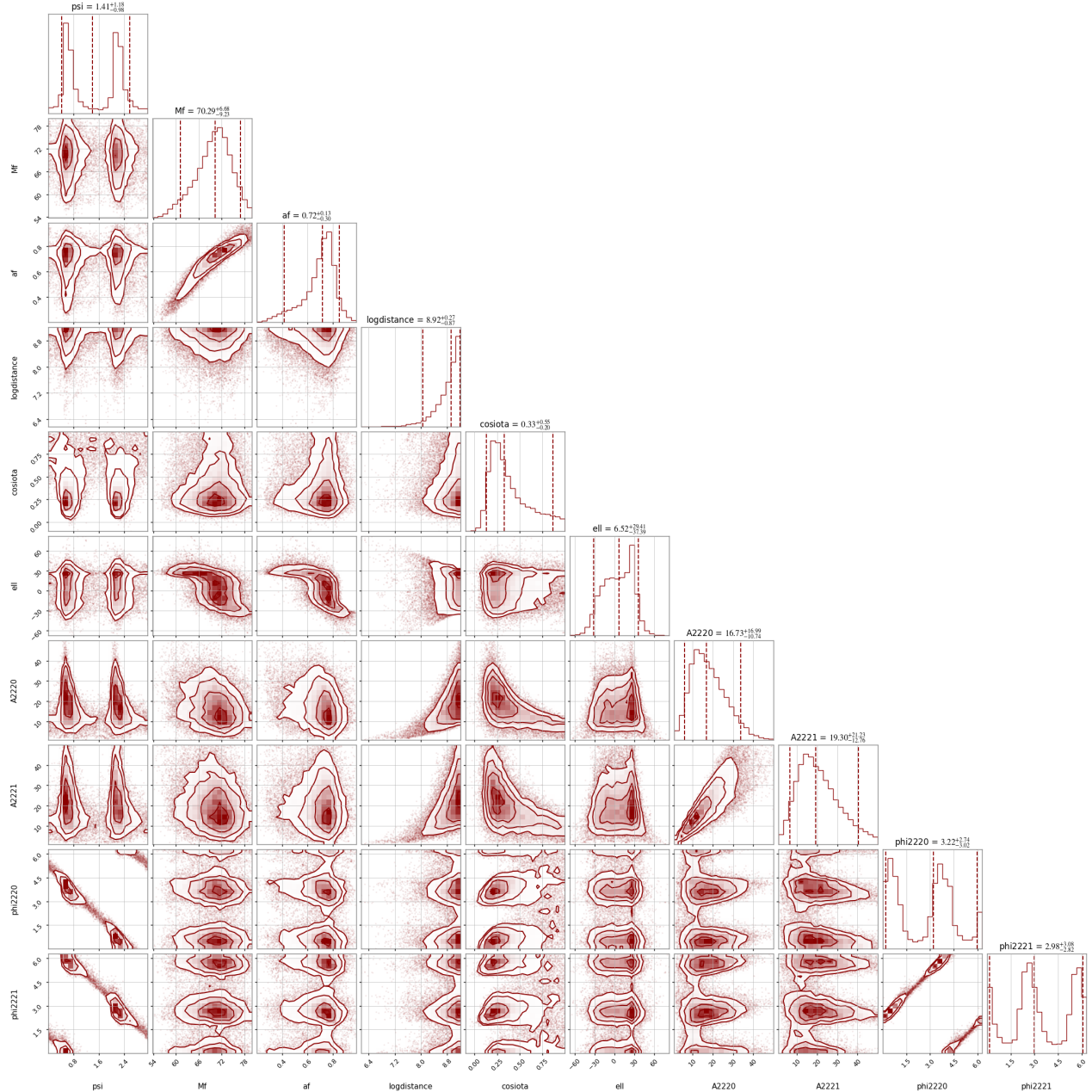


Figure 4.5: Corner plot of constraining cubic even gravity parameter(s) using GW250114. $\text{sign}(\lambda) \cdot \ell$ (ell) denotes the length scale of the coupling.

During the writing of this thesis, others have studied this event with spin expansion results for cubic even gravity. This means one uses the proper gravitational corrections, but limited in spin range. The remnant spin of this event is $a \approx 0.7$ (as is generally expected

from NR [25, 65]). This means the spin expansion results are somewhat accurate, but lose accuracy when the nested sampler is in the ‘large spin’ part of the parameter space [5, 59]. These results have not been published yet, but constraints are better than the ones here, with $\ell \lesssim 25$ km [222]. This suggests that for accurate tests of EFT scenarios, the gravitational corrections are necessary. However, for rough estimates, scalar proxies are sufficient [8]. One could thus for example use scalar proxies for prospect studies of next-generation detectors.

4.2.2 GW231123

We have analysed the ringdown part of GW231123 in GR, sGB, dCS, and cubic even gravity. The signal has been limited to a length of 0.2 s. Furthermore the signal has been cleaned by removing the glitches from the data [214]. Their presence would otherwise bias the noise characterisation, which uses the signal-free segment before the event [111, 126, 214]. The following parameters have been fixed for the priors [214]:

```
mf-time=297.81734744639897 (the mass used in the time after merger)
ra=3.236646265583545 (right ascension sky coordinate)
dec=0.24590662887894865 (declination sky coordinate)
phi=0 (azimuthal orientation GW)
M ∈ [120, 400] M⊙ (mass of remnant)
a/M ∈ [0.2, 0.99] (spin of remnant)
```

The first four parameters are fixed to the values of the LVK analysis [214]. The mass and spin range have been limited to be around the values expected from inspiral-merger-ringdown data. By doing so, unphysical values are avoided. The ranges are however taken sufficiently broad such that the posteriors fall off within the prior boundaries [214]. The starting time has been chosen as $t_0 = 6M$, following the analysis of LVK [214].

Figures 4.7, 4.8, and 4.9 visualise the posteriors obtained using GW231123 to probe EFT corrections. Figure 4.6 visualises the posterior within GR. The corner plots have the same structure as the GW250114 ones. Notice first of all that the remnant black hole has a large mass of order $300M_{\odot}$, with a large spin of order $a/M \sim 0.8$. This higher remnant spin and troublesome nature of the signal make it interesting to study this event [214].

Constraints of EFT quantities are less stringent in these analyses, compared to GW250114. For sGB we have a constraint around $\ell \lesssim 200.0$ km, for dCS around $\ell \lesssim 101.8$ km, and for cubic even gravity $\ell \cdot \text{sign}(\lambda) \in [-142.6, 160.3]$ km. When converting to λ and using the median value for the black hole mass, one finds respectively: $\lambda \lesssim 0.024$, $\lambda \lesssim 0.002$, and $\lambda \in [-0.008, 0.012]$. This loss in constraining power can be explained. The remnant black hole is 4 – 5 times more massive than the black hole measured in GW250114. As corrections scale as $\lambda \sim M^{-4}$, increasing the mass suppresses EFT corrections [5]. If one looks at the λ constraints for GW231123, one notices that they are a little tighter than the GW250114 observation for dCS and cubic even gravity, and only slightly less stringent for sGB. This means that the constraining power on λ is roughly similar, but that the scaling with the mass suppresses the deviations. This comparable constraining power on λ originates from two effects that compete with each other. First, the SNR is less in this scenario (estimated ringdown SNR of 17.4 compared to 25.8, at $t_0 = 6M$). There

is no confirmed overtone because of this [11, 214]. This makes ringdown-only tests less powerful, because one needs multiple frequencies to test the Kerr hypothesis [25]. The lack of having a second mode reliably detected strains this analysis. Second, the remnant spin of GW231123 is higher, amplifying EFT corrections [2, 3, 214]. This makes the constraining power of GW231123 stronger, offsetting the lower SNR.

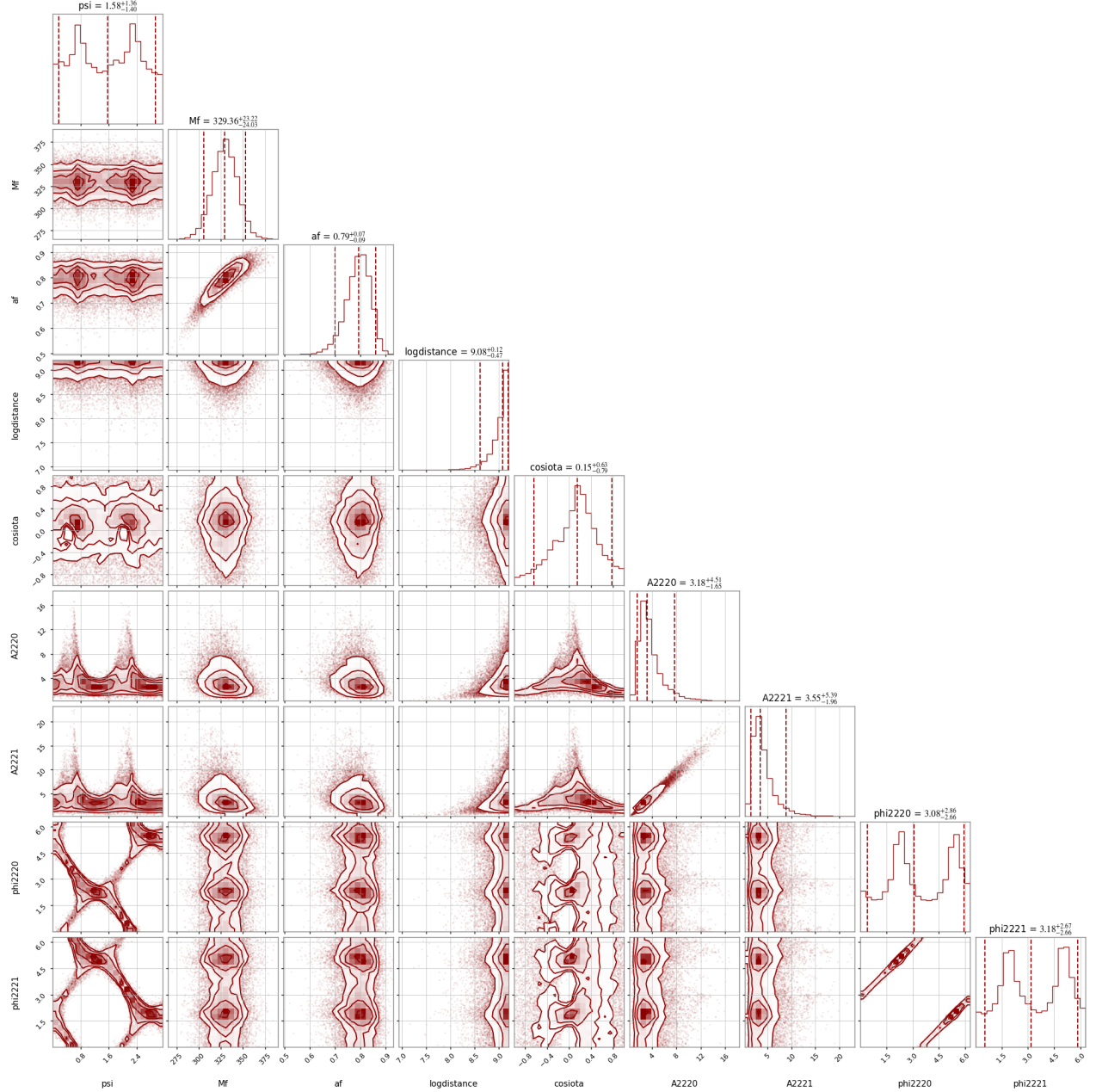


Figure 4.6: Corner plot of GW231123, within GR, constraining astrophysical quantities.

Despite the constraints being worse than those from GW250114, the posterior is well supported around $\ell = 0$, corresponding to no possible deviation from GR being observed. This is once again confirmed by the Bayes factors, which prefer GR, as their logarithms are negative. The Bayes factors for sGB, dCS and cubic even gravity are, respectively, $\ln(\mathcal{B}_{sGB}) = -1.1$, $\ln(\mathcal{B}_{dCS}) = -1.5$, and $\ln(\mathcal{B}_{cbc}) = -0.9$.

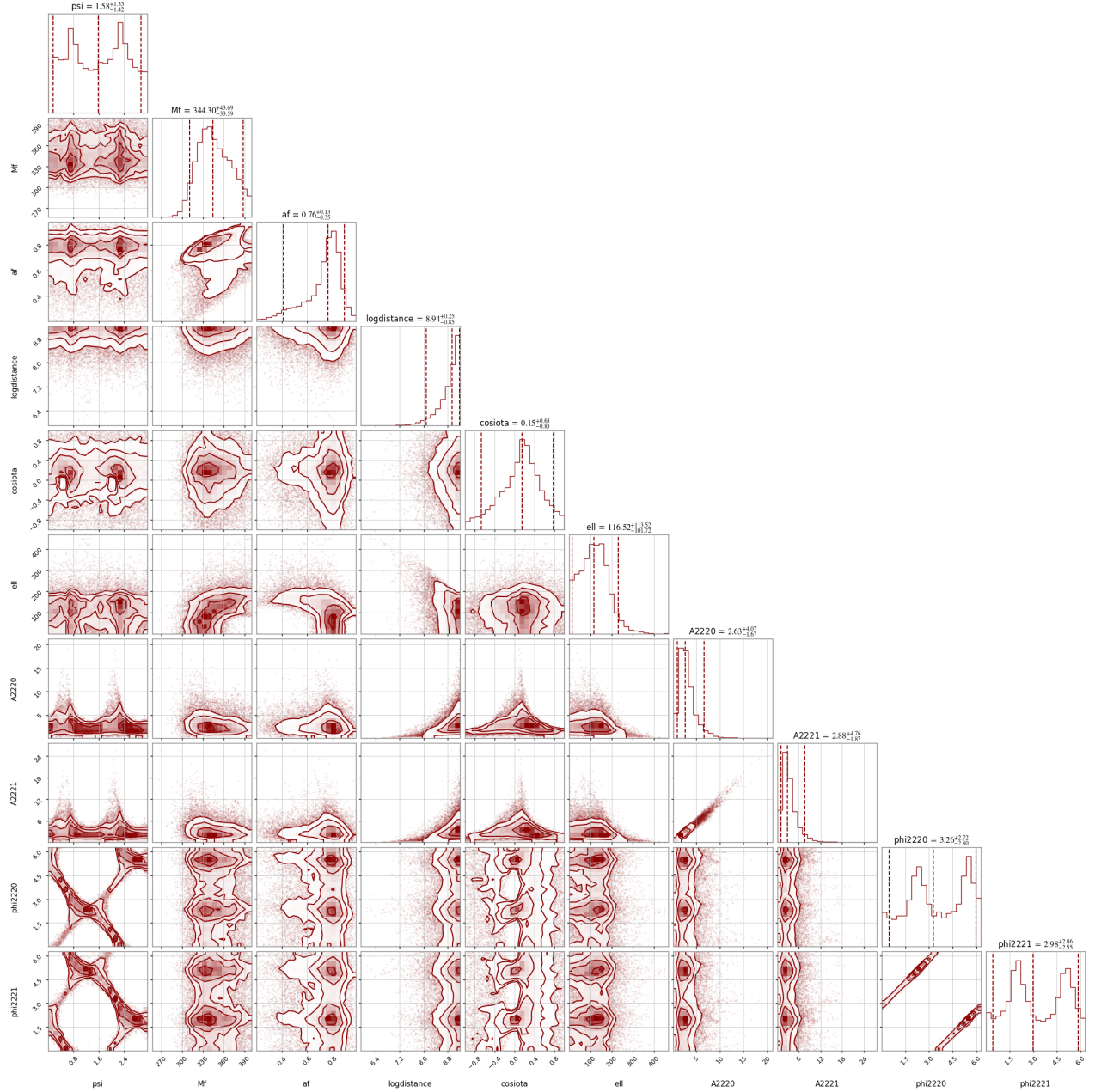


Figure 4.7: Corner plot of constraining scalar Gauss-Bonnet parameter(s) using GW231123. ℓ (ell) denotes the length scale of the coupling.

Combining this observation GW231123 with the previous one GW250114, and if we believe that the scalar proxy is sufficiently accurate, we can constrain beyond GR scenarios, and we find no observational proof of GR being invalid, as expected.

Next-generation detectors are required for high-precision tests, where one can further constrain EFT coupling parameters. Because constraints change notably when using gravitational modes, compared to the scalar proxy (see Section 4.2.1), we can postulate that scalar proxies are insufficient in making very accurate constraints. As such, calculating gravitational modes for high spins would be a useful next step for next-generation detectors.

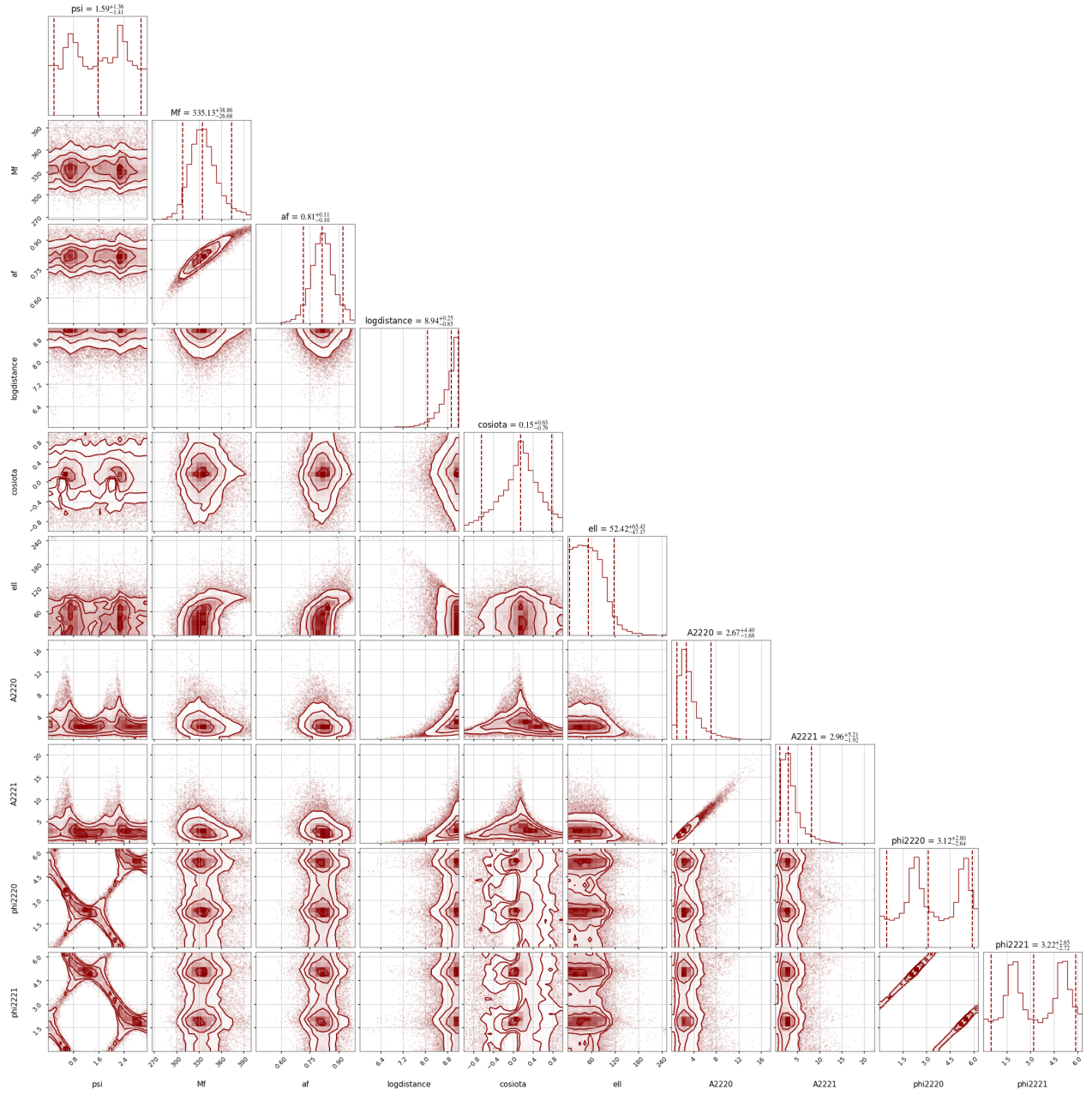


Figure 4.8: Corner plot of constraining dynamical Chern-Simons parameter(s) using GW231123. ℓ (ell) denotes the length scale of the coupling.

We notice in studying these gravitational wave events that the most important aspect of constraining EFT deviations is the mass of the remnant black hole: the lower the mass, the larger the constraining power. The loudness (SNR) of the signal and the spin of the remnant are important as well but subdominant, because of the M^{-4} scaling dominating. If the black hole has high spin and/or the event high SNR, the constraints will be more stringent. The best probes are the events with relatively small mass, which are additionally quite loud and with high remnant spin. There is a minimum requirement for the SNR, in that evidence of an overtone should exist.

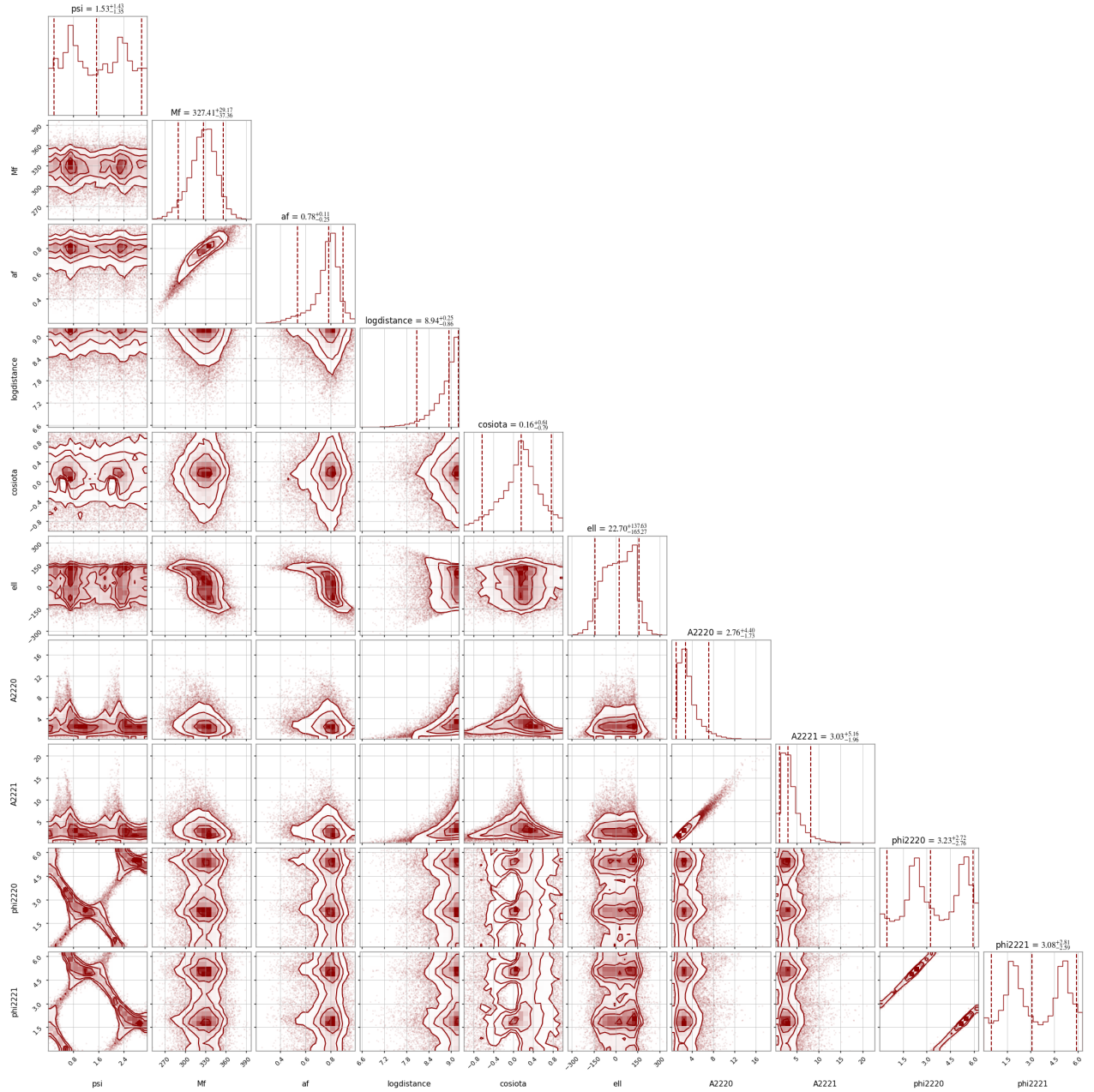


Figure 4.9: Corner plot of constraining cubic even gravity parameter(s) using GW231123. $\text{sign}(\lambda) \cdot \ell$ (ell) denotes the length scale of the coupling.

4.3 Are Scalar Proxies Valid?

In the previous Sections 4.1 and 4.2, we studied the deviation of QNMs from GR in observational data. For this we used a scalar proxy, and justified this following recent literature [8]. However, they remain approximations and it is crucial to evaluate the accuracy of these approximations. Because high spin results are not fully available for gravitational EFT deviations, it is difficult to make accurate statements on the capability of scalar modes to approximate gravitational ones. Nevertheless, Figures 4.1a and 4.1b visualised how close the scalar deviations are to the gravitational ones and already revealed that the

two can differ substantially. Furthermore, in using spin expansion results on GW250114, one found different constraints. Although these results are biased for high spins due to the absence of gravitational mode calculations in this spin regime, this should not be crucial as the spin of the remnant of this event is within the regime of validity of the spin expansion results [11, 51].

The analysis of [8] was based on the idea that agnostic tests could not differentiate between the model and its proxy. The accuracy with which one can measure the deviation of frequencies is $\delta\Delta\omega = X|1 + \Delta\omega|$ [8]. Here X is the precision of the frequency measurement, $\Delta\omega$ is the relative deviation from the GR mode, $\delta\Delta\omega$ is the absolute error in the determination of the relative deviation from GR. For GW250114, we have $X = 4\%$ for the real part and $X = 10\%$ for the imaginary part [8]. If one uses this, the differences between the scalar proxy and the gravitational corrections fall within the accuracy with which one can measure the frequencies and damping times, with the constraints on λ one gets from these events [8]. In other words, the difference between the scalar proxy and the gravitational mode is smaller than $\delta\Delta\omega$. This follows automatically from the facts that theory-specific searches have more constraining power than agnostic ones, that we have not measured beyond GR corrections, and that the proxy and gravitational corrections are of roughly the same order of magnitude. The first two conditions ensure that EFT corrections remain below the sensitivity of an agnostic search. The third places the scalar corrections below the same threshold, such that the discrepancy between the two also remains undetectable. This analysis however is insufficiently stringent for comparing proxies for EFT scenarios, because this analysis does not properly take into account the benefit of theory-specific searches in multimode analysis. When one measures multiple modes, the dimensionality of the parameter space of the theory-specific search is significantly reduced compared the agnostic search, which studies each mode separately. The requirement of [8] does not take this advantage into account sufficiently. However, for phenomenological metrics, like the Johannsen metric, this approach is more sensible as there are no concrete dynamics for the theory to determine the exact gravitational QNMs with. This is arguably the true use case of this paper [8].

The requirement of [8] for a proxy to be valid, is in a sense equivalent to saying that agnostic tests could not distinguish the two. However, this does in no way guarantee that constraints from the proxy are comparable to the constraints from the gravitational modes, which is, in our view, a requirement for a proxy to be valid. To illustrate this point, imagine that one takes a proxy $v = 0.1u$, with u the proper relative deviation for a certain λ and v the proxy, and with $u < 4\%$ across the spin range, such that it falls within agnostic bounds. v would be seen as a good proxy according to [8], because their difference $u - v = 0.9u < 3.6\% < X|1 + u| = \delta\Delta\omega$ (with $X = 4\%$ or $X = 10\%$) falls within the accuracy of the deviation measurement. However, the constraining power differs by an order of magnitude.

This is what is observed in our analyses as well. The constraining power in λ differs greatly between scalar proxies and gravitational results, as the corrections can be of a different order of magnitude. This effect is diminished by the $\ell \sim \lambda^{1/4}$ scaling, but leads to visibly different constraints on l , as mentioned in Section 4.2.1. Furthermore, the breaking of isospectrality and the scalar proxy having opposite sign relative to the gravitational

corrections over part of the parameter space might also make the proxy less valid. These conclusions slightly nuance the findings of [8]. Scalar proxies are insufficient for high-precision tests of GR, and should be understood as estimates of the constraining power, resulting in indicative constraints. For next-generation detectors, using gravitational results for high spin would be preferred.

Despite the finding that scalar proxies have limited applicability for next-generation detectors and high-precision gravitational wave astronomy, they remain a valuable tool. First and foremost, as [8] alludes to, scalar proxies are well suited for prospect studies, in the absence of gravitational results. It allows for the prediction of what constraining power next-generation detectors can produce (in ℓ). They are a good first test of the properties of the signal and of the capability of the data to measure beyond GR results [8]. Even from a more agnostic point of view, they can be valid. If one measures confidently a non-zero ℓ with scalar proxies, this signals that the data can better be described by beyond GR scenarios, even if not the theory they are a proxy of.

Apart from the observational aspects, scalar quasinormal modes in the EFT of gravity are relevant from a theoretical perspective [2, 3]. They describe many of the general features of QNMs beyond GR. We have also studied the origin of these beyond GR effects, most notably their amplification in the large spin limit, which originates from the phase boundary between ZDMs and DMs. This effect is spin-weight independent and can thus be studied using scalar modes [2–4]. This simple playing ground allows one to study general features of EFT observables.

Chapter 5

Conclusions and Outlook

Conclusion

In this thesis, we have calculated scalar quasinormal modes (QNMs) in scalar Gauss-Bonnet (sGB), dynamical Chern-Simons (dCS), cubic even gravity, axi-dilaton gravity, and cubic odd gravity for spins up to $a/M = 0.99$. The latter two are proven to be identical to the GR spectrum. We have calculated the fundamental $l = 0 - 5$ modes for all m , and first overtone $l = 2 - 5$ modes for cubic even gravity, and first overtone $l = 2$ modes for sGB and dCS, for all m , using pseudospectral methods on a spectral background [2, 3]. Results converge exponentially for all tested modes and spins. The approach presented here is valid in any higher-curvature gravity theory, as exemplified by studying five different theories, and is (in principle) valid for arbitrary spin [2, 3, 47].

As Section 3.3 describes, spin expansion results are shown to be unreliable for high spins (above $a/M \gtrsim 0.7$), where backgrounds calculated with spectral methods become essential [61–63]. The results reveal that effective field theory (EFT) corrections generally increase with spin, independent of the theory, if symmetry allows the corrections to be non-zero [2, 3]. This supports the idea that rapidly rotating black holes amplify higher-curvature effects. In the large spin limit, some modes are shown to have seemingly divergent behaviour; their corrections grow multiple orders of magnitude. This indicates a breakdown in the leading-order expansion in the beyond GR coupling parameter. However, this does not necessarily imply the EFT itself breaking down, as explained in Section 3.4 [2–4].

The growth of leading-order corrections for rapidly rotating black holes can be understood from a shift in the phase boundary between zero damped modes (ZDMs) and damped modes (DMs). This phase boundary separates a region where ZDMs live from a region where only DMs arise, see Figure 3.4. This phase boundary shifts when one goes beyond GR, which changes the QNM spectrum not only quantitatively, but also qualitatively. Modes that became ZDMs in the extremal limit may now become DMs or vice versa [2–4, 25]. This results in a non-perturbative change in the spectrum close to the phase boundary, visible in the amplification of leading-order corrections of the QNMs: if one gets close to the phase boundary, one should expand not only in the EFT coupling parameter, but also in the distance from the boundary [2, 4]. This effect can be seen by studying which mode corrections grow large in the large spin limit. Because this effect is expected to arise whenever there is a phase boundary that shifts, this effect is spin-weight and theory independent [2–4].

We have studied the feasibility and usefulness of these scalar quasinormal modes in the observational search for beyond GR effects. We have implemented scalar QNM corrections as a proxy for gravitational QNM corrections into `pyRing` [7]. In doing so, we neglected breaking of isospectrality. Using the theoretically obtained results, we analysed GW250114 and GW231123 for EFT signatures. GW250114 is the loudest GW signal to date, while GW231123 remains poorly understood and features one of the fastest spinning black hole remnants on record, making both events interesting targets for this study [11, 214]. If the scalar proxy is assumed to be sufficiently accurate, sGB, dCS, and cubic even gravity corrections have been constrained with a 90% interval of $\ell_{sGB} \lesssim 40.0$ km, $\ell_{dCS} \lesssim 22.8$ km and $\ell_{cubic,even} \cdot \text{sign}(\lambda) \in [-30.9, 35.9]$ km.

Scalar corrections are found not to be an excellent proxy. Most critically, the corrections can greatly differ from the gravitational ones, even by more than an order of magnitude. Additionally they can have different signs and scalar proxies neglect isospectrality breaking. This results in different constraints, when comparing to results limited in spin that use gravitational corrections, which tempers the findings of [8], which stated that scalar modes are a sufficiently good proxy for current observations. This strengthens the proposition that gravitational corrections are necessary for high-precision gravitational wave astronomy. Scalar proxies are however useful in obtaining rough estimates, and for rough prospect studies for next-generation detectors. Alongside this, scalar QNMs do capture the high spin amplification of beyond GR effects, emphasising their theoretical importance [2, 3, 8, 177].

Outlook

Future work can extend the results for scalar QNM shifts to gravitational ones. In this sense scalar modes are an important stepping stone towards further research. The gravitational corrections for both fundamental modes and overtones have to be calculated for rapidly rotating black holes in multiple theories, most notably sGB, dCS, cubic even and odd gravity, and quartic gravity. Currently, no numerical solutions for backgrounds of rapidly rotating black holes in quartic gravity exist publicly. Subsequent work should focus on making such spectral backgrounds publicly available and use them to calculate the scalar and gravitational mode corrections. Next steps could include the comparison of these results with earlier results in the eikonal limit of highly spinning black holes in quartic gravity [4].

From an observational perspective, the code adaptations developed in this thesis for using scalar proxies to test GR observationally should be implemented into the `pyRing` codebase of LVK. This would make the methodology available to the entire GW community and would allow for the systematic application of this pipeline to future observations [7, 122].

Last but not least, future work should examine the use of scalar proxies for a prospect analysis of next-generation detectors. This would predict what curvature scales can be probed using next-generation ringdown signals, and it would predict the constraints that could be put on EFT corrections.

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Appendix A

Code of conduct and transparency statement on the use of GenAI

The table below describes the use of GenAI in this thesis.

GenAI assistance used as/for:	Name of the GenAI tool(s) used. If helpful, also describe in which way you were using GenAI related to what is specified as code of conduct.	Code of conduct: For each of the categories below, always take into account the important guidelines and remarks mentioned above (e.g. copyrighted data, sensitive or personal data,...).
As a language assistant for reviewing or improving texts I wrote myself	Yes, ChatGPT, Gemini and Claude. I have used ChatGPT and Claude to check spelling, grammatical structure and language consistency. Additionally I have used mostly Claude, but also ChatGPT, to restructure some sentences I wrote myself. I have also used Gemini to find synonyms. Gemini has only been used in the sense that it shows automatically when using Google.	This use is similar to using spelling and grammar check tools. In general, you do not have to refer to such kind of GenAI use in the text. However, be careful: - When using GenAI tools on texts you did not write yourself to improve the text, you have to refer to the original source or author , otherwise you are committing plagiarism and thus an irregularity.

As a search engine to get information on a topic or to search for existing research on the topic	Yes, ChatGPT, Gemini and Claude. I have used mostly ChatGPT, but also Claude, to get a general overview before going into the literature for a more detailed study and a more scientific understanding. This search often complemented a search on sites like Wikipedia. Gemini has only been used in the sense that it shows automatically when using Google.	This use is similar to e.g. a Google search or checking Wikipedia. If you write your own text based on this information, you do not have to refer to the use of GenAI in the text. You only have to refer to the existing research and references you have checked and used (without such references you would be committing plagiarism and thus committing an irregularity). However, be careful: - Be aware that the output of the GenAI tool cannot be guaranteed as a 100% reliable source of information. The output may not be entirely correct and/or be limited due to the databases it uses. Moreover, knowledge evolves and may change over time; therefore, the database of the GenAI tool may not be up to date. Therefore, verify the information and do not just copy-paste it as you should understand and critically process everything you are writing.
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For literature search	Yes, ChatGPT. I have used ChatGPT to find sources on the topics relevant for the thesis. This was done in addition to a traditional literature search. I additionally used it to make sure I did not miss any important literature results.	This use is comparable to e.g. a Google Scholar search. You do not have to refer to such kind of GenAI use; you only have to refer to the literature references you have checked and used (without such references you would be committing plagiarism and thus committing an irregularity).. However, be careful: - Be aware that the search output is restricted to the database the GenAI tool is built on. After this initial search, look for scientific sources and conduct your own analysis of the source documents. Interpret, analyse and process the information you obtained; verify it and do not just copy-paste it as you should understand and critically process everything you are writing. - Be aware that some GenAI tools may output no or wrong references. As a student you are responsible for further checking and verifying the absence or correctness of references; do not just copy-paste it.
For generating programming code	Yes, ChatGPT. I have used this to help the set-up and installation of necessary packages and tools for the programming code. In addition it has been used for the framework for plotting the results, and saving and extracting data from output files.	Use of GenAI for coding is allowed except when it is prohibited by the teacher or the program of study. If used for coding, correctly mention the use of GenAI assistance in accordance with the instructions on the page on being transparent about the use of GenAI.

Appendix B

GW150914

In this section, one can find the relevant posteriors for the analysis of GW150914. This was the first gravitational wave event ever detected [9]. We have analysed it to constrain beyond GR coupling parameters. Below, one can find the corner plots for GR, sGB, dCS, and cubic even gravity. We used the following parameters for the priors:

```
mf-time=67.92493161247017 (the mass used in the time after merger)
ra=1.1578762186634035 (right ascension sky coordinate)
dec=-1.191080211569962 (declination sky coordinate)
phi=0 (azimuthal orientation GW)
M ∈ [20, 100]M⊙ (mass of remnant)
a/M ∈ [0.1, 0.99] (spin of remnant)
```

The first four parameters are fixed to the values of the LVK analysis [195]. The starting time is fixed to $t_0 = 6M$ after the peak strain, to be consistent with the other events examined in this thesis. The mass and spin priors have been chosen rather broad around the values expected from inspiral-merger-ringdown data. By doing so, unphysical values are avoided. The ranges are however taken sufficiently broad such that the posteriors fall off within the prior boundaries [9, 195].

Once again, no deviations from GR have been observed. The 90% constraints of ℓ are around $\ell \lesssim 42.1$ km, $\ell \lesssim 30.7$ km, and $\ell \cdot \text{sign}(\lambda) \in [-34.8, 37.1]$ km for sGB, dCS and cubic even gravity respectively. Using the median mass value, one finds λ constraints of $\lambda \lesssim 0.028$, $\lambda \lesssim 0.011$, and $\lambda \in [-0.017, 0.022]$ respectively. These constraints are visibly worse (especially for dCS) than for the other events. This is because there is little evidence for an overtone, whilst the remnant spin is not particularly high. The Bayes factors confirm that GR is preferred above beyond GR scenarios, as the Bayes factors are $\ln(\mathcal{B}_{sGB}) = -0.9$, $\ln(\mathcal{B}_{dCS}) = -1.1$, and $\ln(\mathcal{B}_{cbc}) = -1.1$, for sGB, dCS, and cubic even gravity respectively.

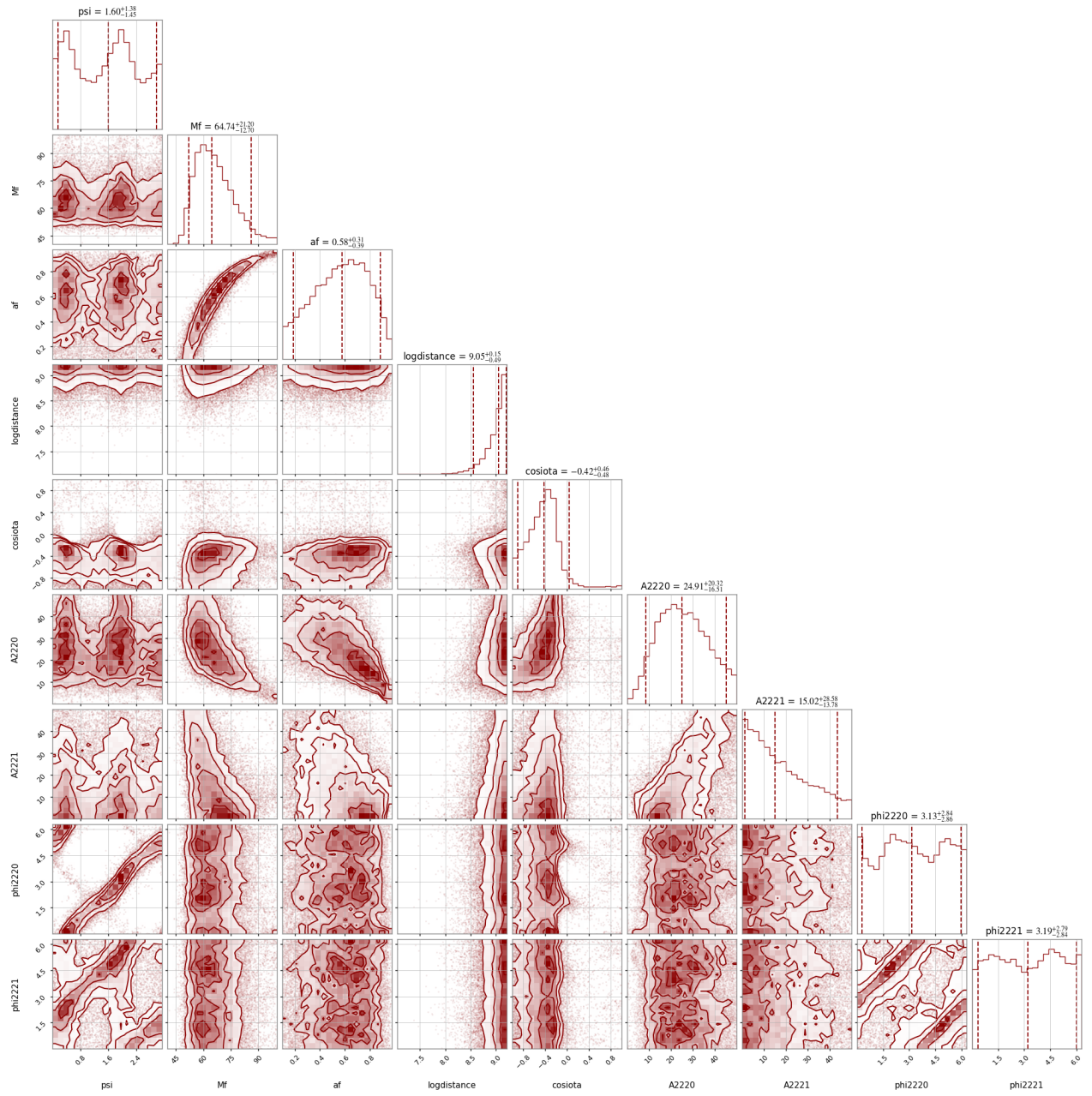


Figure B.1: Corner plot of GW150914, within GR, constraining astrophysical quantities.

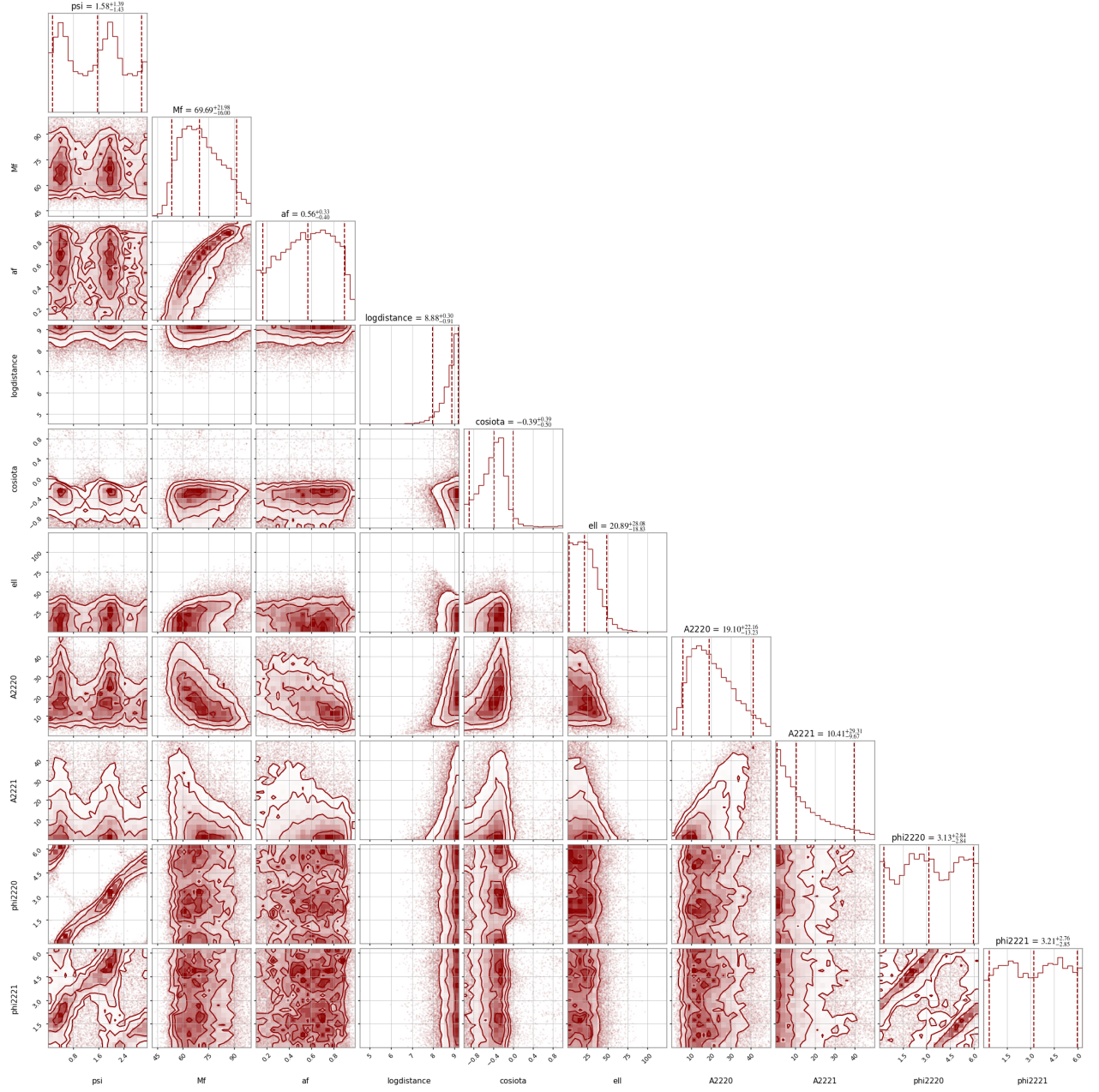


Figure B.2: Corner plot of constraining scalar Gauss-Bonnet parameter(s) using GW150914. ℓ (ℓ) denotes the length scale of the coupling.

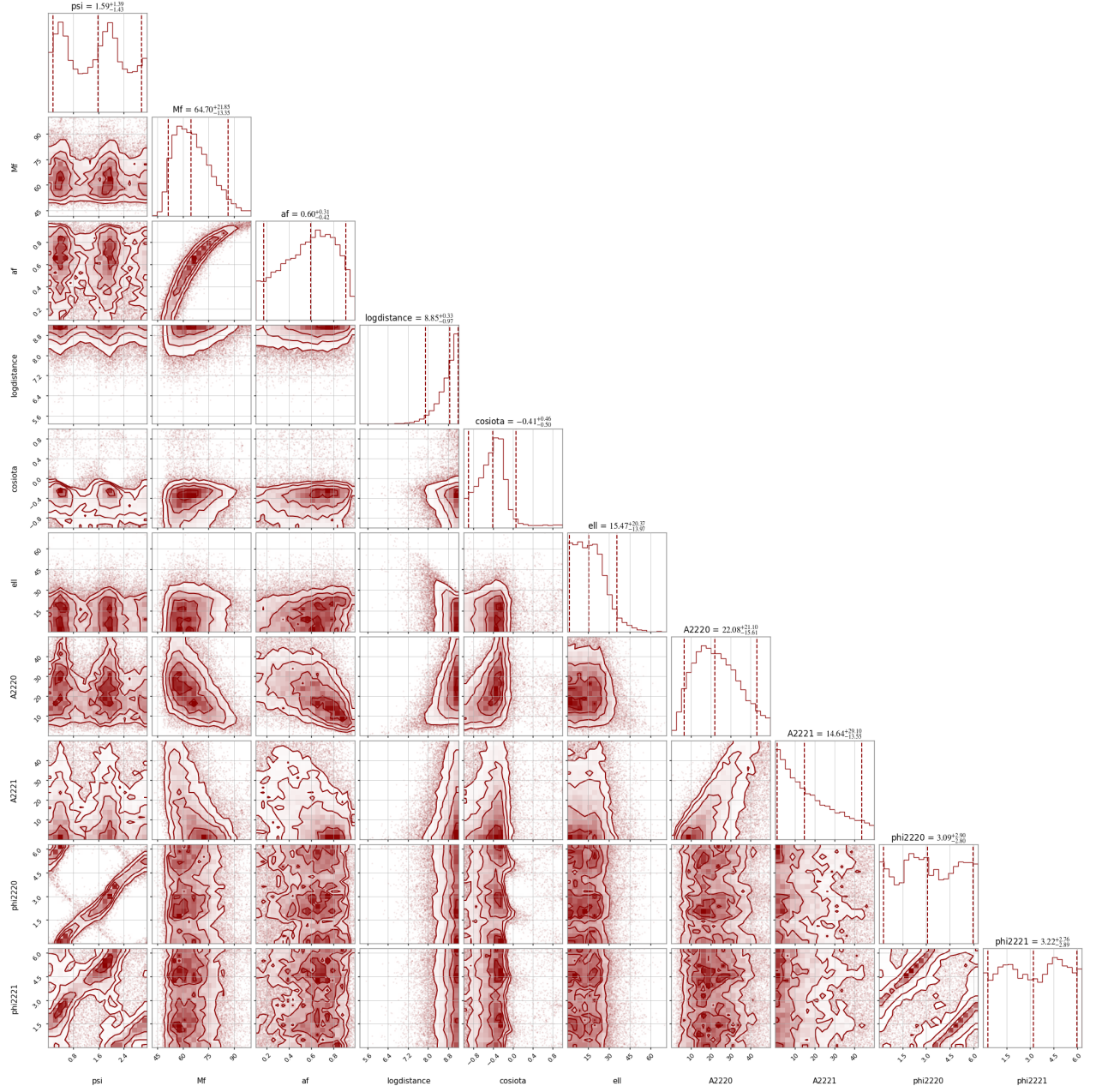


Figure B.3: Corner plot of constraining dynamical Chern-Simons parameter(s) using GW150914. ℓ (ell) denotes the length scale of the coupling.

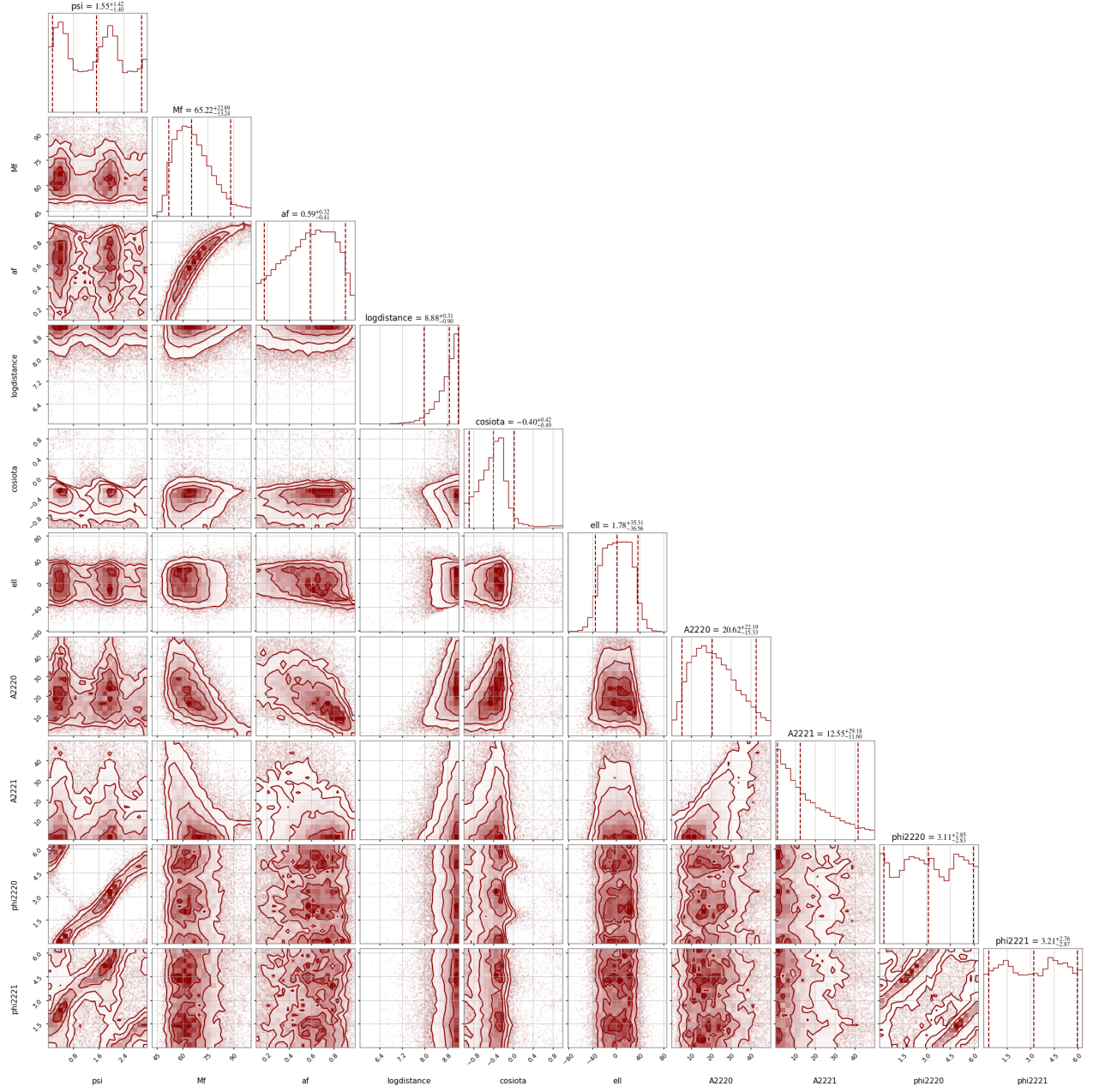


Figure B.4: Corner plot of constraining cubic even gravity parameter(s) using GW150914. ℓ (ell) denotes the length scale of the coupling.

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